The Second-Order Terms Remaining In A Section of Achromatic Beamline

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Outline

Introduction

- We know that some second-order terms also remaining In a Section of achromatic Beamline. Then we do the symbolic computation to find which terms are left.
- Lattice should design carefully to minimize the remaining terms.
- Only first- and second-order terms are considered.

Outline

Background

- Second-order Lie-algebraic map of Sextupole:

$$
\begin{aligned}
x_2 =& -\frac{1}{24}kL^4p_x^2 + \frac{1}{24}kL^4p_y^2 - \frac{1}{6}kL^3xp_x + \frac{1}{6}kL^3yp_y - \frac{1}{4}kL^2x^2 \\
&+ \frac{1}{4}kL^2y^2 + \frac{Lp_tp_x}{\beta_0} \\
p_{x_2} =& -\frac{1}{6}kL^3p_x^2 + \frac{1}{6}kL^3p_y^2 - \frac{1}{2}kL^2xp_x + \frac{1}{2}kL^2yp_y - \frac{1}{2}kLx^2 \\
&+ \frac{1}{2}kLy^2 \\
y_2 =& \frac{1}{12}kL^4p_xp_y + \frac{1}{6}kL^3yp_x + \frac{1}{6}kL^3xp_y + \frac{1}{2}kL^2xy + \frac{Lp_tp_y}{\beta_0} \\
p_{y_2} =& \frac{1}{3}kL^3p_xp_y + \frac{1}{2}kL^2yp_x + \frac{1}{2}kL^2xp_y + kLxy
\end{aligned}
\tag{1}
$$

Background

- Second-order Lie-algebraic map of quartupole:

$$
\begin{aligned}
x_2 =& -\frac{\left(\sqrt{k}l\right)\sin\left(\sqrt{k}l\right)}{2\beta_0}xp_t + \frac{\frac{\sin(\sqrt{k}l)}{2\beta_0} + \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}}{\sqrt{k}}p_xp_t \\
p_{x_2} =& \sqrt{k}\left(\frac{\sin\left(\sqrt{k}l\right)}{2\beta_0} - \frac{\left(\sqrt{k}l\right)\cos\left(\sqrt{k}l\right)}{2\beta_0}\right)xp_t - \frac{\left(\sqrt{k}l\right)\sin\left(\sqrt{k}l\right)}{2\beta_0}p_xp_t \\
y_2 =& \frac{\left(\sqrt{k}l\right)\sinh\left(\sqrt{k}l\right)}{2\beta_0}yp_t + \frac{\frac{\sinh(\sqrt{k}l)}{2\beta_0} + \frac{(\sqrt{k}l)\cosh(\sqrt{k}l)}{2\beta_0}}{\sqrt{k}}p_yp_t \\
p_{y_2} =& \sqrt{k}\left(-\frac{\sinh\left(\sqrt{k}l\right)}{2\beta_0} + \frac{\left(\sqrt{k}l\right)\cosh\left(\sqrt{k}l\right)}{2\beta_0}\right)yp_t + \frac{\left(\sqrt{k}l\right)\sinh\left(\sqrt{k}l\right)}{2\beta_0}p_yp_t
\end{aligned}
\tag{2}
$$

Background

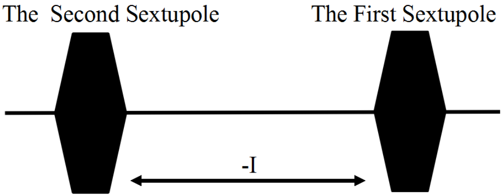
- Propagation of second-Order Terms:

$$\sum_{i=1}^{j} S_{(i)}\mathcal{F}_{(j+1-i)} = r_{ab}T_{bcd} + t_{abc}R_{cd} \quad (3)$$

Outline

Layout

- Layout of two sextupoles placed apart by $-I$ matirx.



Second-Order Terms of The First Sextupole

- Take the x component for an example:

$$
\begin{aligned}
& r_{ab}T_{bcd} \\
= & -I_{ab}T_{bcd} \\
\Rightarrow & -\{-\frac{1}{24}kL^4p_x^2 + \frac{1}{24}kL^4p_y^2 - \frac{1}{6}kL^3xp_x + \frac{1}{6}kL^3yp_y - \frac{1}{4}kL^2x^2 \\
& + \frac{1}{4}kL^2y^2 + \frac{Lp_tp_x}{\beta_0}\}
\end{aligned}
\tag{4}
$$

Second-Order Terms of The Second Sextupole

- Take the x component for an example:

$$
\begin{aligned}
&t_{abc}R_{bd}R_{ce} \\
=&t_{abc}\{(-I_{bd})(-I_{ce})\} \\
\Rightarrow &-\frac{1}{24}kL^4(-p_x)^2+\frac{1}{24}kL^4(-p_y)^2-\frac{1}{6}kL^3(-x)(-p_x)+\frac{1}{6}kL^3(-y)(-p_y) \\
&-\frac{1}{4}kL^2(-x)^2+\frac{1}{4}kL^2(-y)^2+\frac{L(-p_x)p_t}{\beta_0}
\end{aligned}
\tag{5}
$$

Second-Order Terms of The Two Sextupole

- Take the x component for an example:

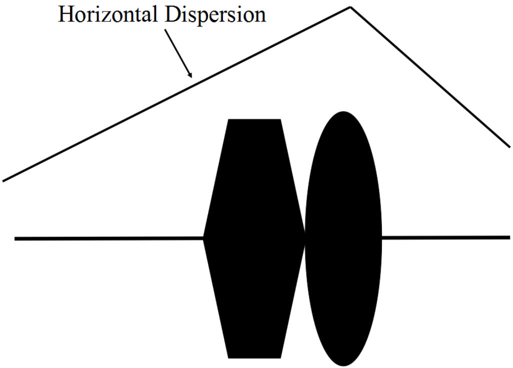
$$\begin{aligned} & r_{ab}T_{bcd} + t_{abc}R_{cd} \\ \Rightarrow & -\frac{2Lp_x p_t}{\beta_0} \end{aligned} \tag{6}$$

- Independent of the strength of sextupole.
- This term is consistent with drift.

Outline

Layout

- Layout of local achromaticity.



Second-Order Terms of Sextupole With Dispersion

- Suppose that dispersion function: D_x, D_{px}.
- Take the x component for an example.
- Second-order terms of sextupole with dispersion:

$$
\begin{aligned}
&-\frac{1}{24}(kl^4)p_x^2 \Rightarrow -\frac{1}{24}(kl^4)(p_x + D_{px}p_t)^2 = -\frac{1}{24}(kl^4)(p_x^2 + D_{px}^2 p_t^2 + 2D_{px}p_x p_t) \\
&-\frac{1}{6}(kl^3)xp_x \Rightarrow -\frac{1}{6}(kl^3)(x + D_x p_t)(p_x + D_{px}p_t) = -\frac{1}{6}(kl^3)(xp_x + D_{px}xp_t + D_x p_x p_t + D_x D_{px} p_t^2) \\
&-\frac{1}{4}(kl^2)x^2 \Rightarrow -\frac{1}{4}(kl^2)(x + D_x p_t)^2 = -\frac{1}{4}(kl^2)(x^2 + D_x^2 p_t^2 + 2D_x x p_t) \\
&\frac{l}{\beta_0}p_x p_t \Rightarrow \frac{l}{\beta_0}(p_x + D_{px}p_t)p_t = \frac{l}{\beta_0}(p_x p_t + D_{px}p_t^2)
\end{aligned} \tag{7}
$$

Second-Order Terms of Quartupole With Dispersion

- Suppose that dispersion function: D_x, D_{px}.
- Take the x component for an example.
- Second-Order terms of quartupole with dispersion:

$$-\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0}xp_t \Rightarrow -\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0}(xp_t + D_x p_t^2)$$

$$\frac{\frac{\sin(\sqrt{k}l)}{2\beta_0} + \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}}{\sqrt{k}}p_x p_t \Rightarrow \frac{\frac{\sin(\sqrt{k}l)}{2\beta_0} + \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}}{\sqrt{k}}(p_x p_t + D_{px} p_t^2) \tag{8}$$

Terms Left In Local Achromaticity

- Classify the same order:

$$xp_t : -\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0} = -\frac{kl^2}{2} + \frac{k^2l^4}{12} + ... = -\frac{1}{2}D_xksls^2 - \frac{1}{6}D_xksls^3$$

$$p_xp_t : \frac{\frac{\sin(\sqrt{k}l)}{2\beta_0} + \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}}{\sqrt{k}} = l - \frac{kl^3}{3} + ... = ls - \frac{1}{6}D_xksls^3 - \frac{1}{12}D_xksls^4$$

$$p_t^2 : -\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0}D_x + \frac{\frac{\sin(\sqrt{k}l)}{2\beta_0} + \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}}{\sqrt{k}}D_{px}$$

$$\{-\frac{1}{4}D_xksls^2\}D_x + \{ls - \frac{1}{6}D_xksls^3 - \frac{1}{24}D_{px}ksls^4\}D_{px} \quad (9)$$

Terms Left In Local Achromaticity

- Left terms in the condition: $-ksD_x = k$.
- Take the x component for an example.

$$
\begin{aligned}
&-\frac{1}{24}(kl^4)p_x^2 \Rightarrow -\frac{1}{24}(kl^4)(p_x + D_{px}p_t)^2 = -\frac{1}{24}(kl^4)(p_x^2 + \underline{\underline{D_{px}^2}}p_t^2 + \underline{\underline{2D_{px}p_xp_t}}) \\
&-\frac{1}{6}(kl^3)xp_x \Rightarrow -\frac{1}{6}(kl^3)(x + D_xp_t)(p_x + D_{px}p_t) = -\frac{1}{6}(kl^3)(xp_x + \underline{\underline{D_{px}xp_t}} + \underline{\underline{D_xp_xp_t}} + \underline{\underline{D_xD_{px}p_t^2}}) \\
&-\frac{1}{4}(kl^2)x^2 \Rightarrow -\frac{1}{4}(kl^2)(x + D_xp_t)^2 = -\frac{1}{4}(kl^2)(x^2 + \underline{\underline{D_x^2p_t^2}} + 2D_xxp_t) \\
&\frac{l}{\beta_0}p_xp_t \Rightarrow \frac{l}{\beta_0}(p_x + D_{px}p_t)p_t = \frac{l}{\beta_0}(p_xp_t + D_{px}p_t^2)
\end{aligned} \tag{10}
$$

Second-Order Terms of Sextupole With Dispersion

- Suppose that dispersion function:D_x, D_{px}.
- Take the p_x component for an example.
- Second-order terms of sextupole with dispersion:

$$
\begin{aligned}
&-\frac{1}{6}(kl^3)p_x^2 \Rightarrow -\frac{1}{6}(kl^3)(p_x + D_{px}p_t)^2 = -\frac{1}{6}(kl^3)(p_x^2 + D_{px}^2 p_t^2 + 2D_{px}p_x p_t) \\
&-\frac{1}{2}(kl^2)xp_x \Rightarrow -\frac{1}{2}(kl^2)(x + D_x p_t)(p_x + D_{px}p_t) = -\frac{1}{2}(kl^2)(xp_x + D_{px}xp_t + D_x p_x p_t + D_x D_{px} p_t^2) \\
&-\frac{1}{2}(kl)x^2 \Rightarrow -\frac{1}{2}(kl)(x + D_x p_t)^2 = -\frac{1}{2}(kl)(x^2 + D_x^2 p_t^2 + 2D_x x p_t)
\end{aligned}
\tag{11}
$$

Second-Order Terms of Quartupole With Dispersion

- Suppose that dispersion function: D_x, D_{px}.
- Take the p_x component for an example.
- Second-Order terms of quartupole with dispersion:

$$\sqrt{k}\left(\frac{\sin(\sqrt{k}l)}{2\beta_0}-\frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}\right)xp_t \Rightarrow \sqrt{k}\left(\frac{\sin(\sqrt{k}l)}{2\beta_0}-\frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}\right)(xp_t+D_x p_t^2)$$
$$-\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0}p_x p_t \Rightarrow -\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0}(p_x p_t + D_{p_x} p_t^2) \tag{12}$$

Terms Left In Local Achromaticity

- Classify the same order:

$$
\begin{aligned}
xp_t &: \sqrt{k}\left(\frac{\sin(\sqrt{k}l)}{2\beta_0} - \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}\right) = \frac{k^2l^3}{6} - \frac{k^3l^5}{60} + \ldots = -kslsD_x - \frac{1}{2}ksls^2D_{px} \\
p_xp_t &: -\frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0} = -\frac{kl^2}{2} + \frac{k^2l^4}{12} + \ldots = -\frac{1}{2}ksls^2D_x - \frac{1}{3}ksls^3D_{px} \\
p_t^2 &: \sqrt{k}\left(\frac{\sin(\sqrt{k}l)}{2\beta_0} - \frac{(\sqrt{k}l)\cos(\sqrt{k}l)}{2\beta_0}\right)D_x - \frac{(\sqrt{k}l)\sin(\sqrt{k}l)}{2\beta_0}D_{px} \\
&\{-\frac{1}{2}D_xksls\}D_x + \{-\frac{1}{2}D_xksls^2 - \frac{1}{6}D_{px}ksls^3\}D_{px}
\end{aligned} \tag{13}
$$

Terms Left In Local Achromaticity

- Left terms in the condition: $-ksD_x = k$.
- In the p_x component:
- Can not cancel each other out.
- Why?

$$
\begin{aligned}
&-\frac{1}{6}(kl^3)p_x^2 \Rightarrow -\frac{1}{6}(kl^3)(p_x + D_{px}p_t)^2 = -\frac{1}{6}(kl^3)(p_x^2 + \underline{\underline{D_{px}^2 p_t^2}} + \underline{\underline{2D_{px}p_x p_t}}) \\
&-\frac{1}{2}(kl^2)xp_x \Rightarrow -\frac{1}{2}(kl^2)(x + D_x p_t)(p_x + D_{px}p_t) = -\frac{1}{2}(kl^2)(xp_x + \underline{\underline{D_{px}xp_t}} + D_x p_x p_t + \underline{\underline{D_x D_{px} p_t^2}}) \\
&-\frac{1}{2}(kl)x^2 \Rightarrow -\frac{1}{2}(kl)(x + D_x p_t)^2 = -\frac{1}{2}(kl)(x^2 + \underline{\underline{D_x^2 p_t^2}} + \underline{\underline{2D_x x p_t}})
\end{aligned}
\tag{14}
$$

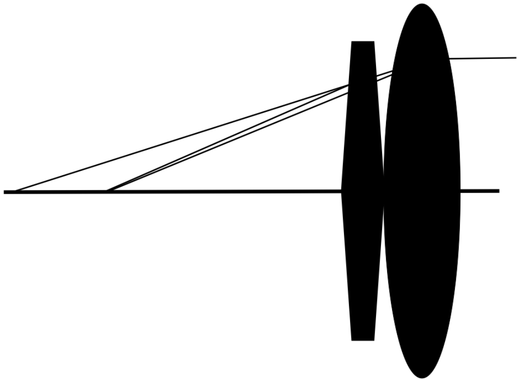
Thin Lens Approximation

- From: Performance Optimization and Optics Design Studies for the e^- e^- Option at the ILC.
- Why we don't have chromaticity term in the Lie-map?

$$
\begin{aligned}
dx' =& \frac{1}{2}k_s(x^2 - y^2)ds = \{ \underbrace{D_x p_t x}_{quadrupole\ field\ in\ sextupole} + \frac{1}{2}D_x^2 p_t^2 \\
&+ \frac{1}{2}(x^2 - y^2)\}k_s ds \\
dx' =& k_q(1 - p_t)xds = (k_q D_x p_t + k_q x - k_q D_x p_t^2 \underbrace{-k_q x p_t}_{Chromaticity})ds
\end{aligned}
\tag{15}
$$

Chromaticity and Achromaticity

- Chromaticity and Achromaticity.



My idea

- My idea: The Nature of the Computational Method.

$$H = -\sqrt{-\frac{c^2 m_0^2}{p_0^2} + (p_t + p_{t_0})^2 - p_x^2 - p_y^2} \underbrace{-\frac{1}{2}\frac{qG}{p_0}}_{k_q\ is\ not\ function\ of\ p_t} (y^2 - x^2) - \frac{p_t + p_{t_0}}{\beta_0} \qquad (16)$$

Does the Canonical Coordinate Matters?

- Does the Canonical Coordinate Matters?

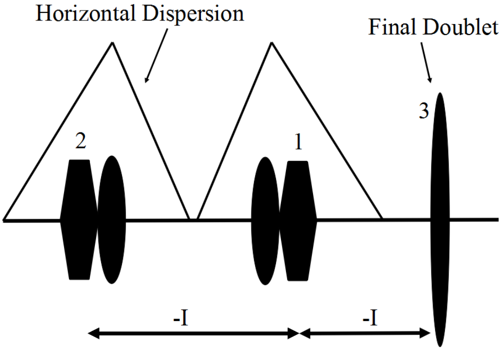
$$x' = \frac{\partial H}{\partial p_x} = \frac{p_x}{\sqrt{-p_y^2 - p_x^2 - \frac{c^2 m_0^2}{p_0^2} + (p_t + p_{t0})^2}} \quad (17)$$

$$p_x = \frac{x'}{\sqrt{1 + x'^2}} \times \sqrt{-p_y^2 - p_x^2 - \frac{c^2 m_0^2}{p_0^2} + (p_t + p_{t0})^2} \Rightarrow x' - \frac{x'^3}{2} + \frac{3x'^5}{8} + \dots \quad (18)$$

Outline

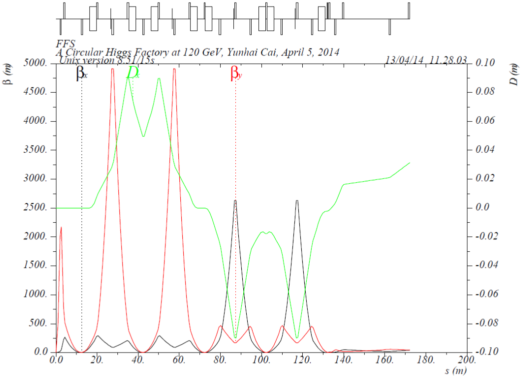
Layout

- Layout of non-local achromaticity.



Example

- From "CHARGED PARTICLE OPTICS IN CIRCULAR HIGGS FACTORY", Y Cai.



Second-Order Terms of Sextupole With Dispersion

- Suppose that dispersion function:$D_{x1} = D_{x2}$, $D_{px1} = -D_{px2}$ and $-I$ Matrix between postion 1 and 3.
- Take the x component for an example.
- Terms left in two sextupoles placed apart by $-I$ matirx :

$$
\begin{aligned}
&-\frac{1}{24}(kl^4)p_x^2 \Rightarrow -\frac{1}{24}(kl^4)(p_x + D_{px}p_t)^2 = -\frac{1}{24}(kl^4)(p_x^2 + D_{px}^2 p_t^2 + 2D_{px}p_x p_t) \\
&-\frac{1}{6}(kl^3)xp_x \Rightarrow -\frac{1}{6}(kl^3)(x + D_x p_t)(p_x + D_{px}p_t) = -\frac{1}{6}(kl^3)(xp_x + D_{px}xp_t + \underline{\underline{D_x p_x p_t}} + \underline{\underline{D_x D_{px} p_t^2}}) \\
&-\frac{1}{4}(kl^2)x^2 \Rightarrow -\frac{1}{4}(kl^2)(x + D_x p_t)^2 = -\frac{1}{4}(kl^2)(x^2 + D_x^2 p_t^2 + \underline{\underline{2D_x x p_t}}) \\
&\frac{l}{\beta_0}p_x p_t \Rightarrow \frac{l}{\beta_0}(p_x + D_{px}p_t)p_t = \frac{l}{\beta_0}(\underline{\underline{p_x p_t}} + D_{px}p_t^2)
\end{aligned}
\tag{19}
$$

Discussion

- The assumption maybe impossible.

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