Lattice calculation for the light-by-light hadronic contribution to muon g-2

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Contents

- Hadronic Light-by-Light (HLbL) on Lattice
  since [T. Blum et al 2005]
  Riken-BNL-Columbia (RBC) Collaboration
  Mainz

- (if time left) topics to g-2 HVP
  since [T. Blum 2003]

  HPQCD
  RBC/UKQCD
  Mainz
  ETMC
  BMW
  Regensburg
  FNAL
  PACS
  :
  :

Collaborators / Machines

- HLbL

Tom Blum, Norman Christ, Masashi Hayakawa, <u>Luchang Jin</u>, Chulwoo Jung, Christoph Lehner, ...

- DWF simulations including HVP
  RBC/UKQCD Collaboration
  HVP: Christoph Lehner,
  Matt Spraggs, ...

Part of related calculation are done by resources from
USQCD (DOE), XSEDE, ANL BG/Q Mira (DOE, ALCC), Edinburgh BG/Q,
BNL BG/Q, RIKEN BG/Q and Cluster (RICC, HOKUSAI)

Support from US DOE, RIKEN, BNL, and JSPS

The RBC & UKQCD collaborations

BNL and RBRC

Mattia Bruno
Tomomi Ishikawa
Taku Izubuchi
Chulwoo Jung
Christoph Lehner
Meifeng Lin
Taichi Kawanai
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni
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CERN

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Luchang Jin
Christopher Kelly
Bob Mawhinney
Greg McGlynn
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Guido Cossu
Luigi Del Debbio
Richard Kenway
Julia Kettle
Ava Khamseh
Brian Pendleton
Antonin Portelli
Oliver Witzel
Azusa Yamaguchi

KEK

Julien Frison

Peking University

Xu Feng

Plymouth University

Nicolas Garron

University of Southampton

Jonathan Flynn
Vera Guelpers
James Harrison
Andreas Juettner
Andrew Lawson
Edwin Lizarazo
Chris Sachrajda
Francesco Sanfilippo
Matthew Spraggs
Tobias Tsang

York University (Toronto)

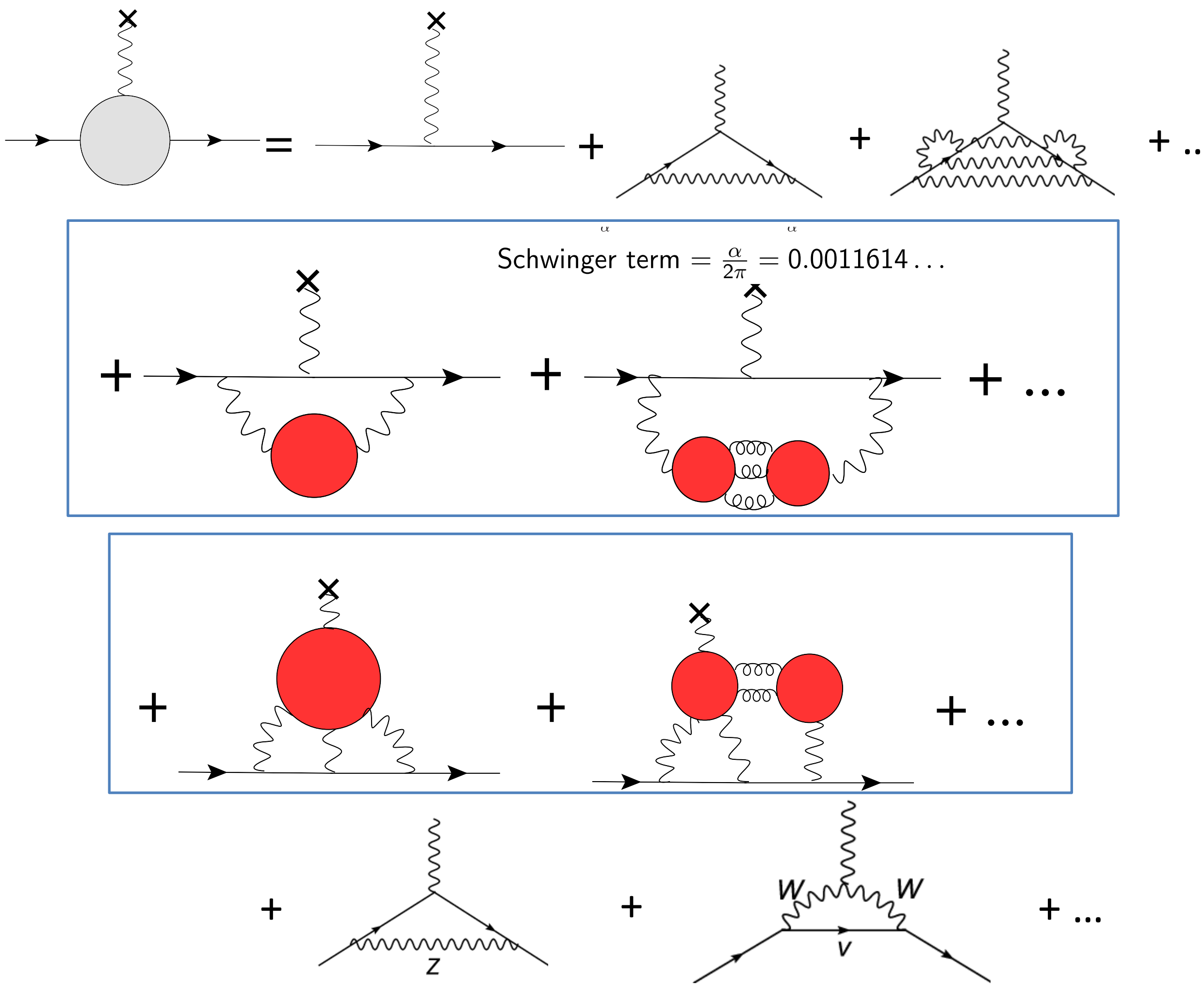
Renwick Hudspith

SM Theory

[T. Teubner's talk]

$$\gamma^\mu \;\rightarrow\; \Gamma^\mu(q) = \left(\gamma^\mu F_1(q^2) + \frac{i\,\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)\right)$$

- QED, hadronic, EW contributions



QED (5-loop)
Aoyama et al.
PRL109,111808 (2012)

Hadronic vacuum polarization (HVP)

[B. Chakraborty]

Hadronic light-by-light (Hlbl)

Electroweak (EW)
Knecht et al 02
Czarnecki et al. 02

$(g\text{-}2)_\mu$ SM Theory prediction

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

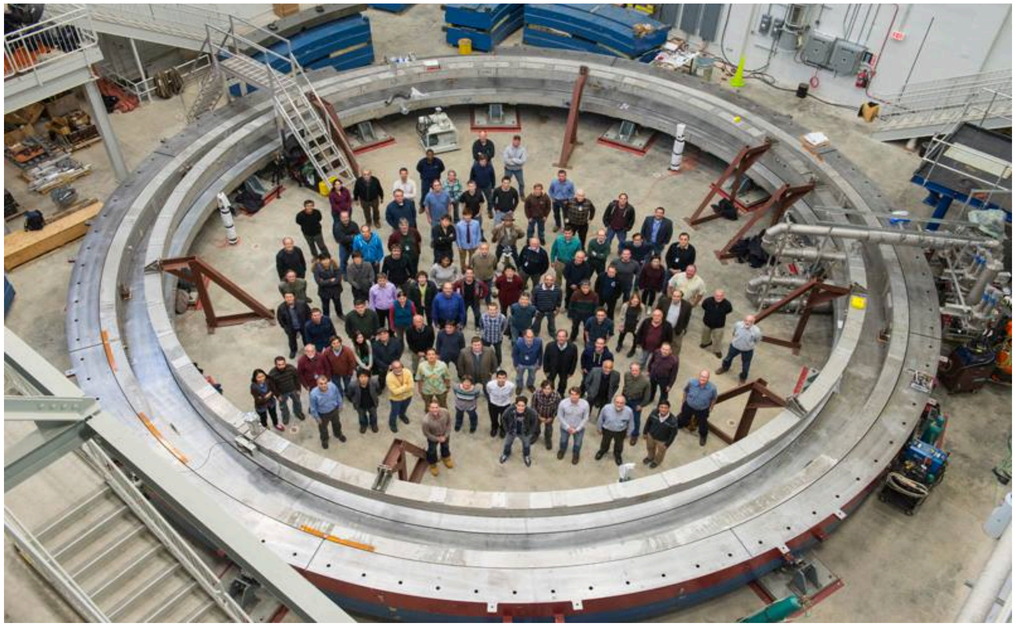
$$a_\mu^{\mathrm{SM}} = (11\ \ 659\ \ 182.8\ \ \pm 4.9) \times 10^{-10}$$

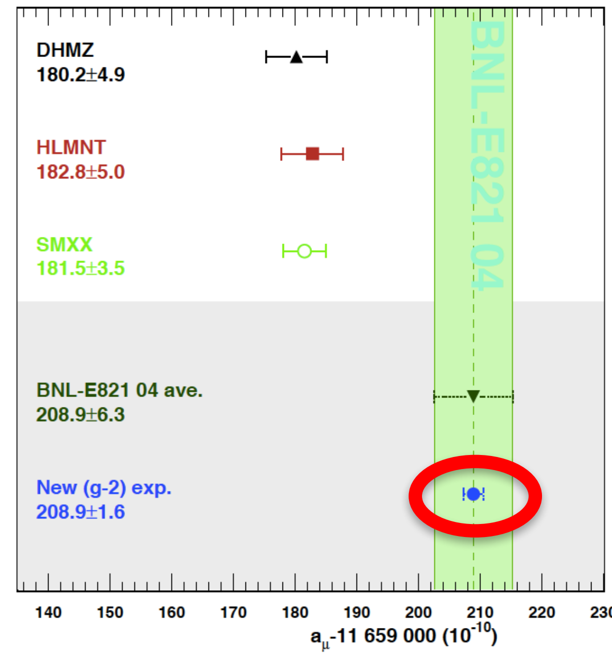
$$
\begin{aligned}
a_\mu^{\mathrm{QED}} &= (11\ \ 658\ \ 471.808\ \ \pm 0.015) \times 10^{-10} \\
a_\mu^{\mathrm{EW}} &= (15.4\ \ \pm 0.2) \times 10^{-10} \\
a_\mu^{\mathrm{had,LOVP}} &= (694.91\ \ \pm 4.27) \times 10^{-10} \\
a_\mu^{\mathrm{had,HOVP}} &= (-9.84\ \ \pm 0.07) \times 10^{-10} \\
a_\mu^{\mathrm{had,lbl}} &= (10.5\ \ \pm 2.6) \times 10^{-10}
\end{aligned}
$$

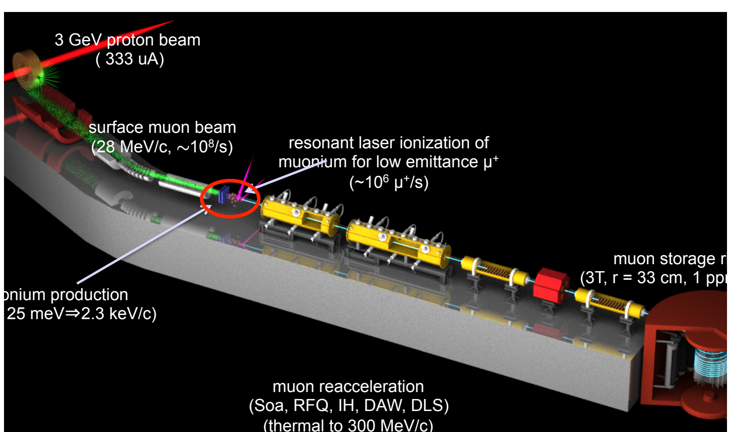
$$\boxed{a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}} = 28.8(6.3)_{\mathrm{exp}}(4.9)_{\mathrm{SM}} \times 10^{-10} \quad [3.6\sigma]}$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- x4 or more accurate experiment FNAL , J-PARC
- Goal : sub 1% accuracy for HVP, and
  → **10% accuracy for HLbL**

Near Future experiments



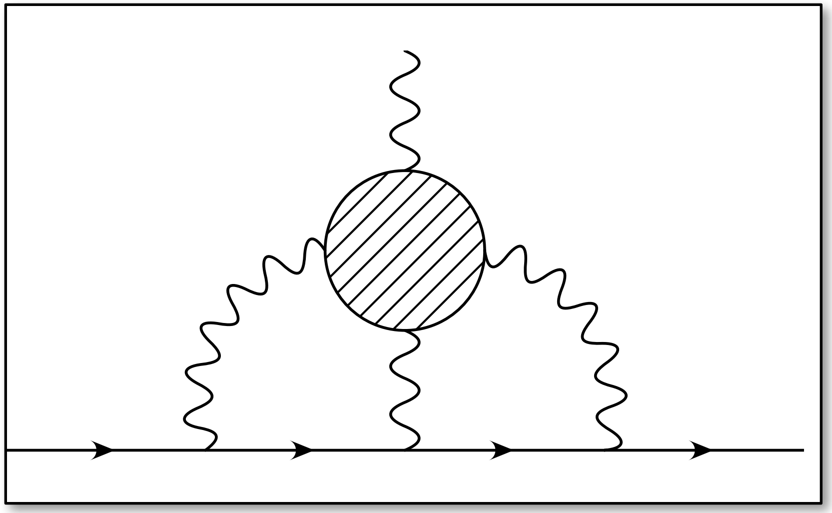




FNAL E989 (2019-) [J. Mott's talk]
- move storage ring from BNL
- x4 more precise results, 0.14ppm

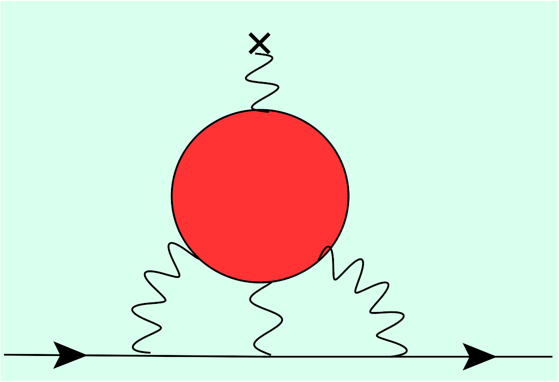
J-PARC E34 [Y. Sato's talk]
- ultra-cold muon beam
- table top storage ring

Hadronic Light-by-Light (HLbL) contributions



Hadronic Light-by-Light

- 4pt function of EM currents
- No direct experimental data available
- Dispersive approach



$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2,p_1) = ie^6\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2)}{k_1^2\,k_2^2\,k_3^2}$$
$$\times\gamma_\nu S^{(\mu)}(\not p_2+\not k_2)\gamma_\rho S^{(\mu)}(\not p_1+\not k_1)\gamma_\sigma$$

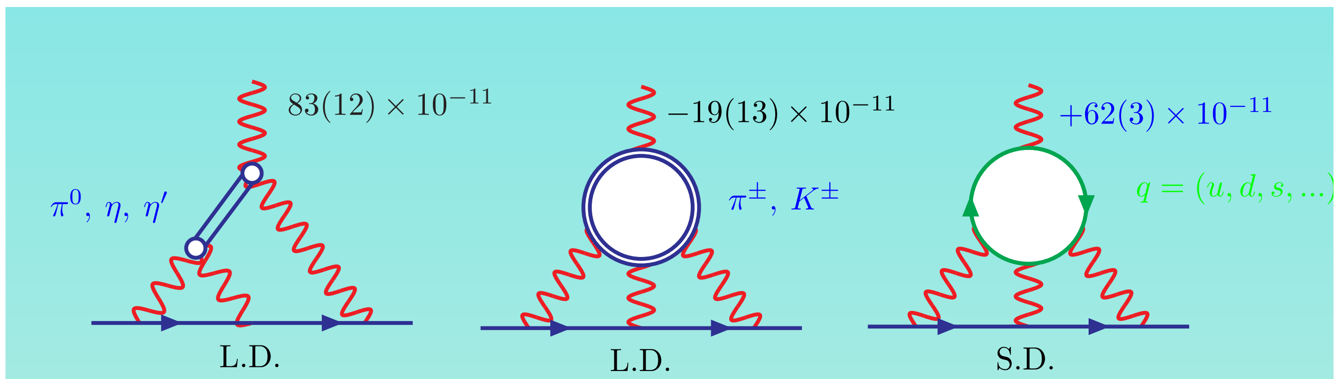
$$\Pi^{(4)}_{\mu\nu\rho\sigma}(q,k_1,k_3,k_2) = \int d^4x_1\,d^4x_2\,d^4x_3\,\exp[-i(k_1\cdot x_1+k_2\cdot x_2+k_3\cdot x_3)]$$
$$\times\langle 0|T[j_\mu(0)j_\nu(x_1)j_\rho(x_2)j_\sigma(x_3)]|0\rangle$$

$$\text{Form factor}:\ \Gamma_\mu(q)=\gamma_\mu\,F_1(q^2)+\frac{i\sigma^{\mu\nu}q_\nu}{2\,m_l}\,F_2(q^2)$$

HLbL from Models

- Model estimate with non-perturbative constraints at the chiral / low energy limits using anomaly : (9—12) x 10^{-10} with 25-40% uncertainty

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$



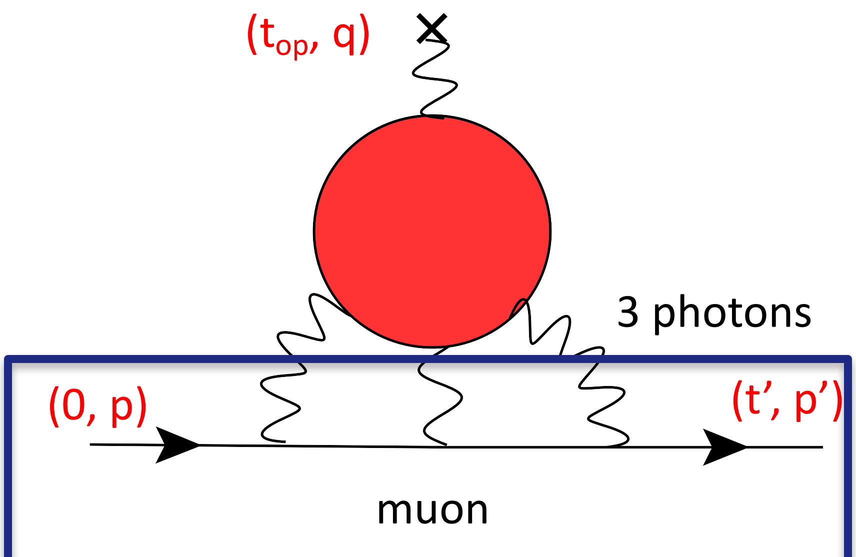
F. Jegerlehner , x 10^{11}

Contribution	BPP	HKS	KN	MV	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	0±10	−19±19	−19±13
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	15±10	22± 5
scalars	−6.8±2.0	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	105±26	116±39

Our Basic strategy : Lattice QCD+QED system

- 4pt function has too much information to parameterize (?)
- Do Monte Carlo integration for QED two-loop with 4 pt function $\pi^{(4)}$ which is sampled in lattice QCD with chiral quark (Domain-Wall fermion)
- Photon & lepton part of diagram is derived either in lattice QED+QCD [Blum et al 2014] (stat noise from QED), or exactly derive for given loop momenta [L. Jin et al 2015] (no noise from QED+lepton).

$$\Gamma_\mu^{(\mathrm{Hlbl})}(p_2, p_1) = ie^6 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \Pi^{(4)}_{\mu\nu\rho\sigma}(q, k_1, k_2, k_3)$$
$$\times \left[S(p_2)\gamma_\nu S(p_2+k_2)\gamma_\rho S(p_1+k_1)\gamma_\sigma S(p_1) + (\mathrm{perm.})\right]$$



- set spacial momentum for
  - external EM vertex q
  - in- and out- muon p, p'

  q = p-p'

- set time slice of muon source(t=0), sink(t') and operator (t_{op})

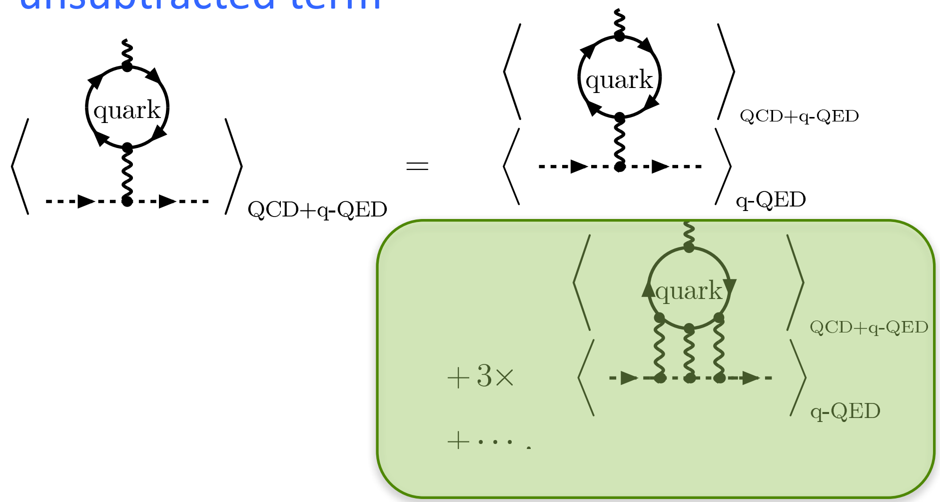
- take large time separation for ground state matrix element

QCD+QED method [Blum et al 2015]

- One photon is treated analytically
- other two sampled stochastically
- needs subtraction
- use AMA for error reduction
- use Furry's theoretm to reduce α^2 noise

unsubtracted term

Subtraction term

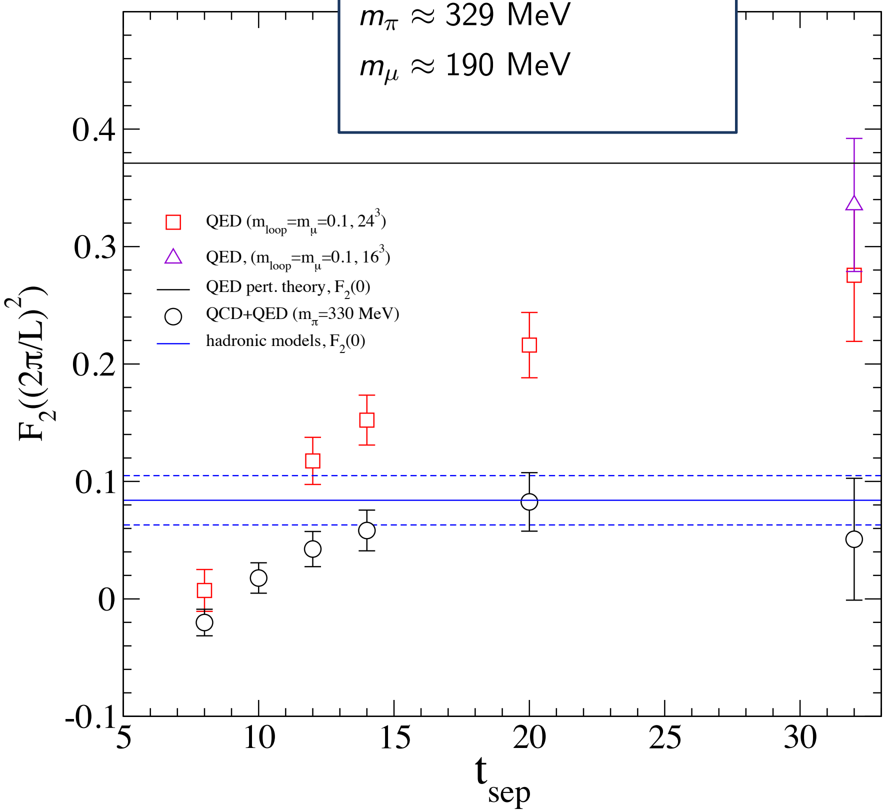


24^3 lattice size

$Q^2 = 0.11$ and $0.18\ \mathrm{GeV}^2$

$m_\pi \approx 329$ MeV

$m_\mu \approx 190$ MeV



- Connected part only

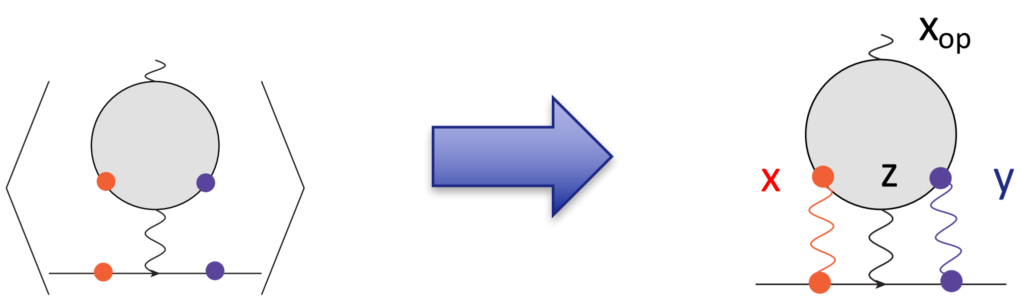
- QED only calculation consistent with QED loop calculation for larger volume

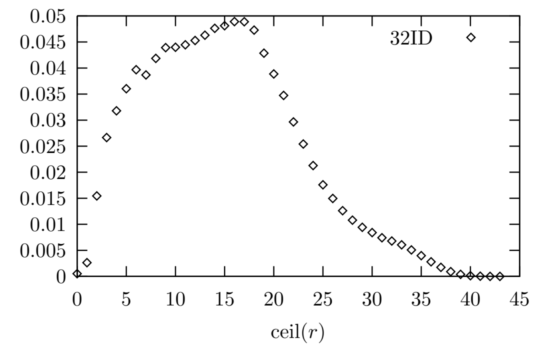
- QED+QCD
  - ball park of model values
  - significant exited state effects ?

Coordinate space Point photon method

[Luchang Jin et al. , PRD93, 014503 (2016)]

- Treat all 3 photon propagators exactly (3 analytical photons) , which makes the quark loop and the lepton line connected :
  disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, two of quark-photon vertex location x,y, z and x_{op} is summed over space-time exactly

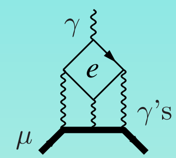




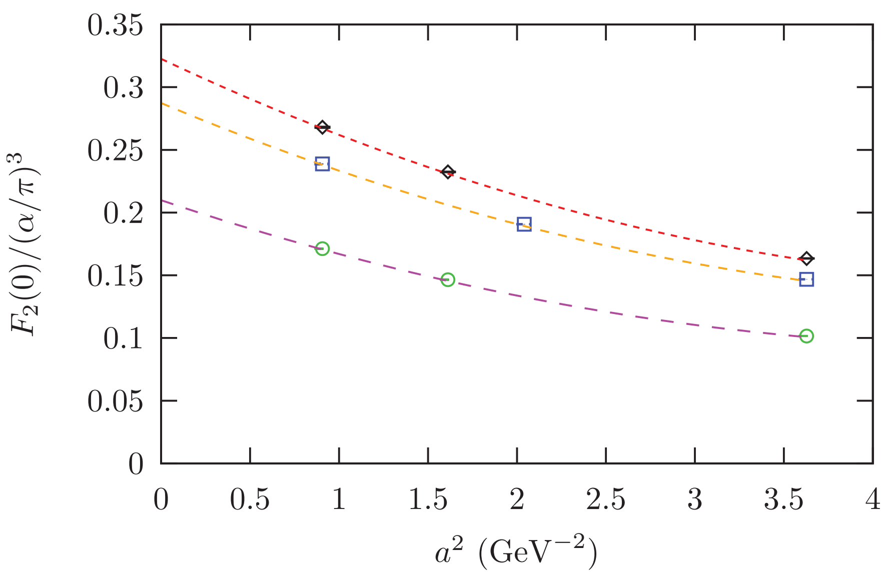
- Short separations, Min[|x-z|,|y-z|,|x-y|] < R ~ O(0.5) fm, which has a large contribution due to confinement, are summed for all pairs
- longer separations, Min[|x-z|,|y-z|,|x-y|] >= R, are done stochastically with a probability shown above (Adaptive Monte Carlo sampling)

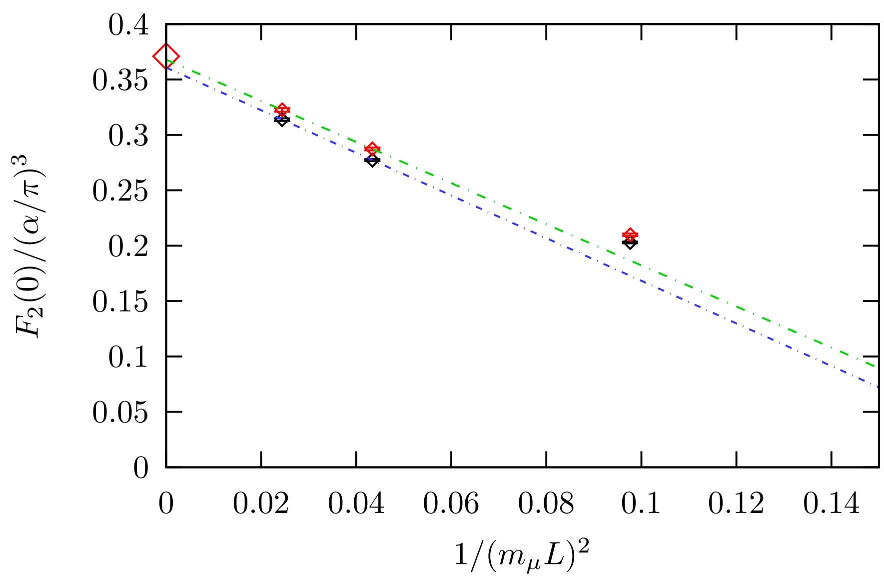
Systematic effects in QED only study

- muon loop, muon line
- a = a m_μ / (106 MeV)
- L= 11.9, 8.9, 5.9 fm
- known result : F2 = 0.371 (diamond) correctly reproduced (good check)



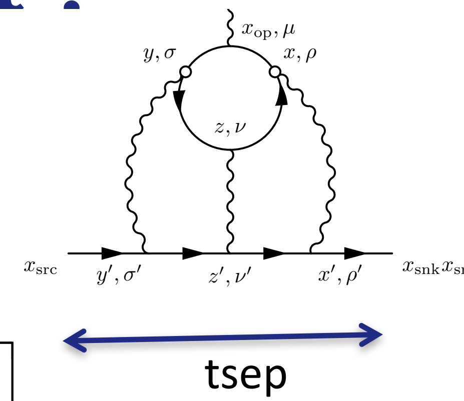
$$a_\mu^{(6)}(\mathrm{lbl}, e) = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \frac{59}{270}\pi^4 - 3\,\zeta(3) - \frac{10}{3}\pi^2 + \frac{2}{3} + O\left(\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^3$$



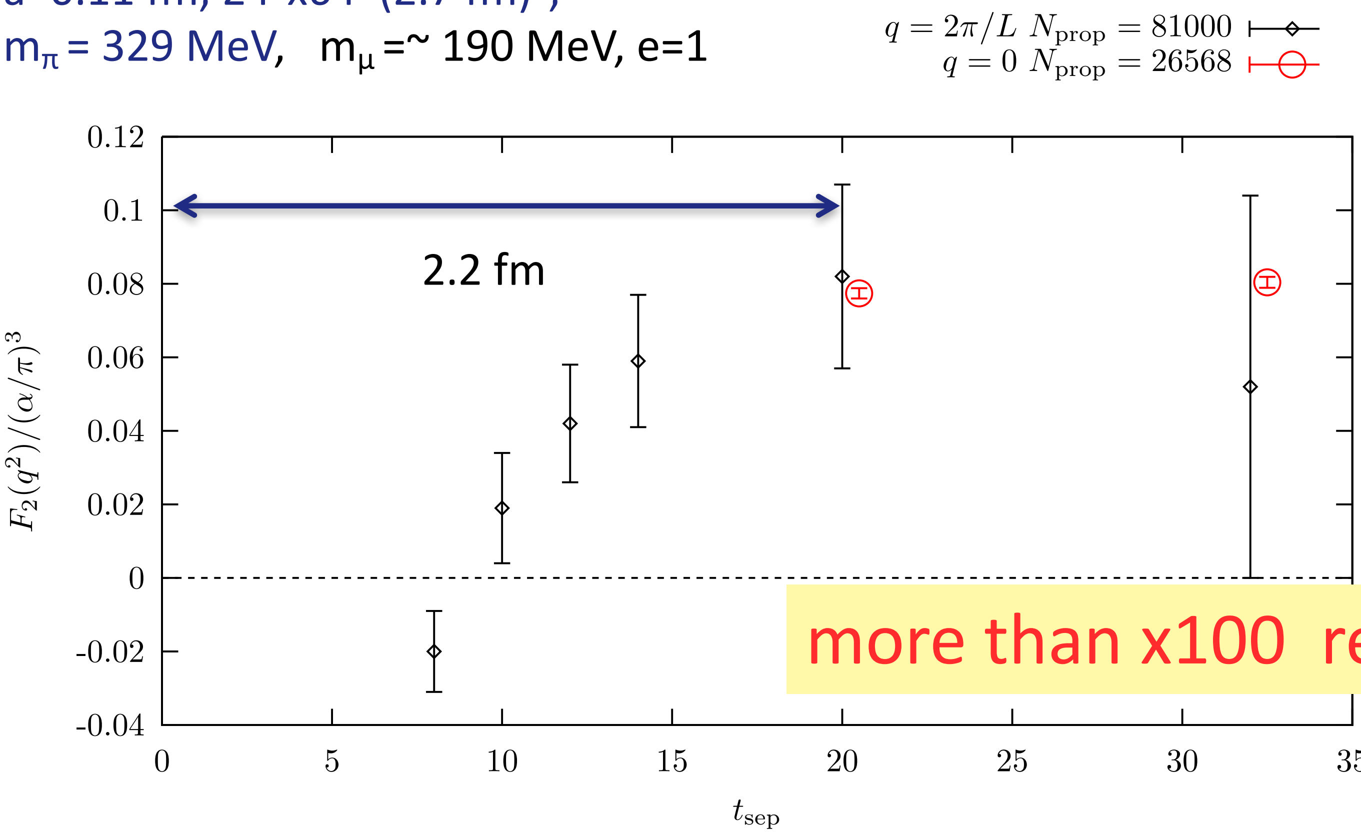


FV and discretization error could be as large as 20-30 %, similar discretization error seen from QCD+QED study

Dramatic Improvement !
Luchang Jin



a=0.11 fm, 24^3x64 $(2.7 \text{ fm})^3$,
m_π = 329 MeV, m_μ =~ 190 MeV, e=1

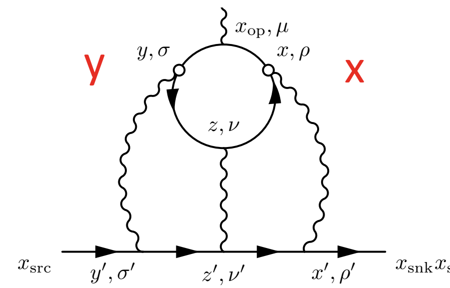


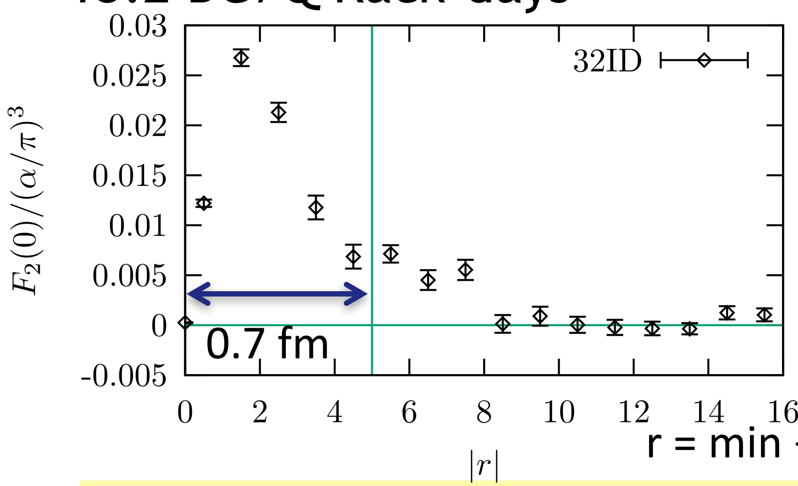
more than x100 reduced cost !

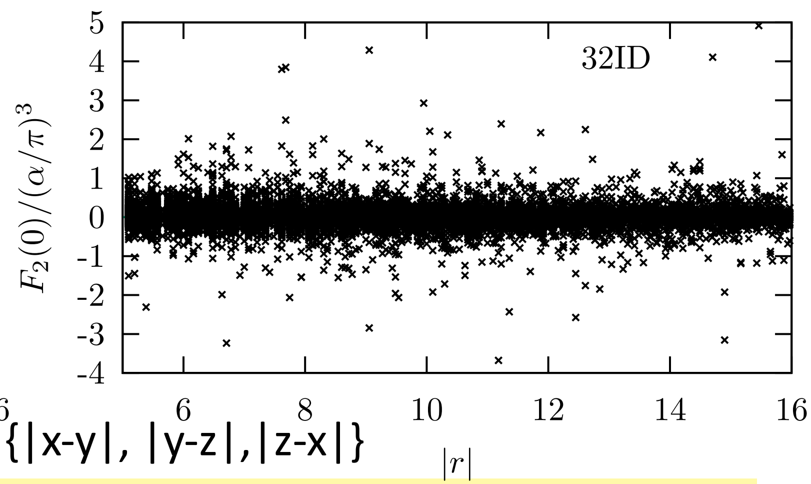
Method	$F_2/(\alpha/\pi)^3$	N_{conf}	N_{prop}	$\sqrt{\text{Var}}$
Conserved	0.0825(32)	12	$(118 + 128) \times 2 \times 7$	0.65
Mom.	0.0804(15)	18	$(118 + 128) \times 2 \times 3$	0.24

M_π=170 MeV cHLbL result
[Luchang Jin et al., PRD93, 014503 (2016)]

- V=(4.6 fm)^3, a = 0.14 fm, m_μ=130 MeV, 23 conf
- pair-point sampling with AMA (1000 eigV, 100CG) ,
  > 6000 meas/conf
  - |x-y| <= 0.7fm, all pairs, x2-5 samples
    217 pairs (10 AMA-exact)
  - |x-y| > 0.7fm, 512 pairs (48 AMA-exact)
- 13.2 BG/Q Rack-days







r = min {|x-y|, |y-z|,|z-x|}

$$\frac{(g_\mu - 2)_{\rm cHLbL}}{2} = (0.1054 \pm 0.0054)(\alpha/\pi)^3 = (132.1 \pm 6.8) \times 10^{-11}.$$

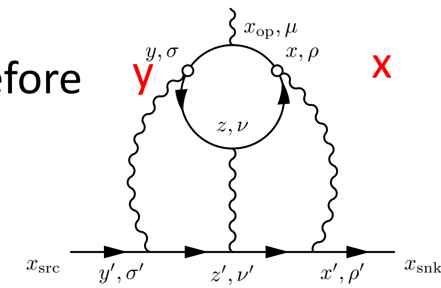
Strange contribution : (0.0011$\pm$ 0.005) $(\alpha/\pi)^3$

physical M_π=140 MeV cHLbL result

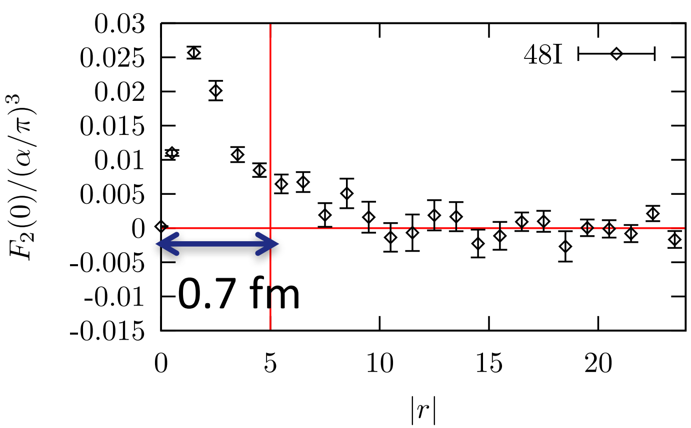
[Luchang Jin et al. , preliminary]

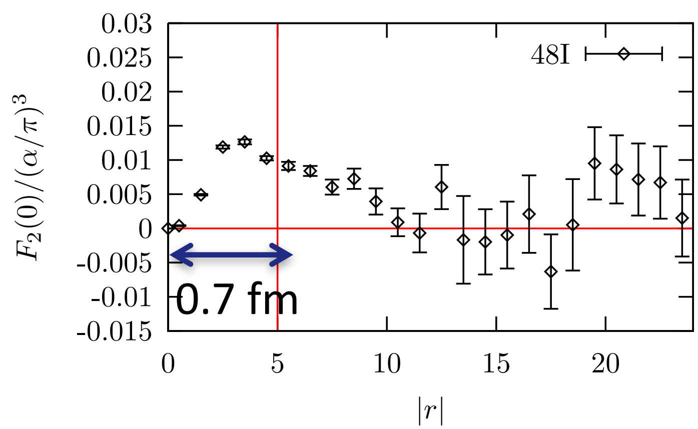
- V=(5.5 fm)^3, a = 0.11 fm, m_μ=106 MeV, 69 conf [RBC/UKQCD]
- Two stage AMA (2000 eigV, 200CG and 400 CG) using zMobius, ~4500 meas/conf
- 160 BG/Q Rack-days

integrand safely suppressed before reaching r ~ L/2



0.6 fm





r = min {|x-y|, |y-z|,|z-x|}

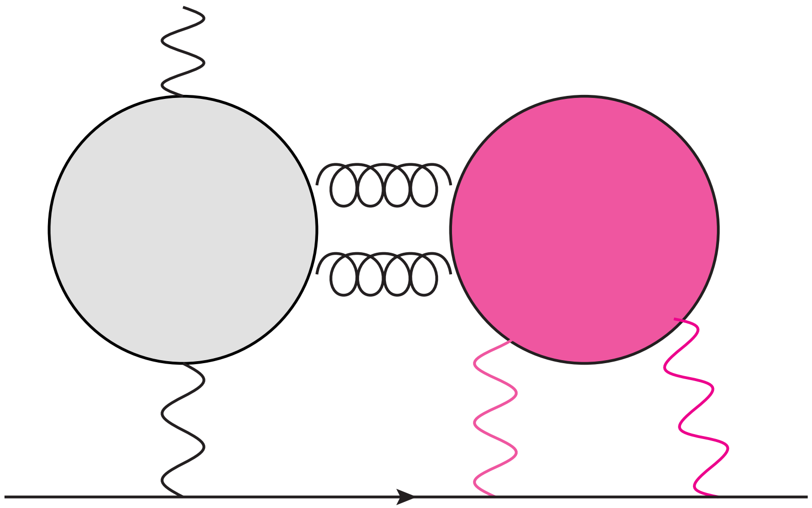
r = max{|x-y|, |y-z|,|z-x|}

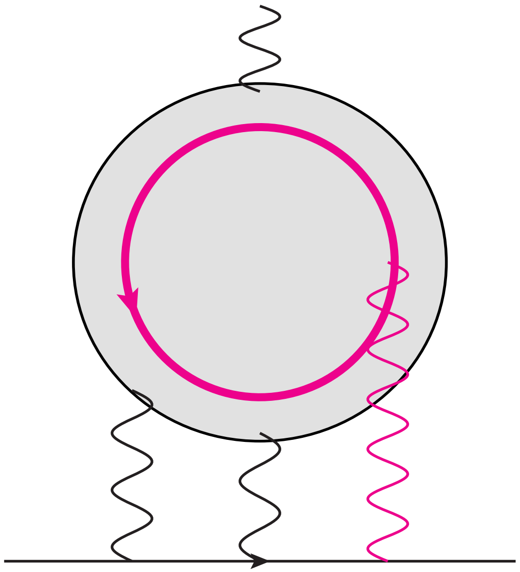
$$a_\mu^{\rm LbL,\ con} = (0.0926 \pm 0.0077) \times \left(\frac{\alpha}{\pi}\right)^3 = (11.60 \pm 0.96) \times 10^{-10},$$

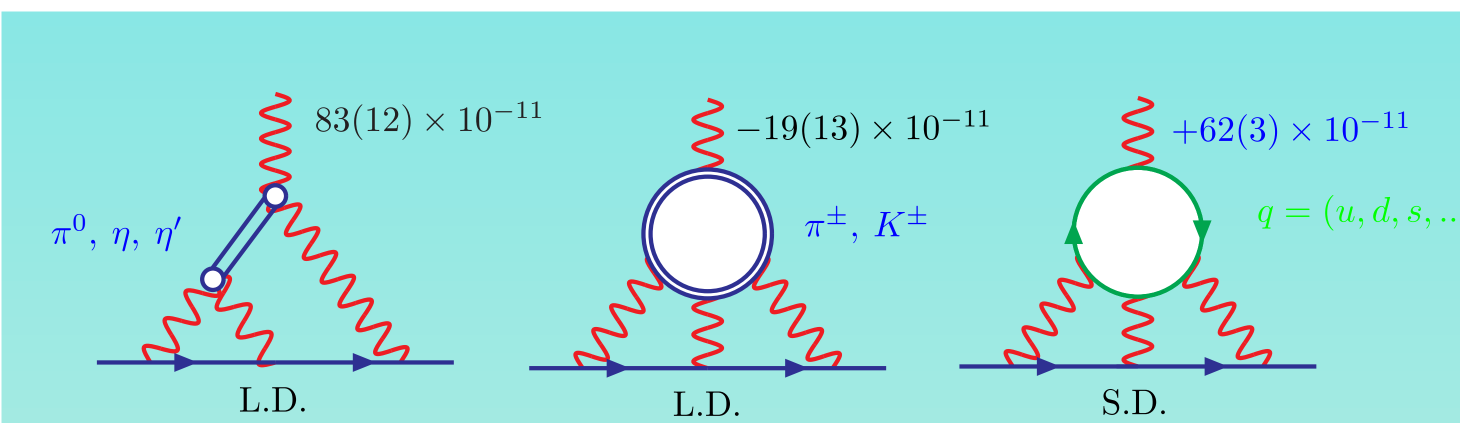
(preliminary, stat err only)

Disconnected diagrams in HLbL

- Disconnected diagrams

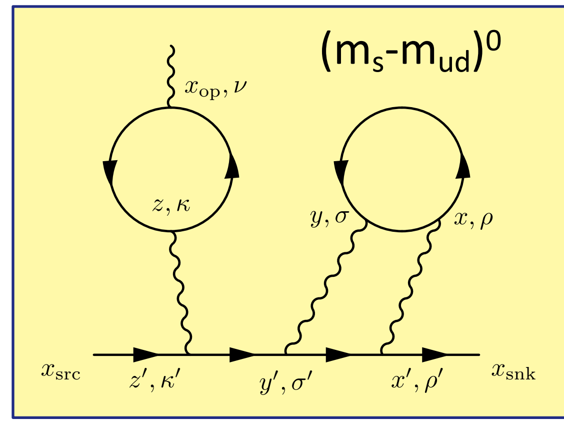




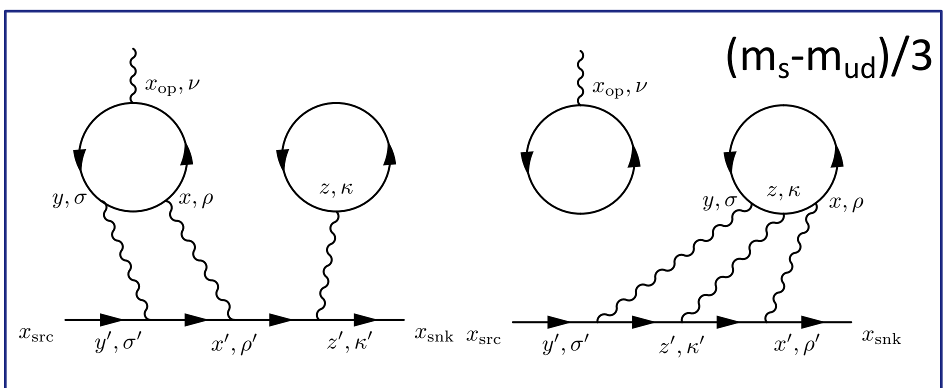


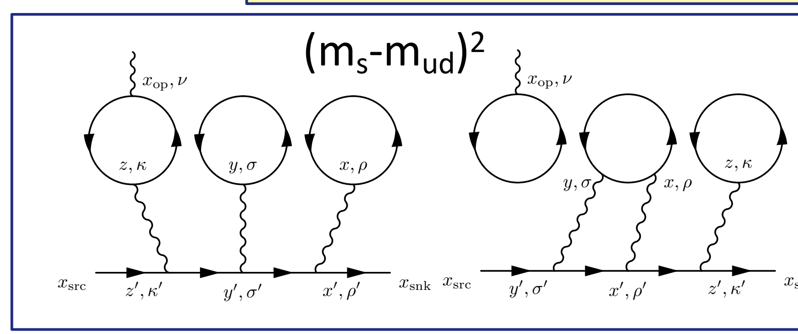
SU(3) hierarchies for d-HLbL

- At $m_s = m_{ud}$ limit, following type of disconnected HLbL diagrams survive $Q_u + Q_d + Q_s = 0$
- Physical point run using similar techniques to c-HLbL.



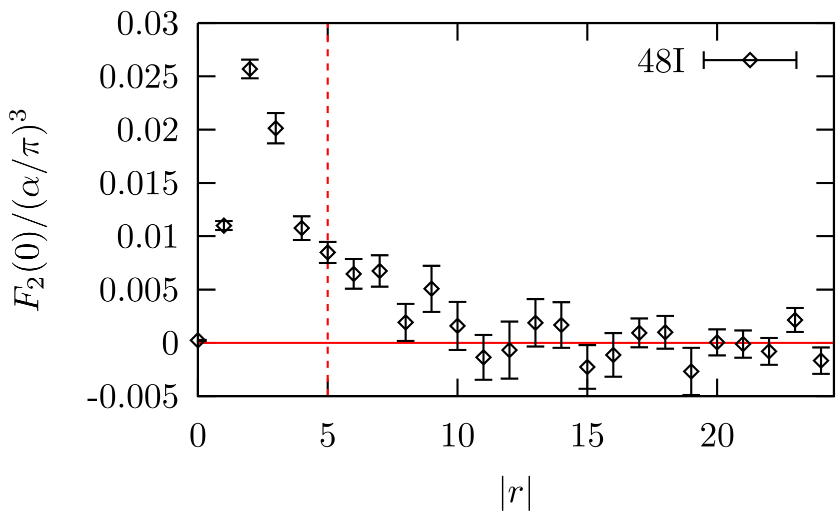
- other diagrams suppressed by
  $O(m_s - m_{ud})\ /3$ and $O(\ (m_s - m_{ud})^2\ )$

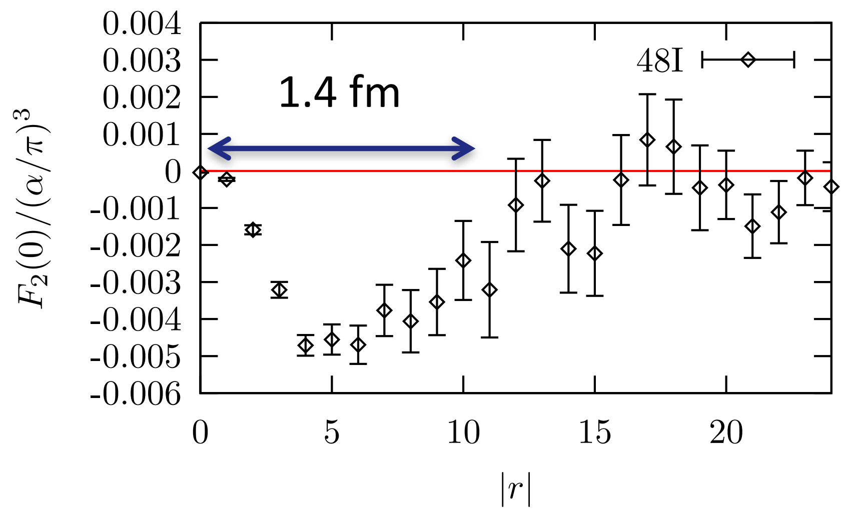




139 MeV Pion, connected and disconnected LbL results (preliminary)

- left: connected, right : leading disconnected





- Using AMA with 2,000 zMobius low modes, AMA

(Preliminary, statistical error only)

$$\left.\frac{g_\mu - 2}{2}\right|_{\mathrm{cHLbL}} = (0.0926 \pm 0.0077) \times \left(\frac{\alpha}{\pi}\right)^3 = (11.60 \pm 0.96) \times 10^{-10}$$

$$\left.\frac{g_\mu - 2}{2}\right|_{\mathrm{dHLbL}} = (-0.0498 \pm 0.0064) \times \left(\frac{\alpha}{\pi}\right)^3 = (-6.25 \pm 0.80) \times 10^{-10}$$

$$\left.\frac{g_\mu - 2}{2}\right|_{\mathrm{HLbL}} = (0.0427 \pm 0.0108) \times \left(\frac{\alpha}{\pi}\right)^3 = (5.35 \pm 1.35) \times 10^{-10}$$

Systematic errors

- Missing disconnected diagrams
  → compute them

- Finite volume

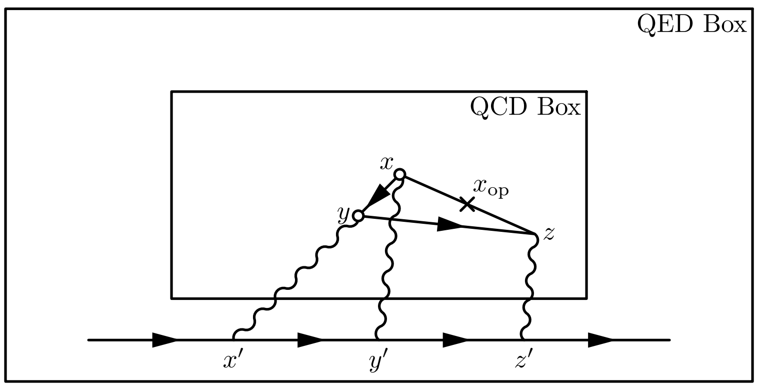
- Discretization error

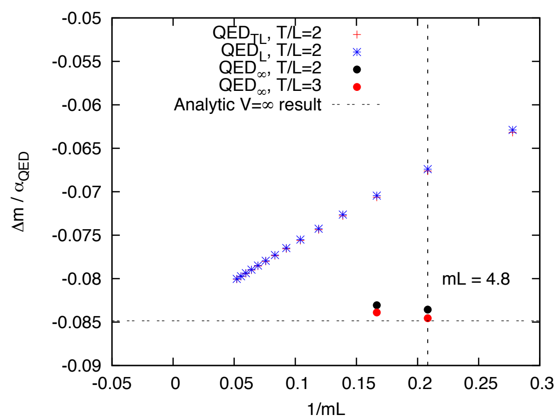
  → a scaling study for 1/a = 2.7 GeV, 64 cube lattice at physical quark mass is proposed to ALCC at Argonne

- ...

QCD box in QED box

- FV from quark is exponentially suppressed ~ $\exp(-M_\pi L_{QCD})$
- Dominant FV effects would be from photon
- Let photon and muon propagate in larger (or infinite) box than that of quark



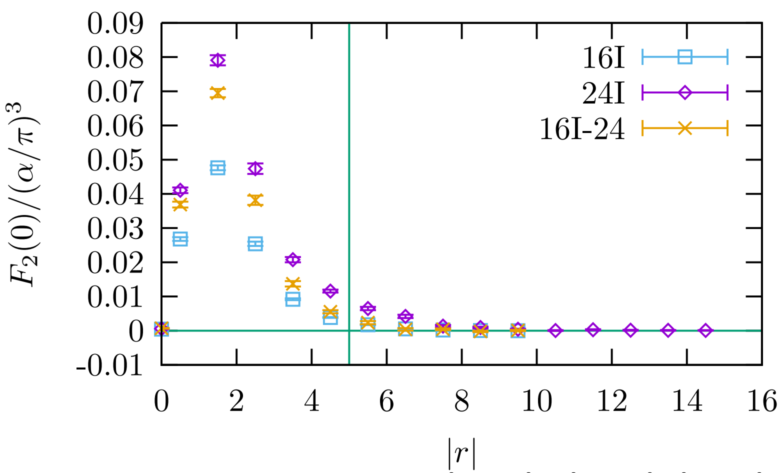


- We could examine different lepton/photon in the off-line manner e.g. QED_L (Hayakwa-Uno 2008) with larger box, Twisting Averaging [Lehner TI LATTICE14] or Infinite Vol. Photon propagators [C. Lehner, L.Jin, TI LATTICE15] [Maintz group, LATTICE16]

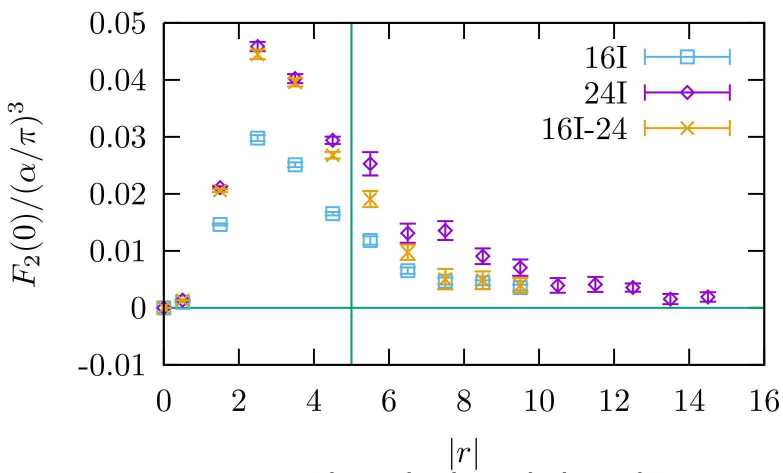
QED box in QCD box (contd.)

- Mπ=420 MeV, mμ=330 MeV, 1/a=1.7 GeV
- $(16)^3$ = $(1.8\text{ fm})^3$ QCD box in $(24)^3$= $(2.7\text{ fm})^3$ QED box

Ensemble	$m_\pi L$	QCD Size	QED Size	$\frac{F_2(q^2=0)}{(\alpha/\pi)^3}$
16I	3.87	$16^3\times 32$	$16^3\times 32$	0.1158(8)
24I	5.81	$24^3\times 64$	$24^3\times 64$	0.2144(27)
16I-24		$16^3\times 32$	$24^3\times 64$	$\boxed{0.1674(22)}$



r = min {|x-y|, |y-z|,|z-x|}



r = max{|x-y|, |y-z|,|z-x|}

Summary

- Lattice calculation for g-2 calculation is improved very rapidly
- HLbL including leading disconnected diagrams :
  Many orders of magnitudes improvements
  -> 8 % stat error in connected, 13 % stat error in leading disconnected
  - coordinate-space integral using analytic photon propagator with adaptive probability (point photon method)
  - config-by-config conserved external current
  - take moment of relative coordinate to directly take q→0
  - AMA, zMobius, 2000 low modes

(preliminary, con+L-discon, stat err only)

$$a_\mu^{\text{LbL, con}} = (11.60 \pm 0.96) \times 10^{-10}, \quad a_\mu^{\text{LbL, L-dcon}} = (-6.25 \pm 0.80) \times 10^{-10}$$

$$a_\mu^{\text{LbL, c+Ld}} = (0.0427 \pm 0.0108) \times \left(\frac{\alpha}{\pi}\right)^3 = (5.35 \pm 1.35) \times 10^{-10}$$

- Still large systematic errors (missing disconnected, FV, discr. error, ...)
- Also direct 4pt method [Mainz group] and Dispersive analysis [Colangelo et al. 2014, 2015, Pauk&Vanderhaeghen 2014]
- Goal : HVP sub 1%, HLbL 10% error

Future plans

- (discretization error) Nf=2+1 DWF/ Mobius ensemble at physical point, L=5.5 fm, a=0.083 fm, $(64)^3$ at Mira, ALCC @Argonne started to run
- (FV study)   QCD box in QED box at physical point
- Subleading Disconnected diagrams

[T. Blum PRL91 (2003) 052001]

HVP from Lattice

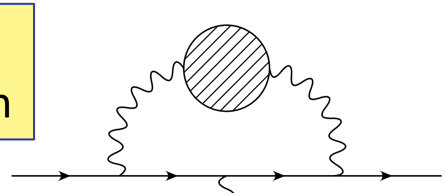
- Analytically continue to Euclidean/space-like momentum $K^2 = - q^2 > 0$
- Vector current 2pt function

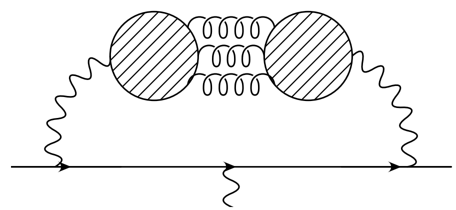
$$a_\mu = \frac{g-2}{2} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2) \hat{\Pi}(K^2)$$

$$\Pi^{\mu\nu}(q) = \int d^4x e^{iqx} \langle J^\mu(x) J^\nu(0) \rangle$$

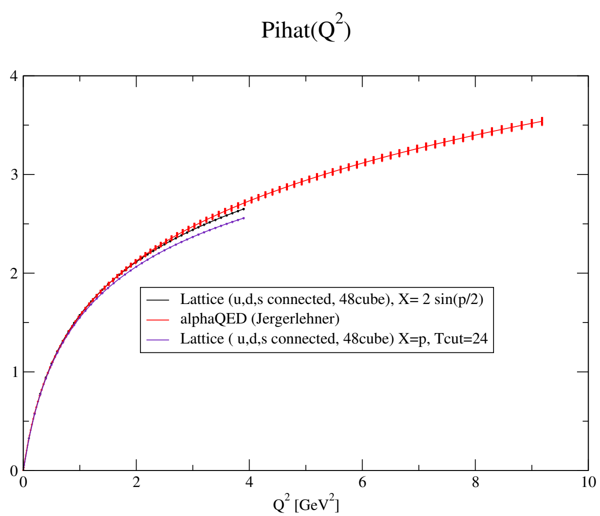
- Low Q2, or long distance, part of Π (Q2) is relevant for g-2

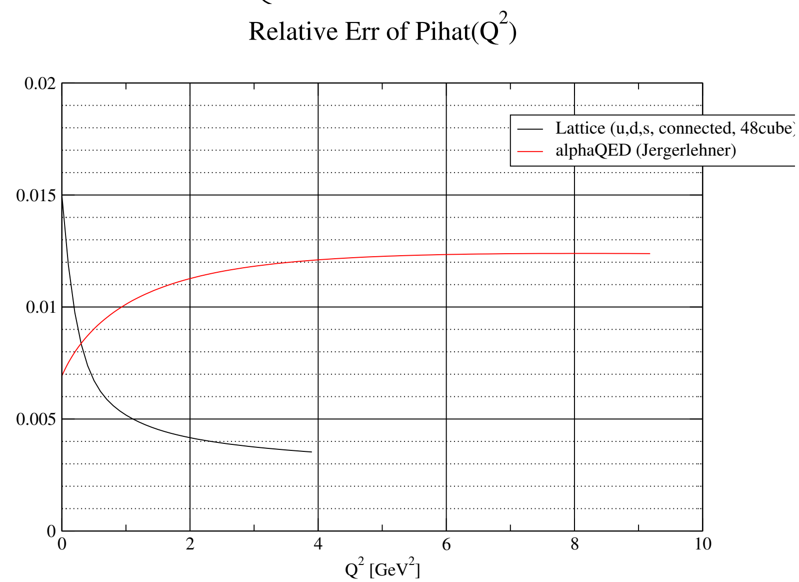
connected contribution





disconnected contribution





Current conservation, subtraction, and coordinate space representation

- Current conservation => transverse tensor

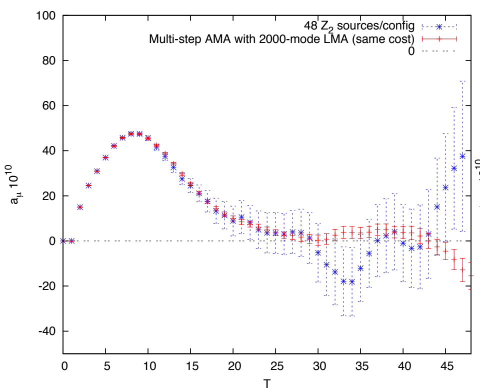
$$\sum_x e^{iQx}\langle J_\mu(x)J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

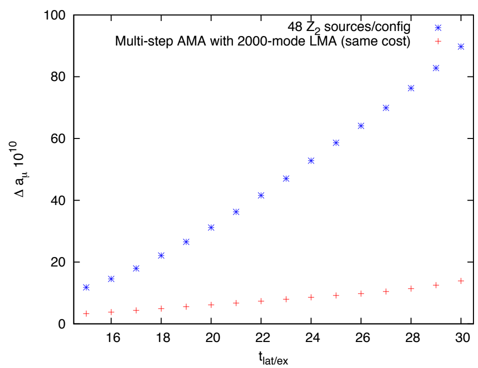
- Coordinate space vector 2 pt Green function C(t) is directly related to subtracted Π (Q2) [Bernecker-Meyer 2011, …]

$$\Pi(Q^2) - \Pi(0) = \sum_t \left(\frac{\cos(qt) - 1}{Q^2} + \frac{t^2}{2} \right) C(t)$$

- g-2 value is also related to C(t) with know kernel w(t) from QED.

$$\boxed{a_\mu^{\mathrm{HVP}} = \sum_t w(t)C(t), \quad w(t) \propto t^4 \cdots}$$





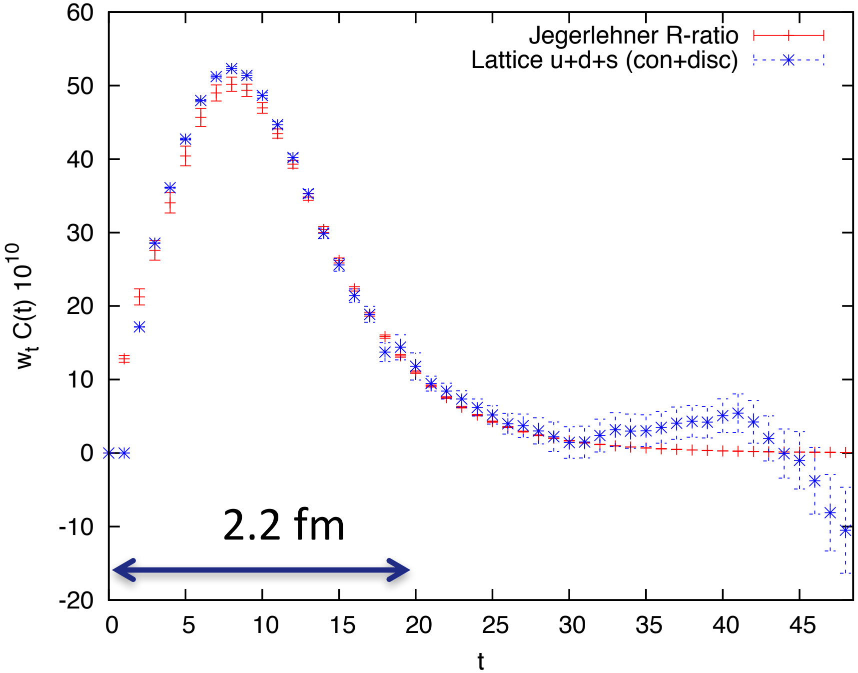
RBC/UKQCD
Chiral Lattice quark DWF
physical point
Quark Propagator Low Mode (A2A)
using All-Mode Averaging (AMA)

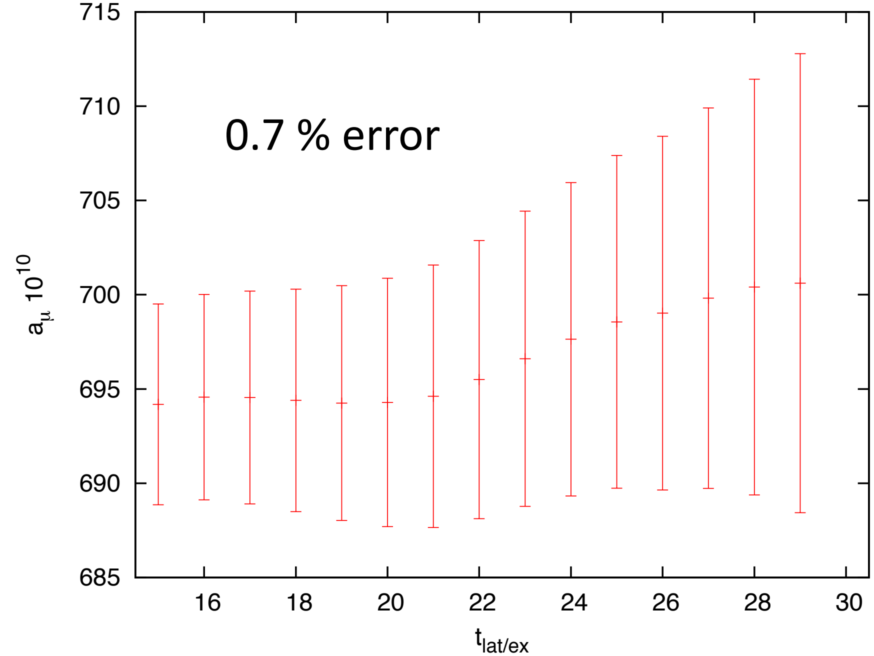
(plan B)   Interplay between Lattice and Experiment

- Check consistency between Lattice and R-ratio
- Short distance from Lattice, Long distance from R-ratio :

  error <= 1 % at $t_{lat/exp}$ = 2fm

$$a_\mu^{\rm HVP} = \left[\sum_{t=0}^{t_{\rm lat/exp}} w(t)C(t)\right]^{\rm LAT} + \left[\int_{t_{\rm lat/exp}}^{\infty} dt\, w(t)C(t)\right]^{\rm EXP}$$





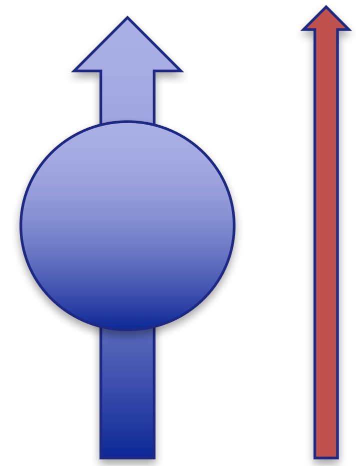
2016 : Disconnected, charm, QED, isospin breaking effects are being included
(RBC/UKQCD C. Lehner et al, also other collaborations)

Backup slides

Anomalous magnetic moment

- Fermion's energy in the external magnetic field:

$$V(x) = -\vec{\mu}_l \cdot \vec{B}$$

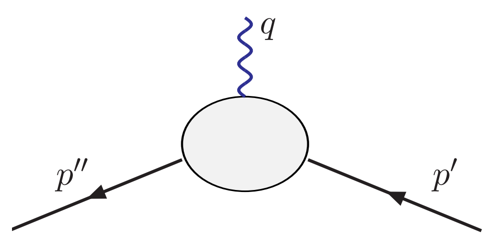


- Magnetic moment and spin g_l : Lande g-factor

  g_l's deviation from tree level value, 2 :

$$\vec{\mu}_l = g_l \frac{e}{2m_l} \vec{S}_l \qquad a_l = \frac{g_l - 2}{2}$$

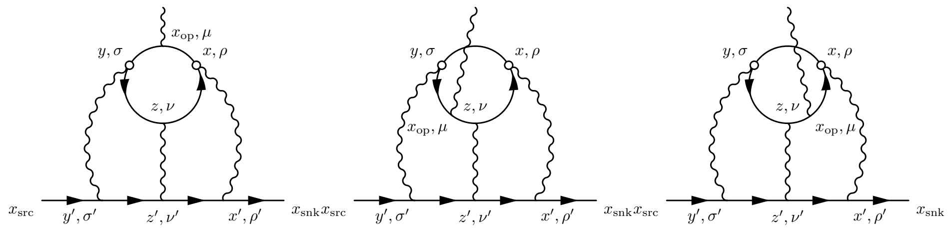
- Form factor : $\Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} \boxed{F_2(q^2)}$



After quantum correction $\Rightarrow a_l = F_2(0)$

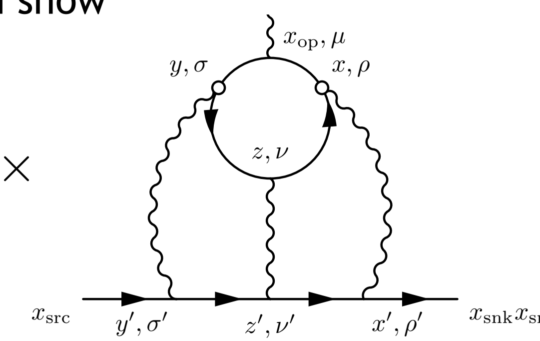
Conserved current & moment method

- [conserved current method at finite q2] To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external currents config-by-config.



- [moment method , q2→0] By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q->0 limit value is directly computed via the first moment of the relative coordinate, xop – (x+y)/2, one could show

$$\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{\rm op}} (x_{\rm op} - (x+y)/2)_i \times$$



to directly get $F_2(0)$ without extrapolation.

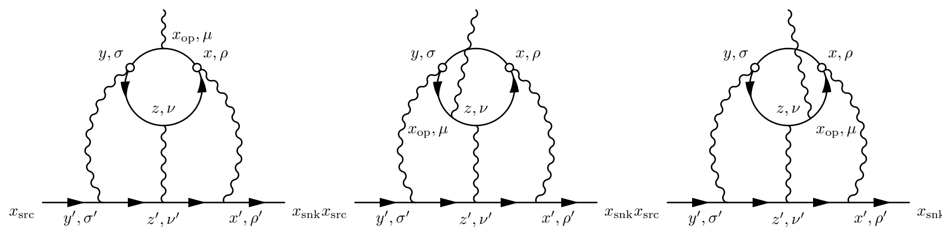
$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2\,m_l} F_2(q^2)$$

M_π=170 MeV cHLbL result (contd.)

"Exact" … q = 2pi / L,

"Conserved (current)" … q=2pi/L, 3 diagrams

"Mom" … moment method q->0, with AMA



Method	$F_2/(\alpha/\pi)^3$	$N_{\rm conf}$	$N_{\rm prop}$	$\sqrt{\rm Var}$	$r_{\max}$	SD	LD	ind-pair
Exact	0.0693(218)	47	$58 + 8 \times 16$	2.04	3	−0.0152(17)	0.0845(218)	0.0186
Conserved	0.1022(137)	13	$(58 + 8 \times 16) \times 7$	1.78	3	0.0637(34)	0.0385(114)	0.0093
Mom. (approx)	0.0994(29)	23	$(217 + 512) \times 2 \times 4$	1.08	5	0.0791(18)	0.0203(26)	0.0028
Mom. (corr)	0.0060(43)	23	$(10 + 48) \times 2 \times 4$	0.44	2	0.0024(6)	0.0036(44)	0.0045
Mom. (tot)	0.1054(54)	23						

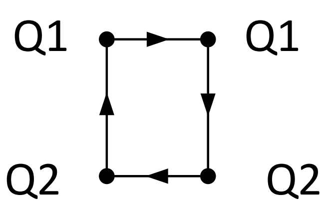
Direct 4pt calculation for selected kinematical range

[J. Green et al. Mainz group, Phys. Rev. Lek 115, 222003(2015)]

- Compute connected contribution of 4 pt function in momentum space
- Forward amplitudes related to γ*(Q1)γ*(Q2) -> hadron cross section via dispersion relation

$$\begin{array}{ll} & \mathcal{M}_{\rm had}\left(\gamma^*(Q_1)\gamma^*(Q_2) \to \gamma^*(Q_1)\gamma^*(Q_2)\right) \\ \leftrightarrow & \sigma_{0,2}\left(\gamma^*(Q_1)\gamma^*(Q_2) \to \text{had.}\right) \end{array}$$

$$\nu = -Q_1 \cdot Q_2$$



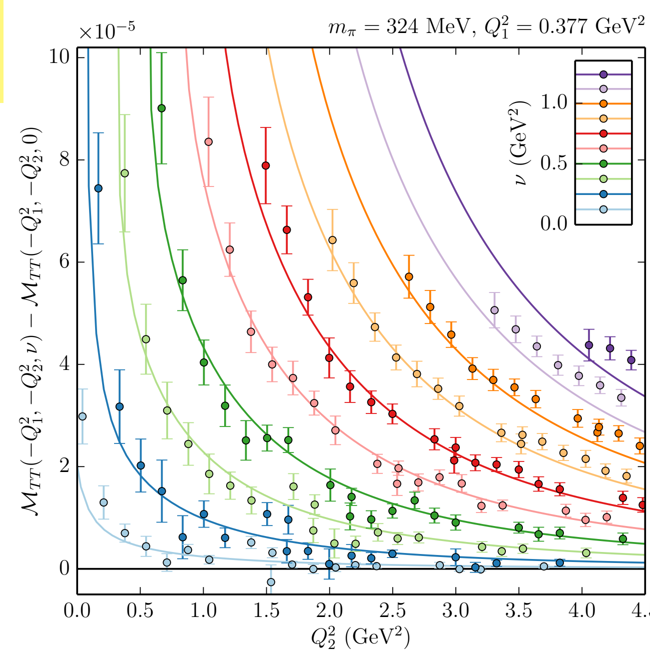


FIG. 3. The forward scattering amplitude $\mathcal{M}_{\rm TT}$ at a fixed virtuality $Q_1^2 = 0.377\text{GeV}^2$, as a function of the other photon virtuality Q_2^2, for different values of ν. The curves represent the predictions based on Eq. (10), see the text for details.

- solid curve: model prediction
- π0 exchange is seen to be not dominant, possibly due to heavy quark mass in the simulation (Mπ = 324 MeV)
- disconnected quark diagram loop

in progress in 2016

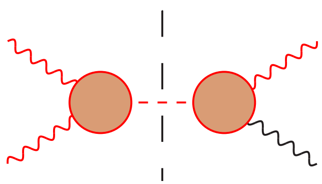
Dispersive approach for HLbL

[Colangelo et al. 2014, 2015, Pauk&Vanderhaeghen 2014]

- Using crossing symmetry, gauge invariance, 138 form factors are reduced 12 relevant for HLbL

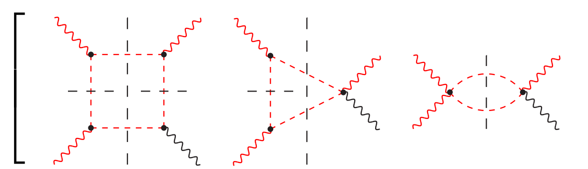
$$a_\mu^{\rm HLbL} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \frac{1}{(p+q_1)^2 - m_\mu^2} \frac{1}{(p-q_2)^2 - m_\mu^2} \times \sum_{j=1}^{12} \xi_j \hat{T}_{i_j}(q_1, q_2; p) \hat{\Pi}_{i_j}(q_1, q_2, -q_1 - q_2),$$

- π0, η,η' exchange, pion-loop (exactly scalar QED with pion Form factor)



$$= \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, q_4^2)}{s - M_\pi^2}$$

$$\Pi^{\rm FsQED}_{\mu\nu\lambda\sigma} = F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2) \times$$



- other contribution is neglected

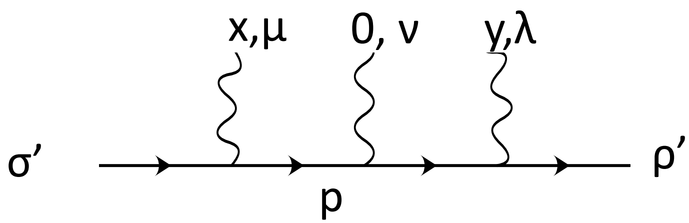
Continuum Infinite Volume (a.k.a HVP way)

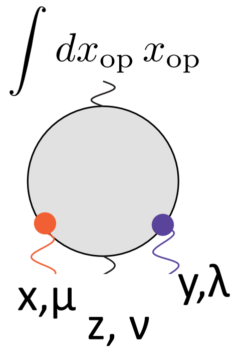
$$a_\mu^{\mathrm{HVP}} = \sum_t w(t) C(t), \quad w(t) \propto t^4 \cdots$$

- One could also use infinite volume/continuum lepton&photon diagram in coordinate space

  [J. Green et al. Mainz group, LAT16 proceedings]

$$\mathcal{L}_{\mu\nu\lambda\sigma\rho}(x, y; p)$$





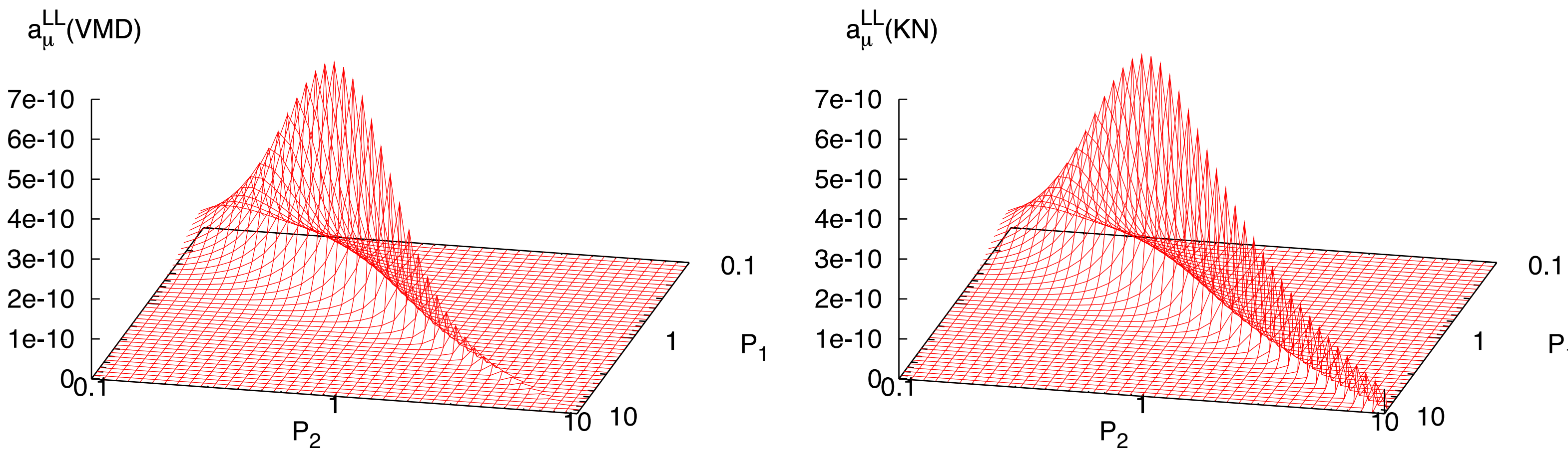
- Techniques in continuum model calculation [Knect Nyffeler 2002; Jegerlehner Nyffeler 2009] : angle average over muon momentum, and carry out angle of two virtual photons

$$L(x_1, x_2) = \sum_{m,l} \sum_{\substack{k=|l-m| \\ \text{step}=2}}^{l+m} (-1)^k C_k(\hat{x}_1 \hat{x}_2)$$
$$\times \int dQ_1 dQ_2 \frac{4 Z_1 Z_2}{m^2 Q_1 Q_2 X_1 X_2} \frac{(-Z_1 Z_2)^l}{l+1} J_{k+1}(Q_1 X_1) J_{k+1}(Q_2 X_2)$$
$$\times \left[\frac{\theta(1 - Q_2/Q_1)}{Q_1^2} \left(\frac{Q_2}{Q_1}\right)^m + \frac{\theta(1 - Q_1/Q_2)}{Q_2^2} \left(\frac{Q_1}{Q_2}\right)^m \right]$$

Can Lattice produce a counter part ?

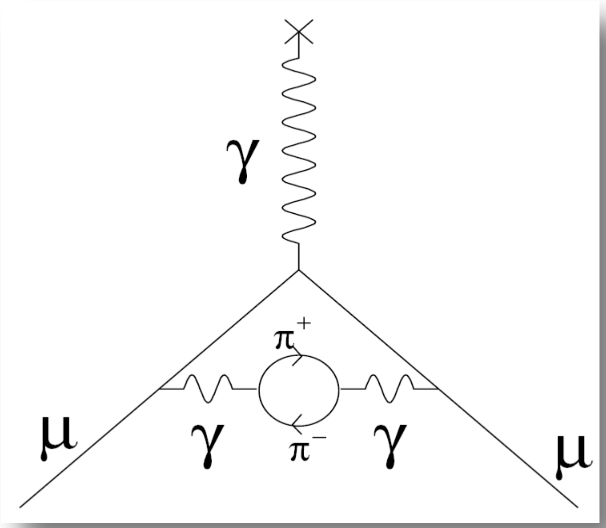
[J. Bijnens]

- Which momentum regimes important studied: JB and J. Prades, Mod. Phys. Lett. A **22** (2007) 767 [hep-ph/0702170]
- $a_\mu = \int dl_1 dl_2 a_\mu^{LL}$ with $l_i = \log(P_i/GeV)$

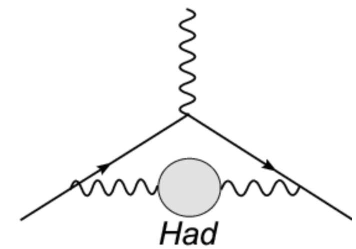


Which momentum regions do what:
volume under the plot $\propto a_\mu$

Hadronic Vacuum Polarization (HVP) contribution to g-2



Leading order of hadronic contribution (HVP)



- Hadronic vacuum polarization (HVP)

V_μ  V_ν $= (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_V(q^2)$

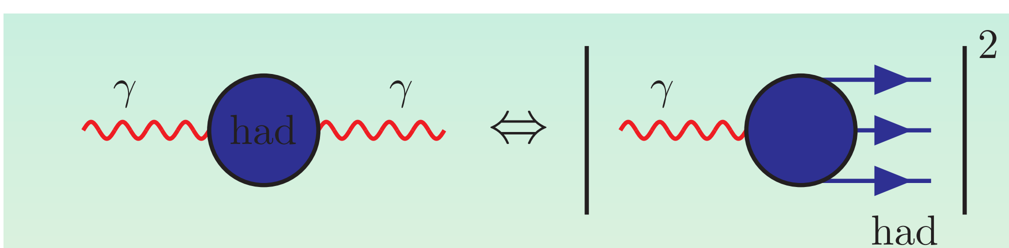
quark's EM current : $V_\mu = \sum_f Q_f \bar{f}\gamma_\mu f$

- Optical Theorem

$$\mathrm{Im}\Pi_V(s) = \frac{s}{4\pi\alpha}\sigma_{\mathrm{tot}}(e^+e^- \to X)$$

- Analycity

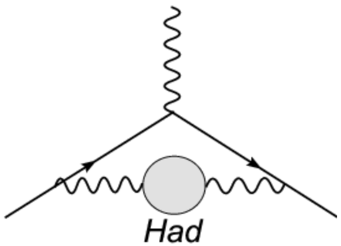
$$\Pi_V(s) - \Pi_V(0) = \frac{k^2}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}\Pi_V(s)}{s(s-k^2-i\epsilon)}$$



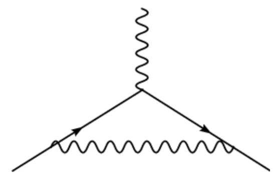
F. Jegerlehner's lecture

Leading order of hadronic contribution (HVP)

- Hadronic vacuum polarization (HVP)



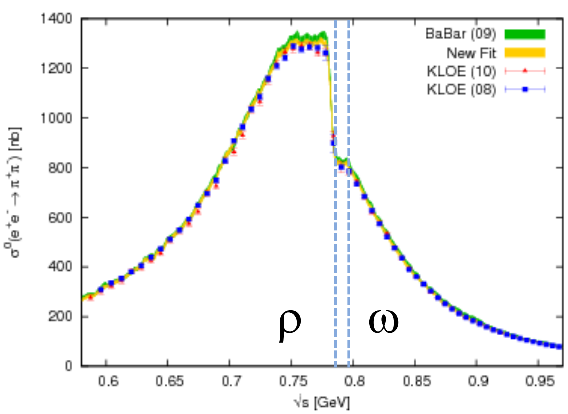
$$a_\mu^{\rm had} = \int ds$$

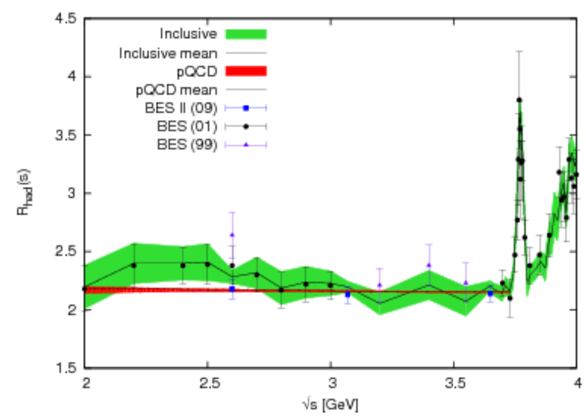


×

$$= \frac{\alpha}{\pi^2}\int_{m_\pi^2}^{\infty}\frac{ds}{s}\mathrm{Im}\Pi(s)K(s) \qquad K(s) = \int_0^1 dx\frac{x^2(1-x)}{x^2+(s/m_\mu^2)(1-x)}$$

$$= \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\left[\int_{m_\pi^2}^{s_{\rm cut}} ds\frac{K(s)}{s}R_{\rm had}^{\rm data}(s) + \int_{s_{\rm cut}}^{\infty} ds\frac{K(s)}{s}R_{\rm had}^{\rm pQCD}(s)\right]$$



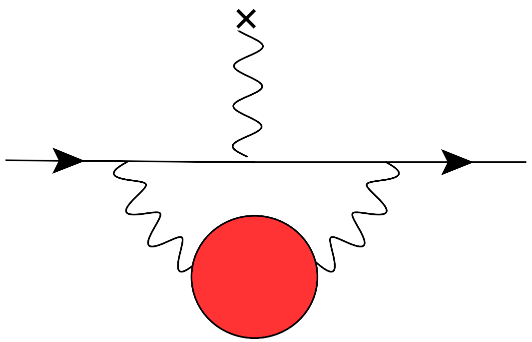


Hagiwara, et al. J.Phys. G38,085003 (2011)

HVP from experimental data

- From experimental e+ e- total cross section σ_{total}(e+e-) and dispersion relation

$$a_\mu^{\rm HVP} = \frac{1}{4\pi^2}\int_{4m_\pi^2}^{\infty} ds K(s)\sigma_{\rm total}(s)$$

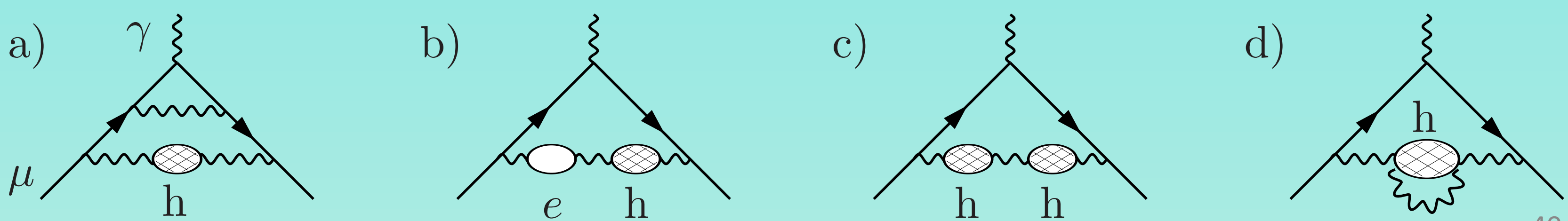


time like $q^2 = s >= 4\, m_\pi^2$

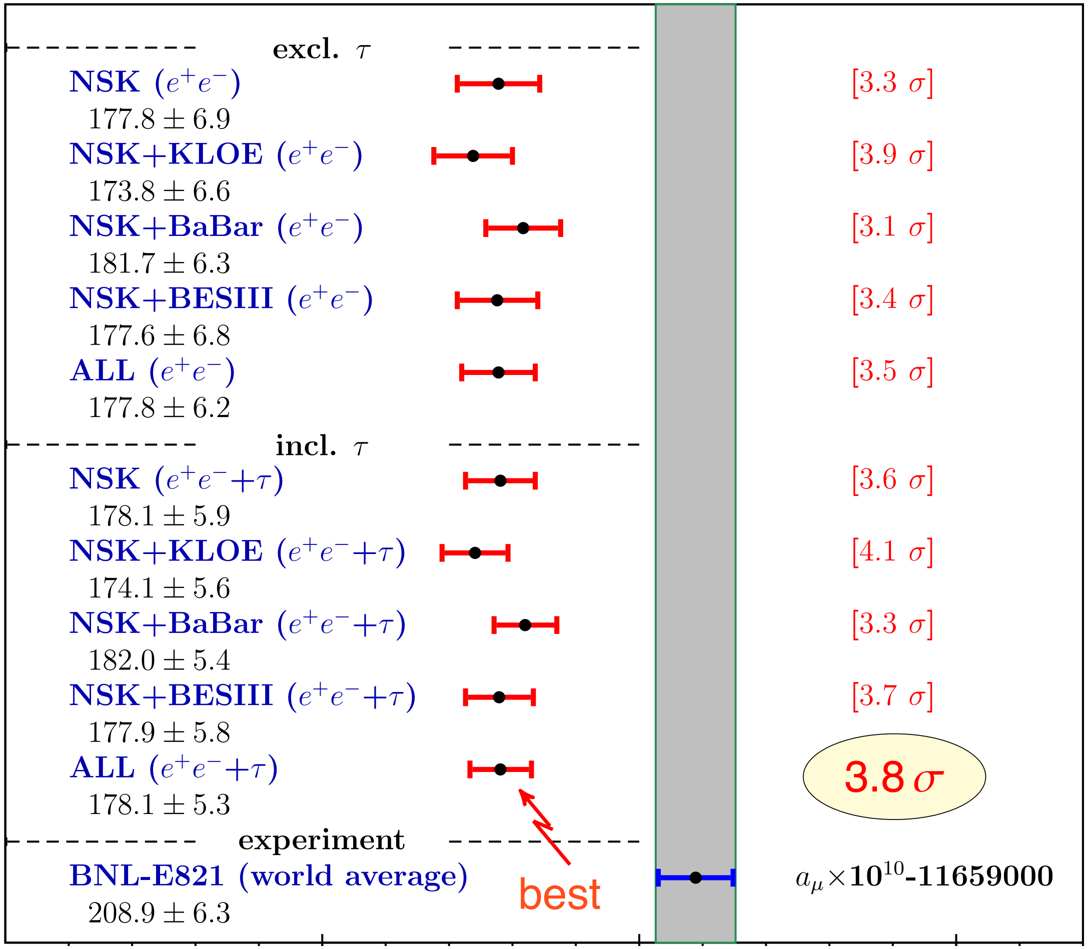
$$a_\mu^{\rm HVP,LO} = (694.91 \pm 4.27) \times 10^{-10}$$

[~ 0.6 % err]

$$a_\mu^{\rm HVP,HO} = (-9.84 \pm 0.07) \times 10^{-10}$$



F. Jegerlehner FCCP2015 summary including BES-III



[T. Blum PRL91 (2003) 052001]

HVP from Lattice

- Analytically continue to Euclidean/space-like momentum $K^2 = - q^2 > 0$
- Vector current 2pt function

$$a_\mu = \frac{g-2}{2} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2) \hat{\Pi}(K^2)$$

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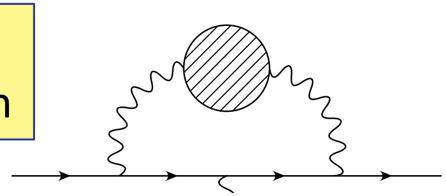
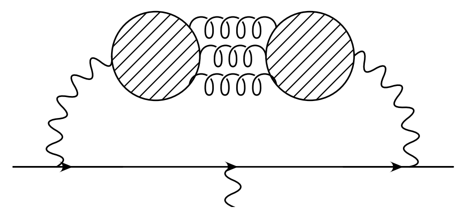
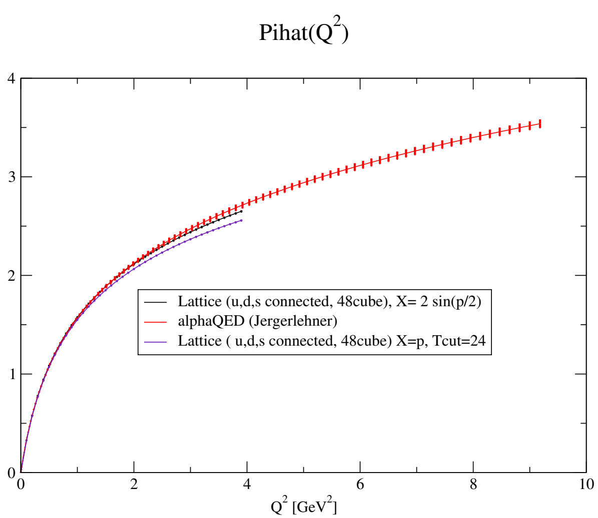
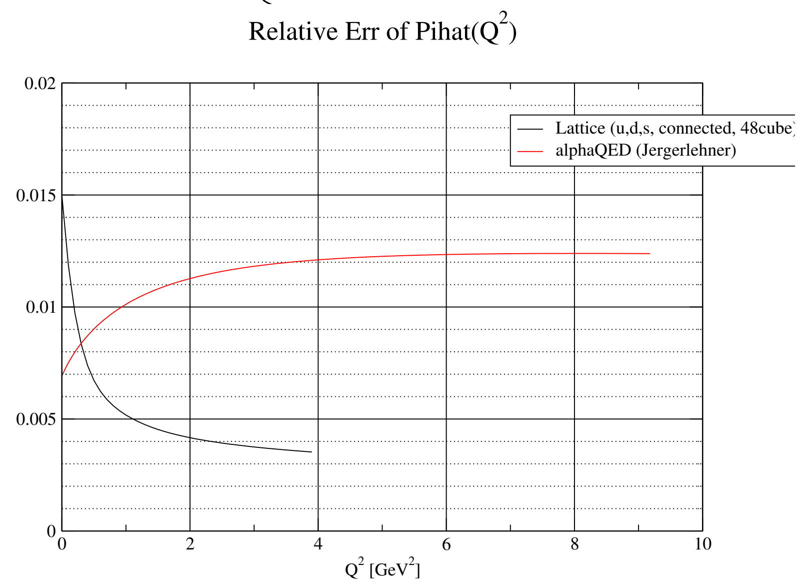
connected contribution





disconnected contribution





Current conservation, subtraction, and coordinate space representation

- Current conservation => transverse tensor

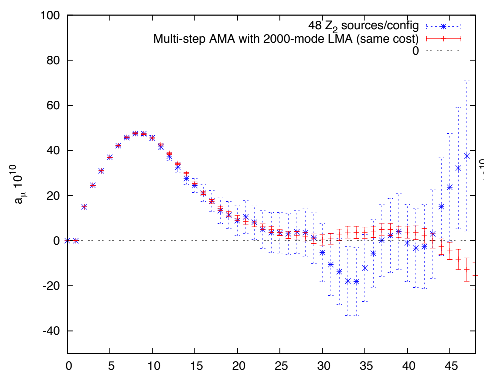
$$\sum_x e^{iQx}\langle J_\mu(x)J_\nu(0)\rangle = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi(Q^2)$$

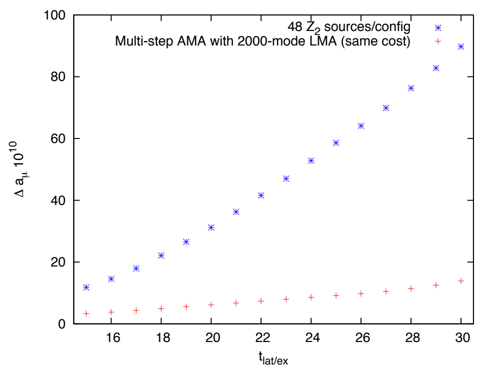
- Coordinate space vector 2 pt Green function C(t) is directly related to subtracted Π (Q2) [Bernecker-Meyer 2011, …]

$$\Pi(Q^2) - \Pi(0) = \sum_t \left(\frac{\cos(qt)-1}{Q^2} + \frac{t^2}{2} \right) C(t)$$

- g-2 value is also related to C(t) with know kernel w(t) from QED.

$$\boxed{a_\mu^{\mathrm{HVP}} = \sum_t w(t)C(t), \quad w(t) \propto t^4 \cdots}$$



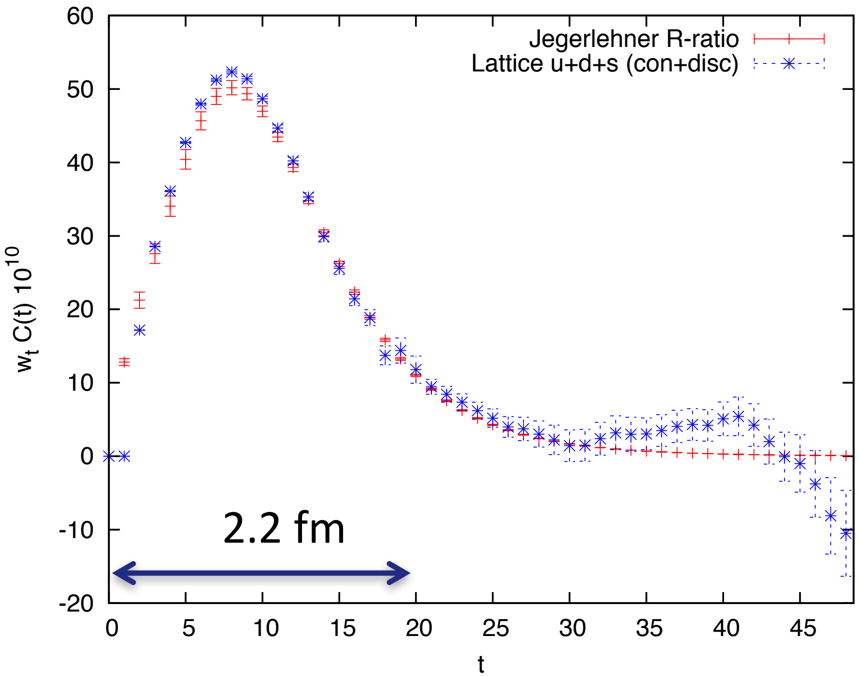


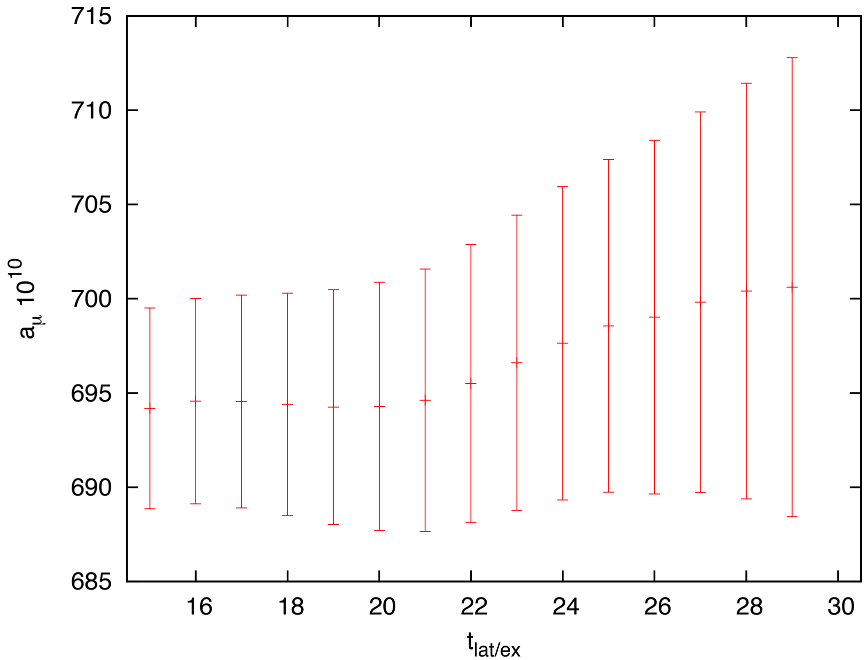
RBC/UKQCD
Chiral Lattice quark DWF
physical point
Quark Propagator Low Mode (A2A)
using All-Mode Averaging (AMA)

(plan B) Interplay between Lattice and Experiment

- Check consistency between Lattice and R-ratio
- Short distance from Lattice, Long distance from R-ratio :
  error <= 1% at $t_{lat/exp}$ = 2fm

$$a_\mu^{\rm HVP} = \left[\sum_{t=0}^{t_{\rm lat/exp}} w(t)C(t)\right]^{\rm LAT} + \left[\int_{t_{\rm lat/exp}}^{\infty} dt\, w(t)C(t)\right]^{\rm EXP}$$



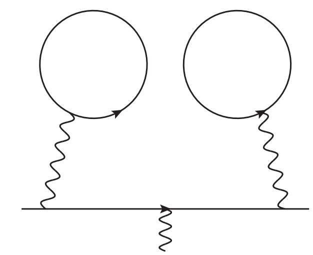


disconnected quark loop contribution

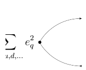
- [C. Lehner et al. (RBC/UKQCD 2015, arXiv:1512.09054, PRL)]
- Very challenging calculation due to statistical noise
- Small contribution, vanishes in SU(3) limit,

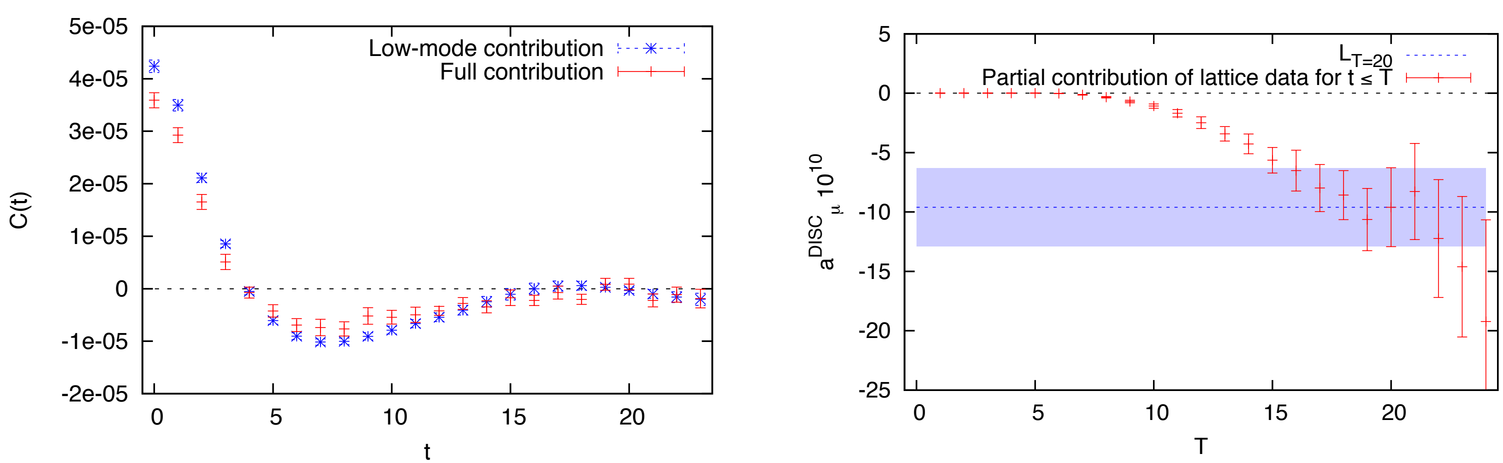
  Qu+Qd+Qs = 0
- Use low mode of quark propagator, treat it exactly

  (all-to-all propagator with sparse random source)
- First non-zero signal



$$a_\mu^{\mathrm{HVP\ (LO)\ DISC}} = -9.6(3.3)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-10}$$





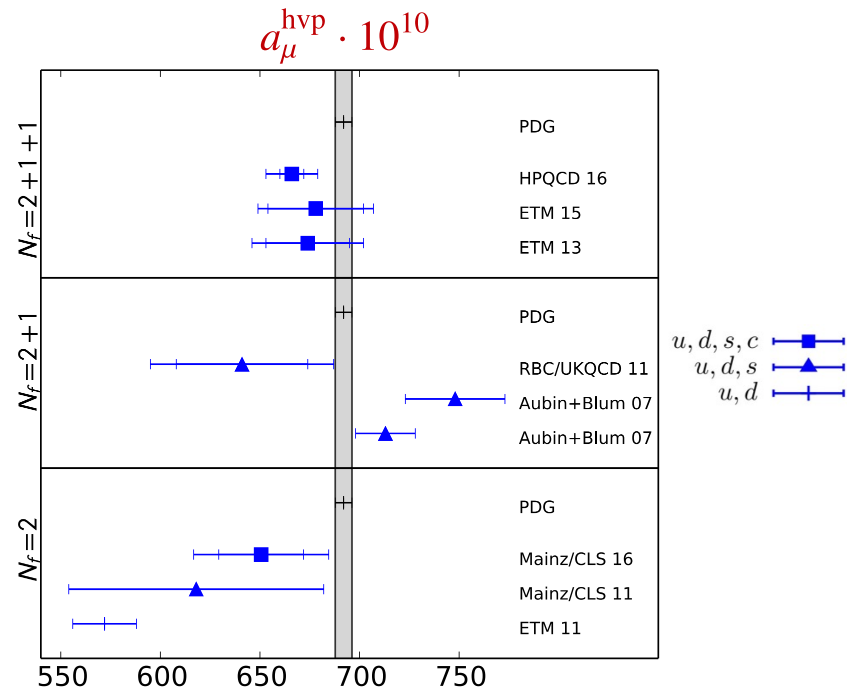
HVP Summary and future prospects

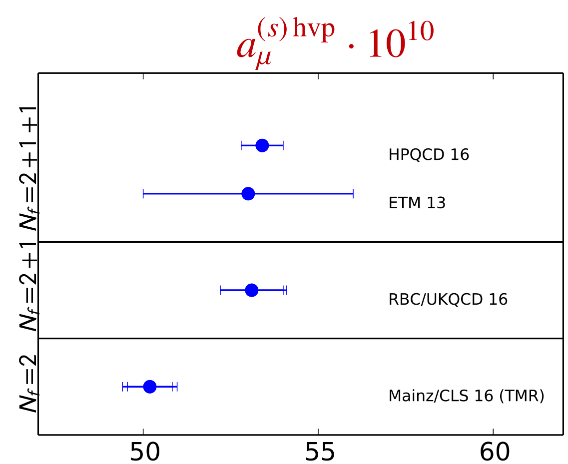
- HVP on Lattice is rapidly progress
- Statistic error is well control
  (low mode, AMA…)
- Disconnected diagram is managed

- Systematic errors

  - *Finite Volume (ππ model ?)*
  - *EM Isospin, ud mass difference*
  - *charm*
  - *discretization error*

- (Plan-B)
  Interplay between Lattice and R-ratio ?



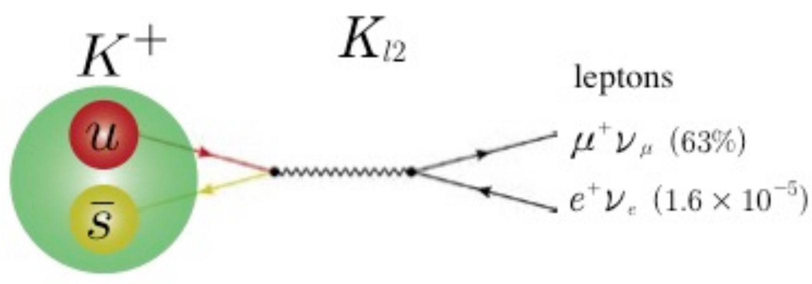


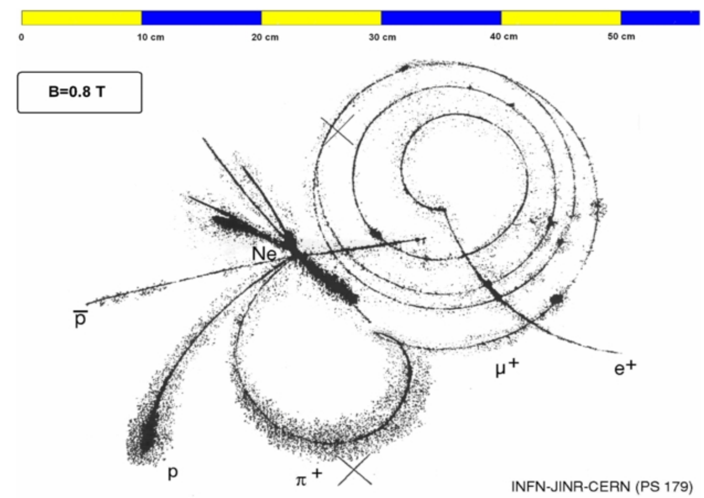
[H. Wittig, LAT16]

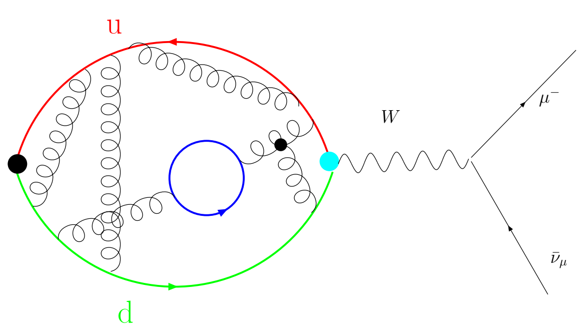
Sub-percent accuracy on Physical point

- now <u>on-physical point (M_π=135 MeV)</u>,
  a few lattice spacing a^{-1} = 1.7 and 2.4 GeV, V~(5.5 fm)^3

$$f_\pi = 0.1298(9)(0)(2)\ \text{GeV}[0.7\%]$$
$$f_K = 0.1556(8)(0)(2)\ \text{GeV}[0.5\%]$$







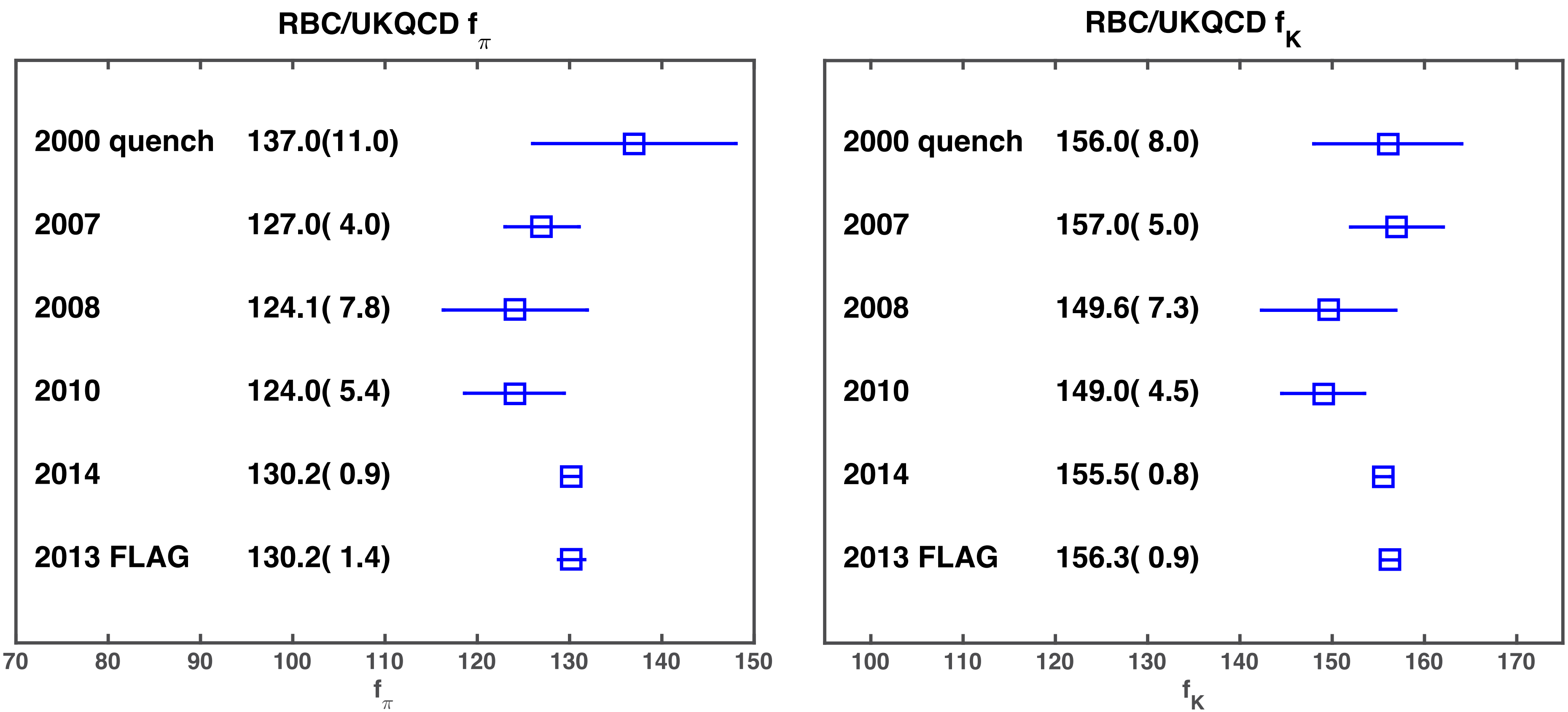
$\langle 0|\bar d\gamma_5 u(0)|\pi\rangle \frac{e^{ipx}}{\sqrt{2E}}\langle\pi|\bar u\gamma_m u\gamma_5 d|0\rangle \times G_F V_{ud} m_\mu \bar\nu(1-\gamma_5)\mu$

$$\mathcal{M}(\pi\to\mu\bar\nu) \sim i f_\pi q_\mu \times G_F V_{ud} m_\mu (\bar\nu\mu)_L$$

$$= \langle\pi(q)|\bar u\gamma_\mu\gamma_5 d(0)|0\rangle \times G_F V_{ud} m_\mu(\bar\nu\mu)_L$$

Sub-percent accuracy on Physical point

- now adding on-physical point (M_π=135 MeV),
  2 lattice spacing a^{-1} = 1.7 and 2.4 GeV, V~(5.5 fm)^3 !

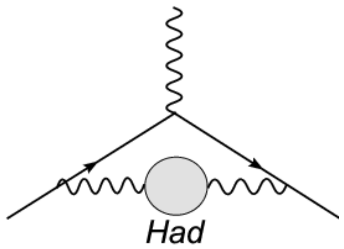


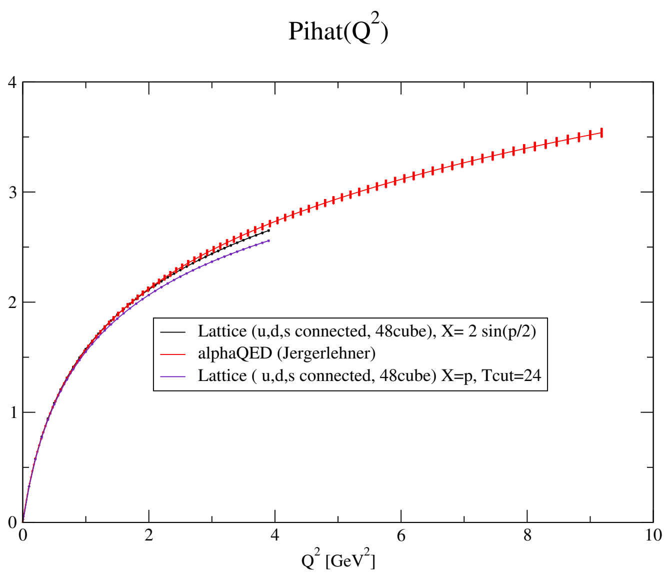
[R. Mawhinney]

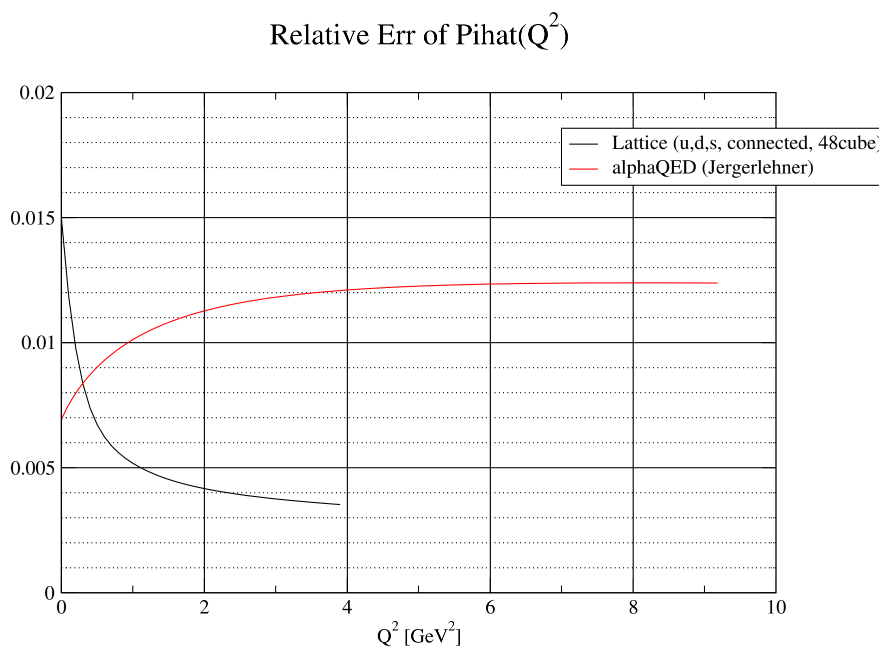
(plan B) Interplays between lattice and dispersive approach g-2

- R-Ratio error ~ 0.6%, HPQCD error ~ 2%
- Goal would be ~ 0.2 %
- Dispersive approach from R-ratio R(s)

$$\hat{\Pi}(Q^2) = \frac{Q^2}{3}\int_{s_0} ds \frac{R(s)}{s(s+Q^2)}$$







also [ETMC, Mainz, ...]

- Can we combine dispersive & lattice and get more precise (g-2)HVP than both ? [2011 Bernecker Meyer]
- Inverse Fourier trans to Euclidean vector correlator
- Relevant for g-2 $Q^2 = (m_\mu/2)^2 = 0.0025\ \mathrm{GeV}^2$
- It may be interesting to think

$$\frac{\hat{\Pi}(Q^2)}{Q^2} = \left[\frac{\hat{\Pi}(Q^2)}{Q^2} - \frac{\hat{\Pi}(P^2)}{P^2}\right]^{\mathrm{Exp}} + \left[\frac{\hat{\Pi}(P^2)}{P^2}\right]^{\mathrm{Lat}}$$

$$a_\mu^{\mathrm{HVP}} = \sum_t w(t)C(t), \quad w(t) \propto t^4 \cdots$$

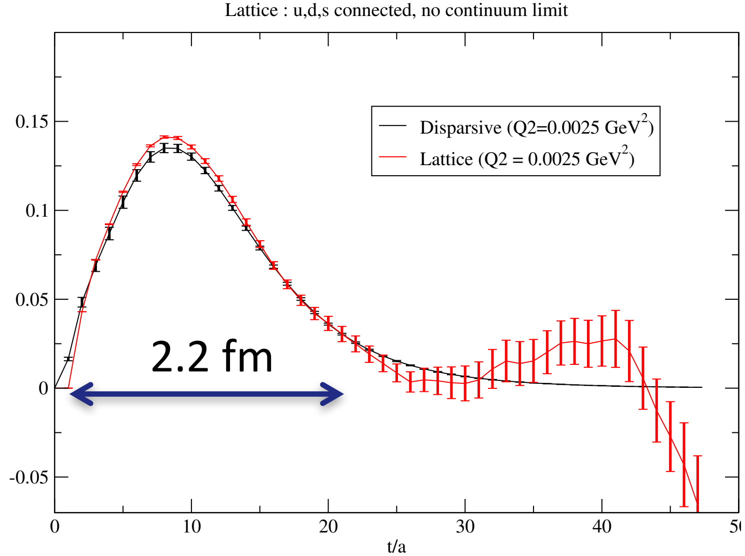
$$\hat{\Pi}(Q^2) = \frac{Q^2}{3}\int_{s_0} ds \frac{R(s)}{s(s+Q^2)}$$

Black : R-ratio , alpha QED (Jegerlehner)

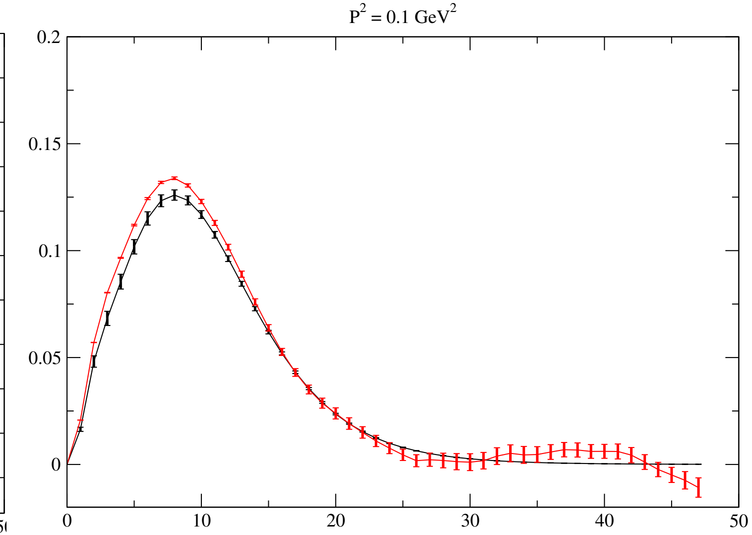
Red : Lattice (DWF)

Pihat(Q2) integrand in coordinate space

Lattice : u,d,s connected, no continuum limit



$P^2 = 0.1\ \mathrm{GeV}^2$

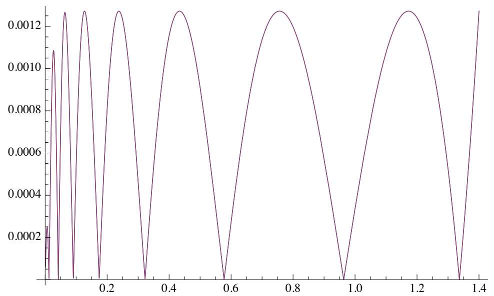


AMA+MADWF(fastPV)+zMobius accelerations

- We utilize complexified 5d hopping term of Mobius action [Brower, Neff, Orginos], zMobius, for a better approximation of the sign function.

$$\epsilon_L(h_M) = \frac{\prod_s^L(1+\omega_s^{-1}h_M) - \prod_s^L(1-\omega_s^{-1}h_M)}{\prod_s^L(1+\omega_s^{-1}h_M) + \prod_s^L(1-\omega_s^{-1}h_M)}, \qquad \omega_s^{-1} = b + c \in \mathbb{C}$$

- 1/a~2 GeV, Ls=48 Shamir ~ Ls=24 Mobius (b=1.5, c=0.5) ~ Ls=10 zMobius (b_s, c_s complex varying) ~5 times saving for cost AND memory



Ls	|eps(48cube) – eps(zMobius)|
6	0.0124
8	0.00127
10	0.000110
12	8.05e-6

- The even/odd preconditioning is optimized (sym2 precondition) to suppress the growth of condition number due to order of magnitudes hierarchy of b_s, c_s [also Neff found this]

$$\texttt{sym2}: \quad 1 - \kappa_b M_4 M_5^{-1} \kappa_b M_4 M_5^{-1}$$

- Fast Pauli Villars (mf=1) solve, needed for the exact solve of AMA via MADWF (Yin, Mawhinney) is speed up by a factor of 4 or more by Fourier acceleration in 5D [Edward, Heller]
- All in all, sloppy solve compared to the traditional CG is 160 times faster on the physical point 48 cube case. And ~100 and 200 times for the 32 cube, Mpi=170 MeV, 140, in this proposal (1,200 eigenV for 32cube) .

$$\underbrace{\frac{20,000}{600}}_{\text{MADWF+zMobius+deflation}} \times \underbrace{\frac{600*32/10}{300}}_{\text{AMA+zMobius}} = 33.3 \times 6.4 = \underline{210 \text{ times faster}}$$

[Blum, TI, Shintani PRD 88 (2013) 094503]

Covariant Approximation Averaging (CAA)
a new class of Error reduction techniques

Original

$$\mathcal{O} = \mathcal{O}^{(\mathrm{appx})} + \mathcal{O}^{(\mathrm{rest})}$$

unbiased imporved

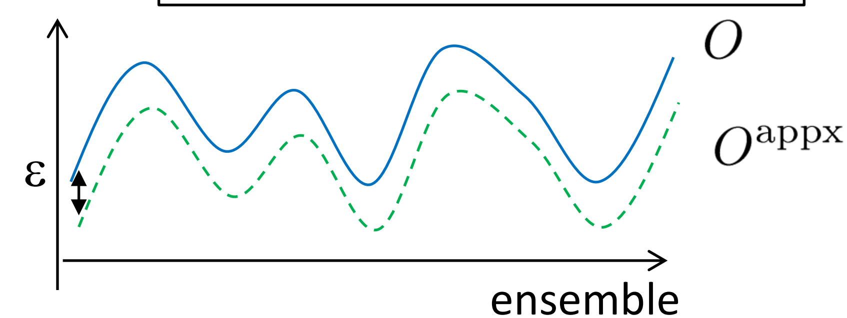
$$\mathcal{O}^{(\mathrm{imp})} = \mathcal{O}^{(\mathrm{rest})} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{(\mathrm{appx}),g}$$

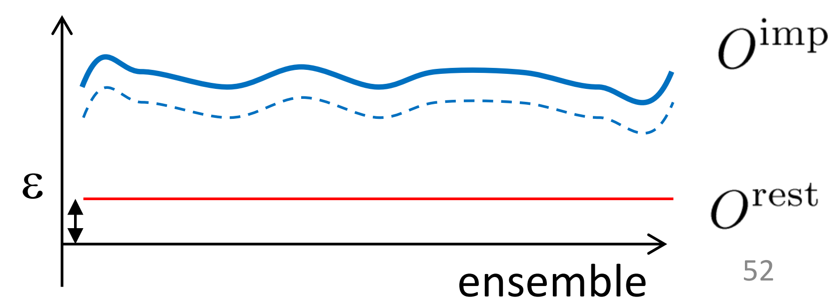
Lattice Symmetry

Expensive : infrequently measured

Cheap : frequently measured

- $O^{(\mathrm{imp})}$ has smaller error
  $O^{(\mathrm{appx})}$ need to be cheap & not to be too accurate
  N_G suppresses the bulk part of noise cheaply





New bias-free estimator even without covariant approximation by a stochastic choice of source location for the exact/rest computation is now available : Appendix D of arXiv:1402.0244

Examples of Covariant Approximations (contd.)

- **All Mode Averaging AMA**

  Sloppy CG or Polynomial approximations

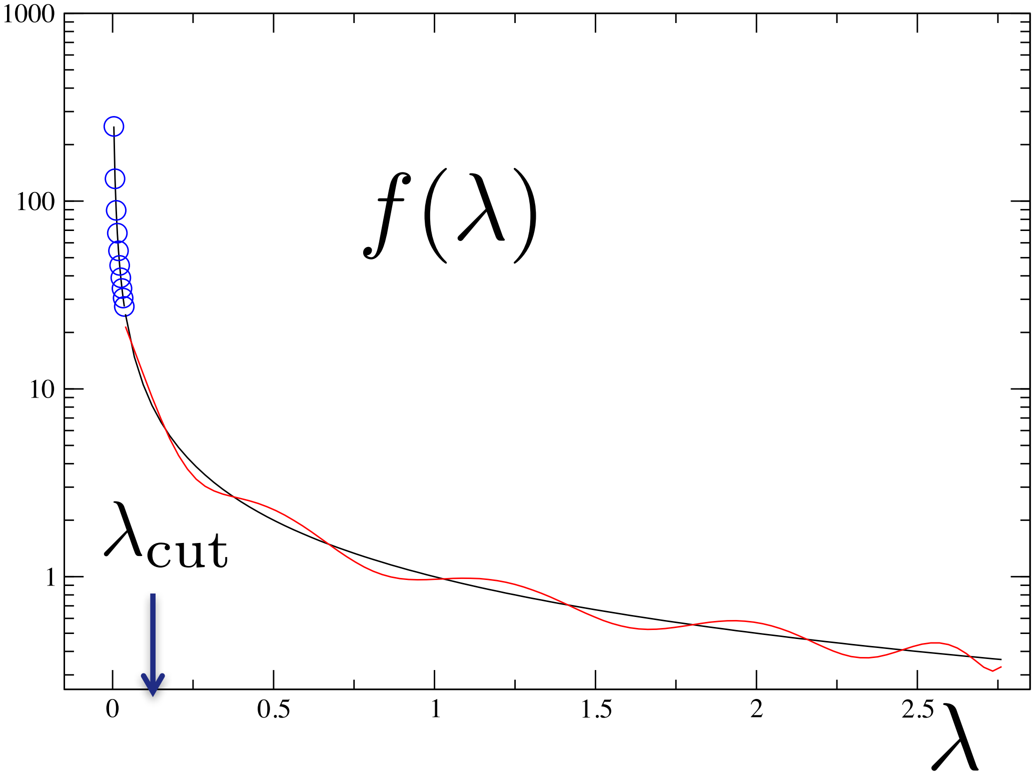
$$\mathcal{O}^{(\text{appx})} = \mathcal{O}[S_l],$$

$$S_l = \sum_\lambda v_\lambda f(\lambda) v_\lambda^\dagger,$$

$$f(\lambda) = \begin{cases} \frac{1}{\lambda}, & |\lambda| < \lambda_{\text{cut}} \\ P_n(\lambda) & |\lambda| > \lambda_{\text{cut}} \end{cases}$$

$$P_n(\lambda) \approx \frac{1}{\lambda}$$

If quark mass is heavy, e.g. ~ strange, low mode isolation may be unneccesary



accuracy control :

- low mode part : # of eig-mode
- mid-high mode : degree of poly.