Five-Loop Running of the QCD Coupling Constant

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with P. Baikov, K. Chetyrkin, arXiv:1606.08.859

I. Generalities

drastic variation of QCD coupling α_s between M_τ and M_Z or M_{Higgs}

example:

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{exp} \pm 0.015_{th}$$

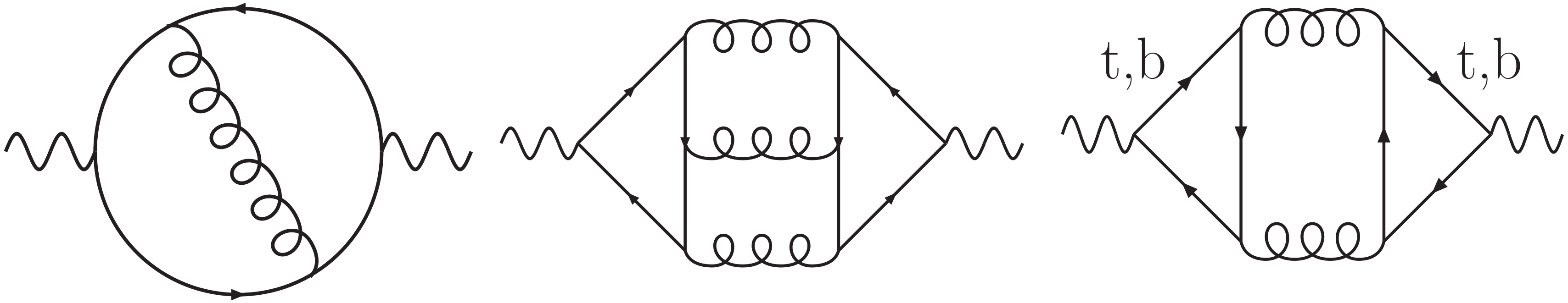
four loop running and matching:

$$\Longrightarrow \alpha_s(M_Z) = 0.1202 \pm 0.0006_{exp} \pm 0.0018_{th} \pm 0.0003_{evol}$$

(evolution error receives contributions from c and b mass, matching scale, four loop truncation of RG equation)

(Baikov, Chetyrkin, JK)

in excellent agreement with direct determination in Z decays (Baikov, Chetyrkin, JK, Rittinger: arXiv:1201.5804)



non-singlet & singlet, vector & axial correlators

$$R^{\rm nc} = 3\Big[\sum_f v_f^2 r_{\rm NS} + \big(\sum_f v_f\big)^2 r_{\rm S}^V + \sum_f a_f^2 r_{\rm NS} + r^A_{\rm S;t,b}\Big] \,,$$

with

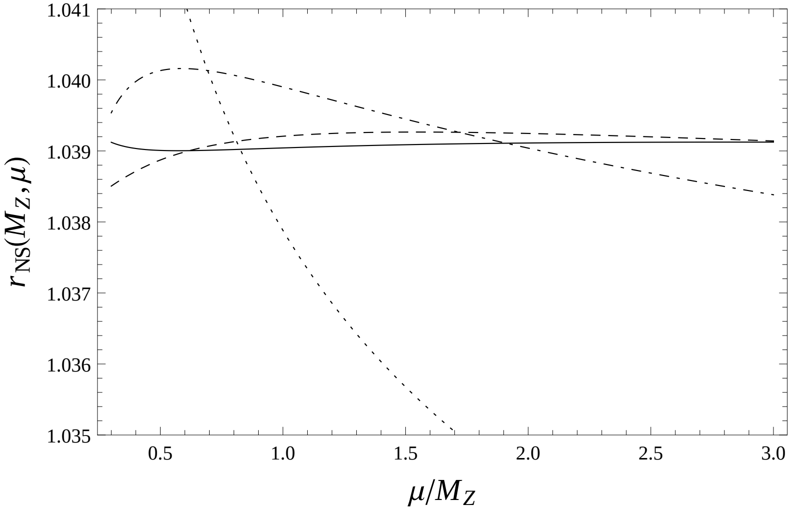
$$r_{\rm NS} = 1 + a_s + 1.40923\, a_s^2 - 12.7671\, a_s^3 - 79.9806\, a_s^4 \,,$$

$$r_{\rm S}^V = -\,0.41318\, a_s^3 - 4.9841\, a_s^4 \,,$$

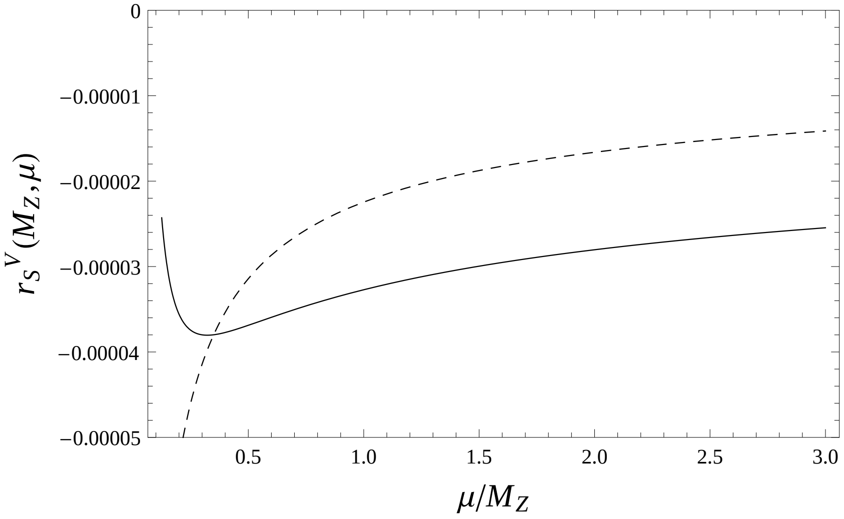
$$r^A_{\rm S:t,b} = (-3.08333 + l_t)\, a_s^2 + (-15.9877 + 3.72222\, l_t + 1.91667\, l_t^2)\, a_s^3$$
$$+ (49.162 - 17.6822\, l_t + 14.7153\, l_t^2 + 3.67361\, l_t^3)\, a_s^4$$

$$\Gamma_Z = \Gamma_0\, R^{\rm nc} = \frac{G_F\, M_Z^3}{24\pi\sqrt{2}}\, R^{\rm nc} \,.$$

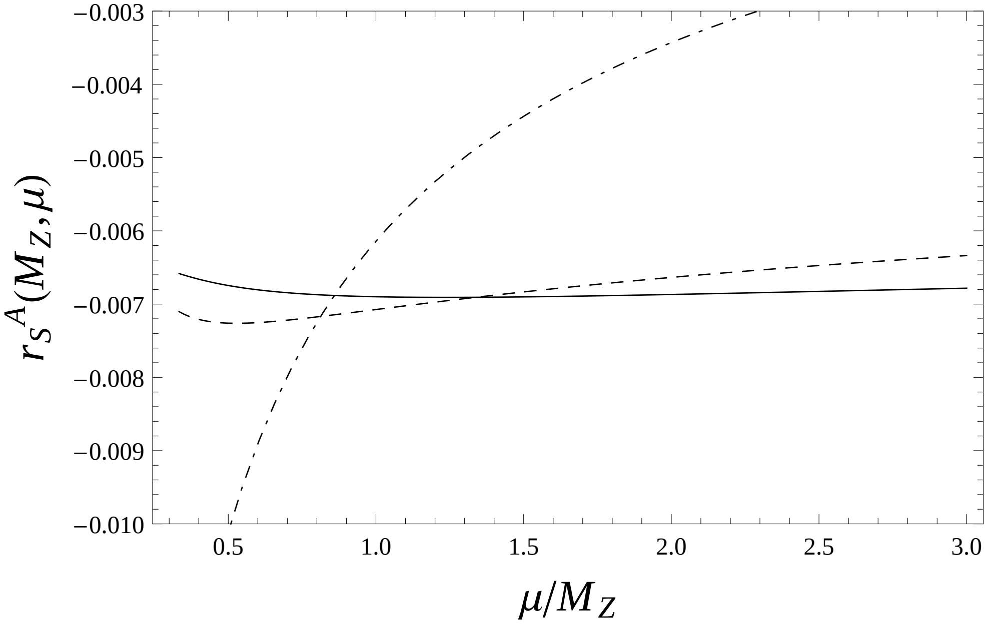
$\Longrightarrow \alpha_s(M_Z) = 0.1190 \pm 0.0026_{exp}$ and small theory error!


$r_{\mathrm{NS}}(M_Z,\mu)$
1.041
1.040
1.039
1.038
1.037
1.036
1.035
0.5
1.0
1.5
2.0
2.5
3.0
μ/M_Z

(a)


$r_S{}^V(M_Z,\mu)$
0
−0.00001
−0.00002
−0.00003
−0.00004
−0.00005
0.5
1.0
1.5
2.0
2.5
3.0
μ/M_Z

(b)


$r_S{}^A(M_Z,\mu)$
−0.003
−0.004
−0.005
−0.006
−0.007
−0.008
−0.009
−0.010
0.5
1.0
1.5
2.0
2.5
3.0
μ/M_Z

(c)

II. Quark Mass and Field Anomalous Dimension

"Running" quark masses depend on renormalization scale

$$\mu^2 \frac{d}{d\mu^2} m|_{g^0, m^0} = m \gamma_m(a_s) \equiv -m \sum_{i \geq 0} \gamma_i \, a_s^{i+1},$$

(with $a_s = \alpha_s/\pi$ and γ_i known from γ_0 to γ_4)
in numerical form:

$$\gamma_m \underset{n_f=4}{=\!=} - \quad a_s - 3.65278\, a_s^2 - 9.94704\, a_s^3 - 27.3029\, a_s^4 - 111.59\, a_s^5,$$

$$\gamma_m \underset{n_f=5}{=\!=} - \quad a_s - 3.51389\, a_s^2 - 7.41986\, a_s^3 - 11.0343\, a_s^4 - 41.8205\, a_s^5$$

n_f	3	4	5	6
$(\gamma_m)_4^{\text{exact}}$	198.899	111.579	41.807	-9.777
$(\gamma_m)_4^{\text{APAP}}$ [Ellis et al]	162.0	67.1	-13.7	-80.0
$(\gamma_m)_4^{\text{APAP}}$ [Elias et al]	163.0	75.2	12.6	12.2
$(\gamma_m)_4^{\text{APAP}}$ [Kataev et al]	164.0	71.6	-4.8	-64.6

exact values for $(\gamma_m)_4$ and predictions based on APAP

III. Higgs decays into quarks

$$\Gamma(H \to \bar{f} f) = \frac{G_F \, M_H}{4\sqrt{2}\pi} \, m_f^2(M_H) \, R^S(s = M_H^2, \mu = M_H)$$

requires (for $a_s(M_H) = 0.0364$)

$$\begin{aligned} R^S(s = M_H^2, \mu = M_H) &= 1 + 5.667 \, a_s + 29.147 \, a_s^2 + 41.758 \, a_s^3 - 825.7 \, a_s^4 \\ &= 1 + 0.2062 + 0.0386 + 0.0020 - 0.00145 \end{aligned}$$

and $m_b(M_H)$. The value of $m_b(M_H)$ is determined by $m_b(10\text{GeV})$ as starting point and its running to $M_H = 125$ GeV.

The shift from the five-loop term

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1 \cdot 10^{-4}$$

is completely negligible

IV. Five-loop β-function

$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i\geq 0} \beta_i a_s^{i+2}$$

$$\beta_0 = \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\},$$

Gross + Wilczek, Politzer

$$\beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\},$$

Caswell, Jones

$$\beta_2 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\},$$

Tarasov + Vladimirov + Zharkov, Larin + Vermaseren

$$\beta_3 = \frac{1}{4^4}\left\{\frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right] n_f + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right] n_f^2 + \frac{1093}{729}n_f^3\right\},$$

van Ritbergen + Vermaseren + Larin, Czakon

$$
\begin{aligned}
\beta_4 = \frac{1}{4^5}\Bigg\{ & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad n_f & \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5 \right] \\
+ \quad n_f^2 & \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right] \\
+ \quad n_f^3 & \left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right] \\
+ \quad n_f^4 & \left[\frac{1205}{2916} - \frac{152}{81}\zeta_3 \right] \Bigg\}
\end{aligned}
$$

Baikov, Chetykin, JK

Absence of ζ_4 and ζ_5 in β_3 term!

Absence of ζ_6 and ζ_7 in β_4 term!

Numerically the terms are surprisingly small!

Consider $\overline{\beta} \equiv \frac{\beta}{-\beta_0\, a_s^2} = 1 + \sum_{i\geq 1} \overline{\beta}_i\, a_s^i$

$$\begin{aligned}
\overline{\beta}(n_f = 3) &= 1 + 1.78\, a_s + 4.47\, a_s^2 + 20.99\, a_s^3 + 56.59\, a_s^4, \\
\overline{\beta}(n_f = 4) &= 1 + 1.54\, a_s + 3.05\, a_s^2 + 15.07\, a_s^3 + 27.33\, a_s^4, \\
\overline{\beta}(n_f = 5) &= 1 + 1.26\, a_s + 1.47\, a_s^2 + 9.83\, a_s^3 + 7.88\, a_s^4, \\
\overline{\beta}(n_f = 6) &= 1 + 0.93\, a_s - 0.29\, a_s^2 + 5.52\, a_s^3 + 0.15\, a_s^4
\end{aligned}$$

very modest growth of coefficients!

qualitative agreement with β_4, as calculated with Asymptotic Pade Approximant (APAP)

$$\beta_4^{APAP} = 740 - 213\, n_f + 20\, n_f^2 - 0.0486\, n_f^3 - \boxed{0.0017993\, n_f^4} \leftarrow \text{input}$$

$$\beta_4^{exact} = 524.56 - 181.8\, n_f + 17.16\, n_f^2 - 0.22586\, n_f^3 - 0.0017993 n_f^4$$

but large cancellations $\Longrightarrow$ numerical disagreement

n_f	0	1	2	3	4	5	6
β_4^{exact}	525	360	228	127	57	15	0.27
β_4^{APAP}	741	548	395	281	205	169	170

excellent agreement between $\alpha_s(M_Z)$ from τ decays (+ running and matching) and direct measurement

$$\alpha_s(m_\tau) = 0.33 \pm 0.014 \Longrightarrow \alpha_s(M_Z) = 0.1198 \pm 0.0015$$

vs $\alpha_s(M_Z) = 0.1197 \pm 0.0028$ **directly from Z-decay**

SUMMARY

- QCD corrections for
  Higgs decay to $f\overline{f}$, Higgs decay to gluons, τ decay to $\nu + had$,
  $\boldsymbol{R = \frac{\sigma_{tot}(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}}$, Z decay to $f\overline{f}$. All are available to $\mathcal{O}({\alpha_s}^4)$
  corresponding to 4 loops

- matched by QCD β-function in 5-loops

- excellent agreement between theory and experiment

- theory prediction is significantly ahead of experiment