Leading twist light-cone distributions amplitudes for S-wave and P-wave quarkonia and Bc mesons

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based on works collaborated
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Contents

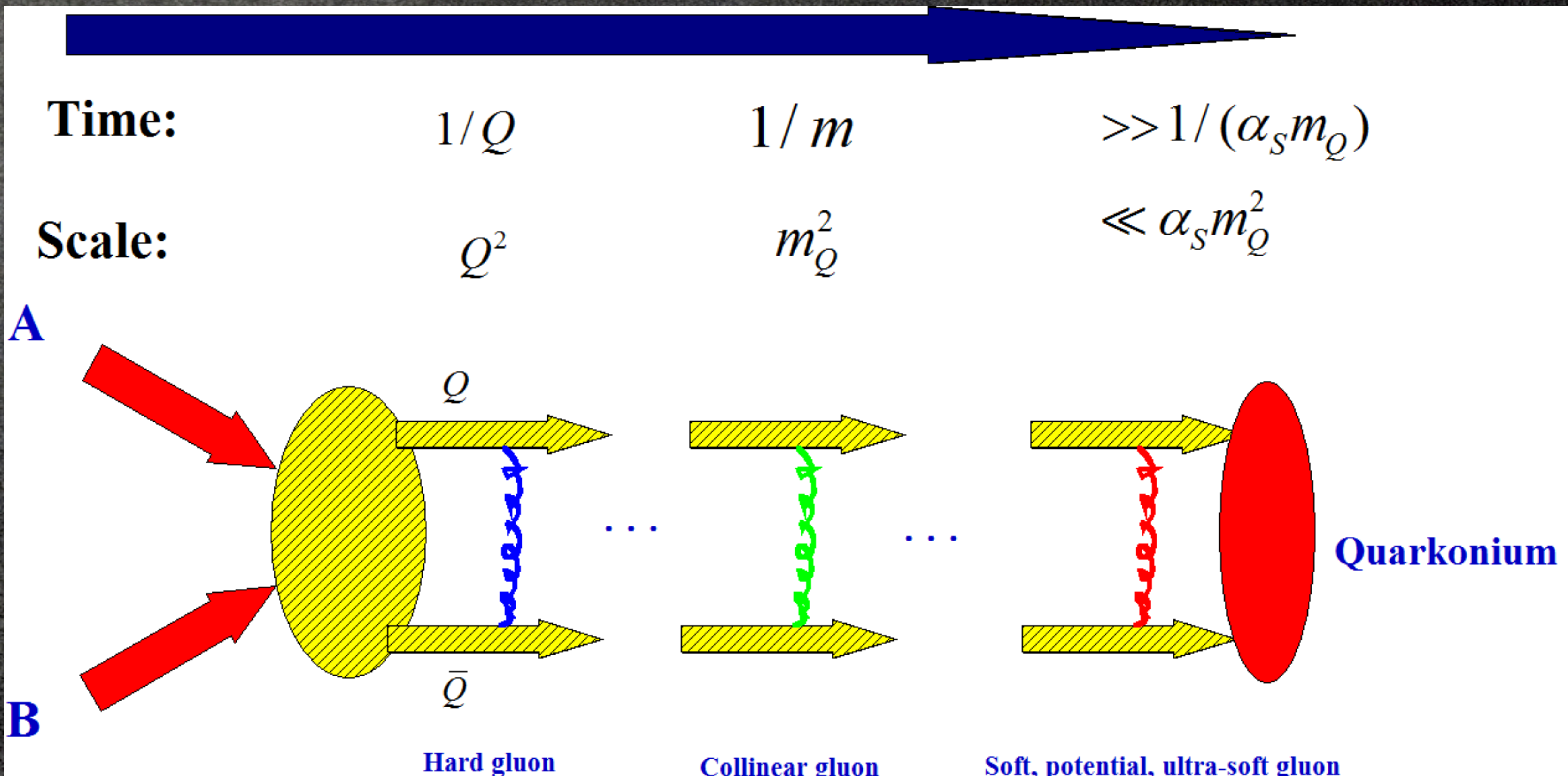
- Hard exclusive processes involving quarkonia
  - NRQCD factorization vs collinear factorization
  - Refactorization of LCDA for quarkonia and Bc meson
- Calculations of the LCDAs
- Some applications
- Summary

Hard exclusive processes

- Multi-scale problem: Hard exclusive processes are important for testing perturbative QCD;
- Collinear Factorization: The amplitude can be factorized into the convolution of the perturbatively calculable hard-kernels and some universal light-cone distribution amplitudes (LCDAs);
- For instance, $\gamma^* \to \pi^0 \gamma$, $Q^2 >> \Lambda_{QCD}$

$$F(Q^2) = f_\pi \int_0^1 dx\, \underbrace{T_H(x; Q^2, \mu)}_{\text{hard-kernel}} \underbrace{\phi_\pi(x;\mu)}_{\text{LCDA}} + \mathcal{O}(m_Q^2/Q^2)$$

Intuitive picture of quarkonia productions -- multi-scale problems



Main difficulties

- Entanglement of many scales

✦ Quarkonia： $m >> mv >> mv^2$

✦ Hard processes： $Q >> m >> \Lambda_{QCD}$

- Two-faces of QCD

✦ Asymptotic freedom at short-distance： perturbatively calculable

✦ Color confinement at long-distance：

completely non-perturbative at scale of $\Lambda_{QCD} \sim 200$ MeV

Solutions

- Factorization： separation of dynamics from different scale regions and different hadrons
    - ✦ Short-distance contributions： perturbative calculations
    - ✦ Long-distance contributions： universal parameters
- Effective field theory： reproduce the low-energy behavior of the underlying theory
    - ✦ simplifying the proof of factorization
    - ✦ matching with clear target
    - ✦ parameterizing long-distance effects by matrix-elements of field operators
    - ✦ stabilizing the perturbation series by RGE resummation

NRQCD factorization

- Short-distance:
  - ✦ short-distance coefficients encoding all effects above heavy quark mass scale;
- Long-distance:
  - ✦ NRQCD matrix-elements/Schrodinger wave-function at origin
- Difficulity: stability of the perturbation series
  - ✦ When Q>>m, log(m/Q) terms in short-distance coefficients cannot be resummed within the frame of NRQCD

Collinear factorization

Example: $\gamma^* \to \eta_Q \gamma$, $Q^2 >> m^2_Q$

$$F(Q^2) = f_{\eta_Q} \int_0^1 dx\, T_H(x; Q^2, \mu) \phi_{\eta_Q}(x; \mu) + \mathcal{O}(m_Q^2/Q^2)$$

- Short-distance： hard-kernel T_H
- Long-distance： LCDA satisfying the ERBL equation

which can be used to resum the large log(m/Q) terms

$$\frac{d}{d \ln \mu^2} f_{\eta_Q} \phi_{\eta_Q}(x; \mu) = \frac{\alpha_s}{2\pi} C_F \int_0^1 V_0(x, y) f_{\eta_Q} \phi_{\eta_Q}(y; \mu)$$

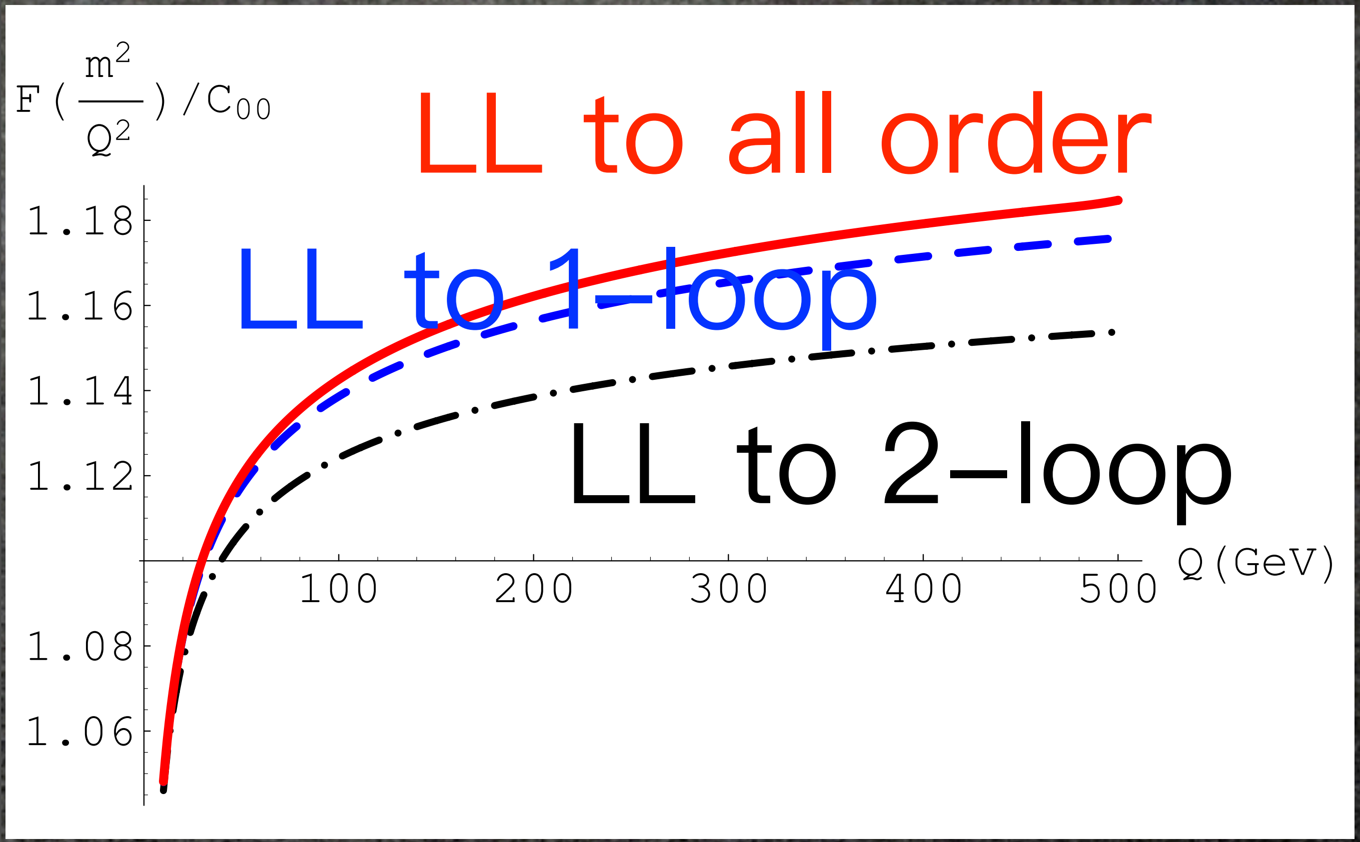
The connections between two factorizations

Ma & Si, PLB, 2006

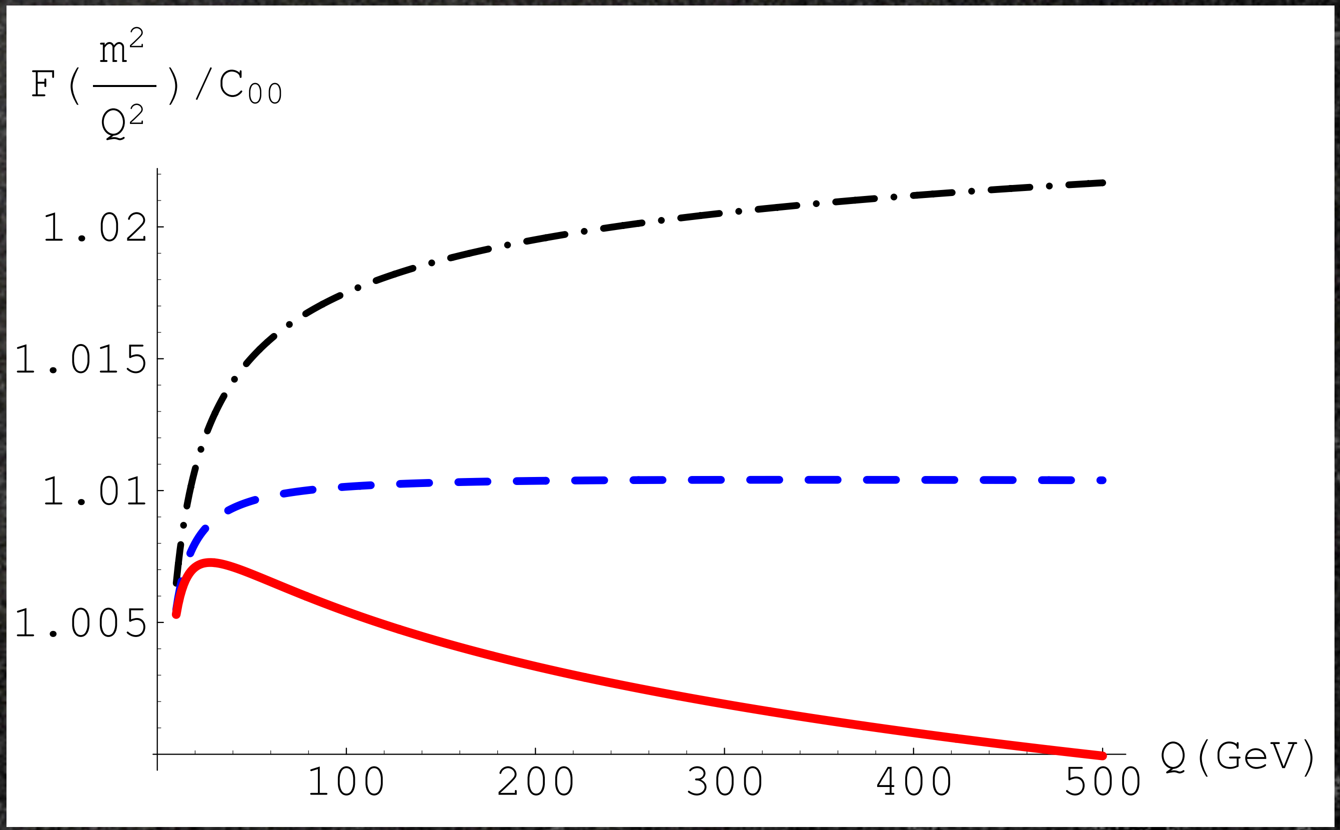
- The LCDA is essential!
- The LCDA for light-mesons are completely non-perturbative
  - ✦ Sum Rules, Lattice QCD, ADS/CFT;
- LCDAs for quarkonia: quarkonia are non-relativistic bound states of heavy quark and anti-quark
  - ✦ constrained by NRQCD matrix-elements;
  - ✦ the distribution part can be calculated through perturbative matching;
  - ✦ reduce the number of non-perturbative parameters.

Resume the large logs by ERBL equation

Y.Jia & D.Yang, NPB 2009



$\gamma^* \to \eta_b \gamma$



$h \to \Upsilon \gamma$

Bc EM form-factor

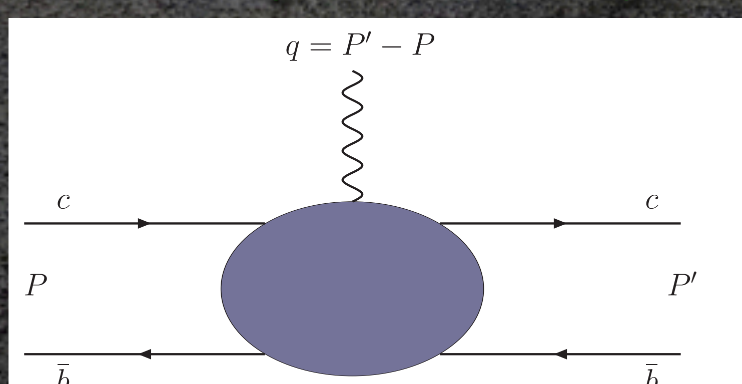
- LCDA for Bc

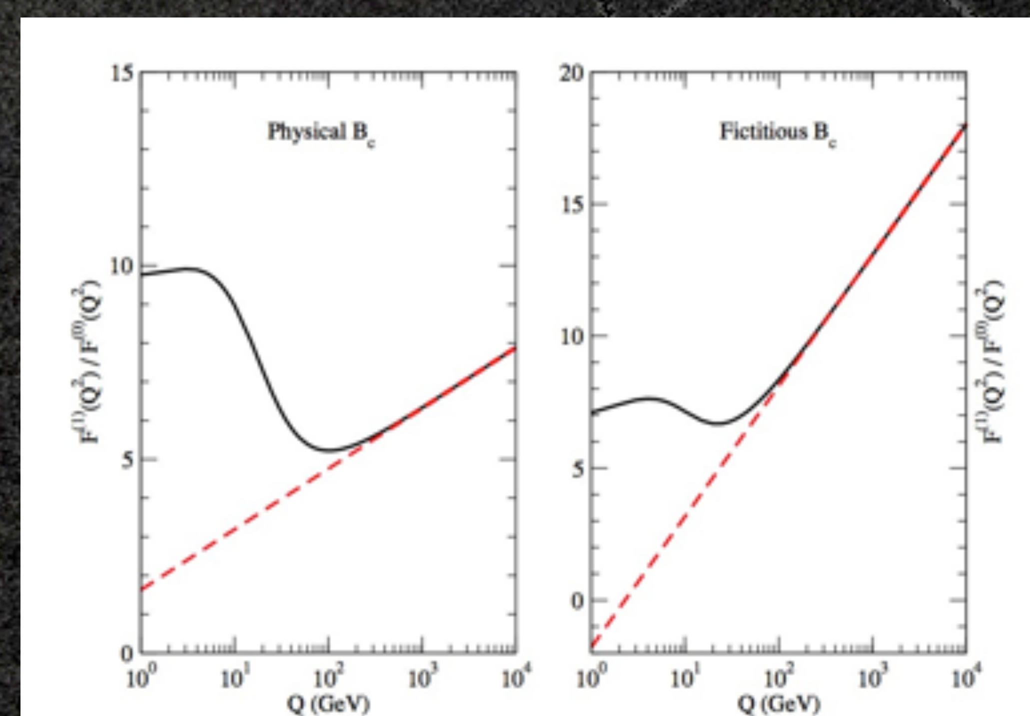
Bell & Feldmann, NPB 2008

$$\hat{\phi}^{(1)}(x,\mu_F^2) = \frac{C_F}{2}\left\{\left(\ln\frac{\mu_F^2}{M_{B_c}^2(x_0-x)^2}-1\right)\left[\frac{x_0+\bar{x}}{x_0-x}\frac{x}{x_0}\theta(x_0-x)+\begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right]\right\}_+ + C_F\left\{\left(\frac{x\bar{x}}{(x_0-x)^2}\right)_{++}+\frac{1}{2}\delta'(x-x_0)\left(2x_0\bar{x}_0\ln\frac{x_0}{\bar{x}_0}+x_0-\bar{x}_0\right)\right\}. \quad (4.15)$$

- Reproduce the asymptotic behavior

Y.Jia, J.X. Wang and D.Yang, JHEP2011





Twist-2 LCDAs for quarkonia

- Defined by the matrix-element of following non-local operators

$$J[\Gamma](\omega) \equiv (\bar{Q}W_c)(\omega n_+/2)\not{n}_+\Gamma(W_c^\dagger Q)(-\omega n_+/2)\,,$$

- LCDAs for S-wave quarkonia

$$\langle H(^1S_0,P)|J[\gamma_5](\omega)|0\rangle = -if_P n_+P\int_0^1 dx\, e^{i\omega n_+P(x-1/2)}\hat{\phi}_P(x;m,\mu)\,,$$

$$\langle H(^3S_1,P,\varepsilon^*)|J[1]|0(\omega)\rangle = -if_V m_V n_+\varepsilon\int_0^1 dx\, e^{i\omega n_+P(x-1/2)}\hat{\phi}_V^{||}(x;m,\mu)\,,$$

$$\langle H(^3S_1,P,\varepsilon^*)|J[\gamma_\perp^\alpha](\omega)|0\rangle = -if_V^\perp n_+P\varepsilon_\perp^{*\alpha}\int_0^1 dx\, e^{i\omega n_+P(x-1/2)}\hat{\phi}_V^\perp(x;m,\mu)\,,$$

Twist-2 LCDAs for quarkonia

- Defined by the matrix-element of following non-local operators

$$J[\Gamma](\omega) \equiv (\bar{Q}W_c)(\omega n_+/2)\not{n}_+\Gamma(W_c^\dagger Q)(-\omega n_+/2)\,,$$

- Wilson Line

$$W_c(x) = \mathrm{P}\ \exp\Big(ig_s \int_{-\infty}^{0} ds\ n_+A(x+sn_+)\Big)$$

- 3 twist-2 LCDAs for S-wave quarkonia

$$\langle H(^1S_0, P)|J[\gamma_5](\omega)|0\rangle = -if_P n_+P \int_0^1 dx\, e^{i\omega n_+P(x-1/2)}\hat{\phi}_P(x;m,\mu)\,,$$

$$\langle H(^3S_1, P, \varepsilon^*)|J[1]|0(\omega)\rangle = -if_V m_V n_+\varepsilon \int_0^1 dx\, e^{i\omega n_+P(x-1/2)}\hat{\phi}_V^{\|}(x;m,\mu)\,,$$

$$\langle H(^3S_1, P, \varepsilon^*)|J[\gamma_\perp^\alpha](\omega)|0\rangle = -if_V^\perp n_+P\varepsilon_\perp^{*\alpha} \int_0^1 dx\, e^{i\omega n_+P(x-1/2)}\hat{\phi}_V^\perp(x;m,\mu)\,,$$

- 7 twist-2 LCDAs for P-wave quarkonia

$$\langle H(^1P_1, P, \varepsilon^*)|J[\gamma_5](\omega)|0\rangle = if_{1A}m_{1A}n_+\varepsilon^* \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}^{\parallel}_{1A}(x; m, \mu)\,,$$

$$\langle H(^1P_1, P, \varepsilon^*)|J[\gamma_\perp^\alpha \gamma_5](\omega)|0\rangle = if^{\perp}_{1A}n_+P\varepsilon_\perp^{*\alpha} \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}^{\perp}_{1A}(x; m, \mu)\,,$$

$$\langle H(^3P_0, P)|J[1](\omega)|0\rangle = f_S n_+P \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}_S(x; m, \mu)\,,$$

$$\langle H(^3P_1, P, \varepsilon^*)|J[\gamma_5](\omega)|0\rangle = if_{3A}m_{3A}n_+\varepsilon^* \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}^{\parallel}_{3A}(x; m, \mu)\,,$$

$$\langle H(^3P_1, P, \varepsilon^*)|J[\gamma_\perp^\alpha \gamma_5](\omega)|0\rangle = if^{\perp}_{3A}n_+P\varepsilon_\perp^{*\alpha} \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}^{\perp}_{3A}(x; m, \mu)\,,$$

$$\langle H(^3P_2, P, \varepsilon)|J[1](\omega)|0\rangle = f_T \frac{m_T^2}{n_+P} n_{+\alpha}n_{+\beta}\varepsilon^{*\alpha\beta} \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}^{\parallel}_T(x; m, \mu)\,,$$

$$\langle H(^3P_2, P, \varepsilon^*)|J[\gamma_\perp^\alpha](\omega)|0\rangle = f^{\perp}_T m_T n_{+\rho}\varepsilon^{*\rho\alpha_\perp} \int_0^1 dx\, e^{i\omega n_+ P(x-1/2)}\hat{\phi}^{\perp}_T(x; m, \mu)\,,$$

Fourier transformed form

- Fourier transformation:

$$Q[\Gamma](x) \equiv \int \frac{d\omega}{2\pi}\, e^{-i(x-1/2)\omega n_+ P}\, J[\Gamma](\omega),$$

- 3 LCDAs for S-wave quarkonia

$$\langle H(^1S_1, P)|Q[\gamma_5](x)|0\rangle = -if_P\hat{\phi}_P(x)\,,$$

$$\langle H(^3S_1, P, \varepsilon^*)|Q[1](x)|0\rangle = -if_V\frac{m_V n_+\varepsilon^*}{n_+P}\hat{\phi}_V^{\parallel}(x)\,,$$

$$\langle H(^3S_1, P, \varepsilon^*)|Q[\gamma_\perp^\alpha](x)|0\rangle = -if_V^\perp\varepsilon_\perp^{*\alpha}\hat{\phi}_V^\perp(x)\,,$$

- 7 LCDAs for S-wave quarkonia

$$\langle H(^1P_1, P, \varepsilon^*)|Q[\gamma_5](x)|0\rangle = -if_{1A}\frac{m_{1A}n_+\varepsilon^*}{n_+P}\hat{\phi}^{\parallel}_{1A}(x)\,,$$

$$\langle H(^1P_1, P, \varepsilon^*)|Q[\gamma^\alpha_\perp\gamma_5](x)|0\rangle = -if^\perp_{1A}\varepsilon^{*\alpha}_\perp\hat{\phi}^\perp_{1A}(x)\,,$$

$$\langle H(^3P_0, P)|Q[1](x)|0\rangle = f_S\hat{\phi}_S(x)\,,$$

$$\langle H(^3P_1, P, \varepsilon^*)|Q[\gamma_5](x)|0\rangle = -if_{3A}\frac{m_{3A}n_+\varepsilon^*}{n_+P}\hat{\phi}^{\parallel}_{3A}(x)\,,$$

$$\langle H(^3P_1, P, \varepsilon^*)|Q[\gamma^\alpha_\perp\gamma_5](x)|0\rangle = -if^\perp_{3A}\varepsilon^{*\alpha}_\perp\hat{\phi}^\perp_{3A}(x)\,,$$

$$\langle H(^3P_2, P, \varepsilon^*)|Q[1](x)|0\rangle = f_T\frac{m_T^2 n_{+\alpha}n_{+\beta}\varepsilon^{*\alpha\beta}}{(n_+P)^2}\hat{\phi}^{\parallel}_T(x)\,,$$

$$\langle H(^3P_2, P, \varepsilon^*)|Q[\gamma^\alpha_\perp](x)|0\rangle = f^\perp_T\frac{m_T n_{+\rho}\varepsilon^{*\rho\alpha_\perp}}{n_+P}\hat{\phi}^\perp_T(x)\,.$$

All 10 LCDAs are defined in a boost invariant way!

Matching to NRQCD

- Matching equations:

$$\langle H|Q[\Gamma](x,\mu)|0\rangle \simeq \sum_{n=0}^{\infty} C_\Gamma^n(x,\mu)\langle H|O_{\Gamma,n}^{\mathrm{NRQCD}}|0\rangle\,.$$

- Necessary NRQCD operators:

$$\mathcal{O}(^1S_0) \equiv \bar\psi_v \gamma_5 \chi_v\,,$$

$$\mathcal{O}^\mu(^3S_1) \equiv \bar\psi_v \gamma_\top^\mu \chi_v\,,$$

$$\mathcal{O}^\mu(^1P_1) \equiv \bar\psi_v \left[\left(-\frac{i}{2}\right)\overleftrightarrow{D}_\top^\mu \gamma_5\right]\chi_v\,,$$

$$\mathcal{O}(^3P_0) \equiv \bar\psi_v \left[-\frac{1}{\sqrt{3}}\left(-\frac{i}{2}\right)\overleftrightarrow{\not{D}}_\top\right]\chi_v\,,$$

$$\mathcal{O}^{\rho\mu\nu}(^3P_1) \equiv \frac{1}{2\sqrt{2}}\bar\psi_v\left(-\frac{i}{2}\right)\overleftrightarrow{D}_\top^\rho\,[\gamma_\top^\mu,\gamma_\top^\nu]\,\gamma_5\chi_v\,,$$

$$\mathcal{O}^\mu(^3P_1) \equiv \frac{1}{2\sqrt{2}}\bar\psi_v\left(-\frac{i}{2}\right)\left[\overleftrightarrow{\not{D}}_\top,\gamma_\top^\mu\right]\gamma_5\chi_v\,,$$

$$\mathcal{O}^{\mu\nu}(^3P_2) \equiv \bar\psi_v\left[\left(-\frac{i}{2}\right)\overleftrightarrow{D}_\top^{(\mu}\,\gamma_\top^{\nu)}\right]\chi_v\,.$$

Each operator corresponds to a given quarkonium!

10 LCDAs→6 NRQCD matrix-elements!

NRQCD matrix-elements

$$
\begin{cases}
\langle \eta_Q | \mathcal{O}(^1S_0) | 0 \rangle = \langle \mathcal{O}(^1S_0) \rangle, \\
\langle \Psi/\Upsilon | \mathcal{O}^\mu(^3S_1) | 0 \rangle = \varepsilon^{*\mu} \langle \mathcal{O}(^1S_0) \rangle, \\
\langle h_Q | \mathcal{O}^\mu(^1P_1) | 0 \rangle = \varepsilon^{*\mu} \langle \mathcal{O}(^3P_0) \rangle, \\
\langle \chi_{Q0} | \mathcal{O}(^3P_0) | 0 \rangle = \langle \mathcal{O}(^3P_0) \rangle, \\
\langle \chi_{Q1} | \mathcal{O}^{\rho\mu\nu}(^3P_1) | 0 \rangle = \frac{1}{2-d} \left(\varepsilon^{*\mu} \left(g^{\rho\nu} - \frac{P^\rho P^\nu}{m_{3A}^2} \right) - \varepsilon^{*\nu} \left(g^{\rho\mu} - \frac{P^\rho P^\mu}{m_{3A}^2} \right) \right) \langle \mathcal{O}(^3P_0) \rangle, \\
\langle \chi_{Q1} | \mathcal{O}^\mu(^3P_1) | 0 \rangle = \varepsilon^{*\mu} \langle \mathcal{O}(^3P_0) \rangle, \\
\langle \chi_{Q2} | \mathcal{O}^{\mu\nu}(^3P_2) | 0 \rangle = \varepsilon^{*\mu\nu} \langle \mathcal{O}(^3P_0) \rangle,
\end{cases}
$$

$$
\langle \mathcal{O}(^1S_0) \rangle = \sqrt{2N_c}\sqrt{2M_{\eta_Q}}\sqrt{\frac{1}{4\pi}} R_{10}(0)\,,
$$

$$
\langle \mathcal{O}(^3P_0) \rangle = \sqrt{2N_c}\sqrt{2M_{\chi_{Q0}}}\,(-i)\sqrt{\frac{3}{4\pi}} R'_{21}(0)\,,
$$

Spin symmetry：

10 LCDAs→6 NRQCD MEs→2 NRQCD MEs!

Tree-level matching

$$
\begin{aligned}
&\langle Q^a(p_1)\bar{Q}^b(p_2)|Q[\Gamma](x)|0\rangle \\
&= \delta^{ab}\int \frac{d\omega}{2\pi}\, e^{-i(x-1/2)\omega n_+P + i\omega n_+\bar{q}}\,\bar{u}(p_1)\not{n}_+\Gamma v(p_2) \\
&= \frac{\delta^{ab}}{n_+P}\delta\left(x - 1/2 - \frac{n_+\bar{q}}{n_+P}\right)\bar{u}(p_1)\not{n}_+\Gamma v(p_2) \\
&= \frac{\delta^{ab}}{n_+P}\left(\left(\delta(x-1/2) - \delta'(x-1/2)\frac{n_+\bar{q}}{n_+P}\right)\bar{u}_v(p_1)\not{n}_+\Gamma v_v(p_2)\right. \\
&\quad \left. + \delta(x-1/2)\frac{1}{2m}\bar{u}_v(p_1)\left\{\not{q}, \not{n}_+\Gamma\right\}v_v(p_2) + \mathcal{O}(v^2)\right),
\end{aligned}
$$

S-wave

P-wave

Tree-level matching cont.

$$\bar{u}_v(p_1)\not{n}_+\gamma_5 v_v(p_2) = n_+v\bar{u}_v(p_1)\gamma_5 v_v(p_2) \sim n_+v\langle Q\bar{Q}|\mathcal{O}(^1S_0)|0\rangle\,,$$

$$\frac{1}{2m}\bar{u}_v(p_1)\left\{\not{q},\not{n}_+\gamma_5\right\}v_v(p_2) = \frac{n_{+\mu}}{2m}\bar{u}_v(p_1)[\not{q},\gamma_\top^\mu]\gamma_5 v_v(p_2) \sim \frac{\sqrt{2}}{m}\langle Q\bar{Q}|n_{+\mu}\mathcal{O}^\mu(^3P_1)|0\rangle\,,$$

$$\frac{n_+\bar{q}}{n_+P}\bar{u}_v(p_1)\not{n}_+\gamma_5 v_v(p_2) = \frac{n_+vn_+\bar{q}}{n_+P}\bar{u}_v(p_1)\gamma_5 v_v(p_2) \sim \frac{n_+v}{n_+P}\langle Q\bar{Q}|n_{+\mu}\mathcal{O}^\mu(^1P_1)|0\rangle\,.$$

$$\langle Q[\gamma_5](x)\rangle \;=\; \delta(x-1/2)\left(\frac{n_+v}{n_+P}\langle\mathcal{O}(^1S_0)\rangle + \frac{\sqrt{2}n_{+\mu}}{mn_+P}\langle\mathcal{O}^\mu(^3P_1)\rangle\right) - \frac{\delta'(x-1/2)}{n_+P}\frac{n_+vn_{+\mu}}{n_+P}\langle\mathcal{O}^\mu(^1P_1)\rangle + \mathcal{O}(v^2)\,.$$

$$\hat{\phi}_P^{(0)}(x) = \hat{\phi}_{3A}^{\parallel(x)(0)} = \delta(x-1/2)\,,\quad \hat{\phi}_{1A}^{\parallel(0)}(x) = -\delta'(x-1/2)/2\,,$$

$$f_P^{(0)} = \frac{i}{m_P}\langle\mathcal{O}(^1S_0)\rangle\,,\quad f_{1A}^{(0)} = i\frac{2}{m_{1A}^2}\langle\mathcal{O}(^3P_0)\rangle\,,\quad f_{3A}^{(0)} = i\frac{\sqrt{2}}{mm_{3A}}\langle\mathcal{O}(^3P_0)\rangle\,,$$

$$\hat{\phi}_V^{\parallel(0)}(x) = \hat{\phi}_V^{\perp}(x) = \hat{\phi}_{1A}^{\perp}(x) = \delta(x - 1/2)\,,$$
$$\hat{\phi}_S^{(0)}(x) = \hat{\phi}_{3A}^{\perp(0)}(x) = \hat{\phi}_T^{\parallel(0)}(x) = \hat{\phi}_T^{\perp(0)}(x) = -\delta'(x - 1/2)/2\,,$$

$$f_V^{(0)} = f_V^{\perp(0)} = \frac{i}{m_V}\langle \mathcal{O}(^1S_0)\rangle\,, \quad f_{1A}^{\perp(0)} = -\frac{i}{m m_{1A}}\langle \mathcal{O}(^3P_0)\rangle\,,$$
$$f_S^{(0)} = -\frac{2}{\sqrt{3}m_S^2}\langle \mathcal{O}(^3P_0)\rangle\,, \quad f_{3A}^{\perp(0)} = -\frac{i\sqrt{2}}{m_{3A}^2}\langle \mathcal{O}(^3P_0)\rangle\,,$$
$$f_T^{(0)} = -f_T^{\perp(0)} = -\frac{2}{m_T^2}\langle \mathcal{O}(^3P_0)\rangle\,.$$

Matching at NLO

- Standard matching procedure: Ma & Si, PLB, 2006
  - ✦ Calculate QCD operator ME to NLO;
  - ✦ Calculate QCD operator ME to NLO;
  - ✦ Obtain short-distance coefficient through matching.
- Threshold expansion: Beneke & Smirnov, NPB, 1998
  - ✦ Compute the QCD loop diagram by threshold expansion;
  - ✦ Low-energy region contributions -> NRQCD ME;
  - ✦ Large-energy region contributions -> short-distance cofficients;
  - ✦ Only large-energy region contr. needed in practice!

More on threshold expansion

For a Feynman integral (in DR) containing small parameters

- Divide the space of the loop momentum into various regions and , in every region, expand the integrand into a Taylor series with respect to the parameters that are considered small there;
- Integrate the integrand, expanded in the appropriate way in every region, over the whole integration domain of the loop momenta;
- Add up all the expanded integrals in all regions, we reproduce the Taylor series of the original Feynman integral with respect to the small parameters exactly;
- Finally, a multiple-scale problem is divided into single (less) scale problems.

NLO Results: S–wave

$$\hat{\phi}_P(x;\mu) = \delta(x-1/2) + \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ 4\left[\left(\ln\frac{\mu^2}{m^2(1-2x)^2} - 1\right)\left(1+\frac{1}{1/2-x}\right) x\theta(1-2x)\right]_+ \right.$$

$$\left. + \left[\frac{16x\bar{x}}{(1-2x)^2}\theta(1-2x)\right]_{++} + (x \leftrightarrow \bar{x}) \right\},$$

$$\hat{\phi}_V^{\parallel}(x;\mu) = \hat{\phi}_P(x;\mu) - \frac{\alpha_s(\mu)}{4\pi} C_F \left[8x\theta(1-2x) + 8\bar{x}\theta(2x-1)\right]_+ ,$$

$$\hat{\phi}_V^{\perp}(x;\mu) = \hat{\phi}_P(x;\mu) - \frac{\alpha_s(\mu)}{4\pi} C_F \left[\left(\ln\frac{\mu^2}{m^2(1-2x)^2} - 1\right)\left(4x\theta(1-2x) + 4\bar{x}\theta(2x-1)\right)\right]_+$$

$$f_P = \left\{1 + \frac{\alpha_s(\mu)}{4\pi} C_F(-6)\right\} \frac{i}{m_P}\langle \mathcal{O}(^1S_0)\rangle ,$$

$$f_V = \left\{1 + \frac{\alpha_s(\mu)}{4\pi} C_F(-8)\right\} \frac{i}{m_V}\langle \mathcal{O}(^1S_0)\rangle ,$$

$$f_V^{\perp} = \left\{1 + \frac{\alpha_s(\mu)}{4\pi} C_F\left(-\ln\frac{\mu^2}{m^2} - 8\right)\right\} \frac{i}{m_V}\langle \mathcal{O}(^1S_0)\rangle ,$$

agree with
Bell&Feldman, Ma&Si

NLO results： P–wave

$$
\begin{aligned}
\hat{\phi}^{\parallel}_{1A}(x;\mu) &= -\delta'(x-1/2)/2 \\
&\quad + \frac{\alpha_s}{4\pi} C_F \left\{ -\left[\left(\ln \frac{\mu^2}{m^2(1-2x)^2} - \frac{5}{2} \right) \frac{4x(5-8x+4x^2)\theta(1-2x)}{(1-2x)^2} \right]_{++} \right. \\
&\quad \left. - \left[\frac{2x(23-10x)\theta(1-2x)}{(1-2x)^2} \right]_{++} - \left[\frac{8x\theta(1-2x)}{(1-2x)^3} \right]_{+++} - (x \leftrightarrow \bar{x}) \right\} , \\
\hat{\phi}^{\perp}_{1A}(x;\mu) &= \hat{\phi}^{\perp}_{V}(x;\mu) - \frac{\alpha_s \mu}{4\pi} C_F \left[\frac{8x\bar{x}\theta(2x-1)}{(1-2x)^2} + \frac{8x\bar{x}\theta(1-2x)}{(1-2x)^2} \right]_{++} , \\
\hat{\phi}_{S}(x;\mu) &= -\delta'(x-1/2)/2 \\
&\quad + \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ -\left[\left(\ln \frac{\mu^2}{m^2(1-2x)^2} - 1 \right) \frac{4x(5-8x+4x^2)\theta(1-2x)}{(1-2x)^2} \right]_{++} \right. \\
&\quad \left. - \left[\frac{8x(7-8x)\theta(1-2x)}{(1-2x)^2} \right]_{++} - \left[\frac{8x\theta(1-2x)}{(1-2x)^3} \right]_{+++} - (x \leftrightarrow \bar{x}) \right\} , \\
\hat{\phi}^{\parallel}_{3A}(x;\mu) &= \hat{\phi}^{\parallel}_{V}(x;\mu) - \frac{\alpha_s \mu}{4\pi} C_F \left[\frac{8x\bar{x}\theta(2x-1)}{(1-2x)^2} + \frac{8x\bar{x}\theta(1-2x)}{(1-2x)^2} \right]_{++} ,
\end{aligned}
$$

NLO results： P−wave

$$\hat{\phi}^{\perp}_{3A}(x) = -\delta'(x-1/2)/2 + \frac{\alpha_s(\mu)}{4\pi}C_F\left\{-\left[\left(\ln\frac{\mu^2}{m^2(1-2x)^2}\right)\frac{16x\bar{x}\theta(1-2x)}{(1-2x)^2}\right]_{++} - \left[\frac{16x\theta(1-2x)}{(1-2x)^2}\right]_{++} - \left[\frac{8x\theta(1-2x)}{(1-2x)^3}\right]_{+++} - (x\leftrightarrow\bar{x})\right\}, \quad (76)$$

$$\hat{\phi}^{\parallel}_{T}(x;\mu) = -\delta'(x-1/2)/2 + \frac{\alpha_s(\mu)}{4\pi}C_F\left\{-\left[\left(\ln\frac{\mu^2}{m^2(1-2x)^2}-4\right)\frac{4x(5-8x+4x^2)\theta(1-2x)}{(1-2x)^2}\right]_{++} - \left[\frac{4x(17-10x)\theta(1-2x)}{(1-2x)^2}\right]_{++} - \left[\frac{8x\theta(1-2x)}{(1-2x)^3}\right]_{+++} - (x\leftrightarrow\bar{x})\right\}, \quad (77)$$

$$\hat{\phi}^{\perp}_{T}(x;\mu) = -\delta'(x-1/2)/2 + \frac{\alpha_s(\mu)}{4\pi}C_F\left\{-\left[\left(\ln\frac{\mu^2}{m^2(1-2x)^2}+2\right)\frac{16x\bar{x}\theta(1-2x)}{(1-2x)^2}\right]_{++} - \left[\frac{32x\theta(1-2x)}{1-2x}\right]_{++} - \left[\frac{8x\theta(1-2x)}{(1-2x)^3}\right]_{+++} - (x\leftrightarrow\bar{x})\right\},$$

NLO results： P-wave

$$f_{1A} = \left\{1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left(\frac{8}{3}\ln\frac{\mu^2}{m^2} + \frac{79}{9}\right)\right\} \frac{2i}{m_{1A}^2} \langle \mathcal{O}(^3P_0)\rangle ,$$

$$f_{1A}^{\perp} = \left\{1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\ln\frac{\mu^2}{m^2} - 4\right)\right\} \frac{2i}{m_{1A}^2} \langle \mathcal{O}(^3P_0)\rangle ,$$

$$f_S = \left\{1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{8}{3}\ln\frac{\mu^2}{m^2} - \frac{2}{9}\right)\right\} \frac{-2}{\sqrt{3}m_S^2} \langle \mathcal{O}(^3P_0)\rangle ,$$

$$f_{3A} = \left\{1 + \frac{\alpha_s(\mu)}{4\pi} C_F (-4)\right\} \frac{\sqrt{2}i}{m m_{3A}} \langle \mathcal{O}(^3P_0)\rangle ,$$

$$f_{3A}^{\perp} = \left\{1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left(3\ln\frac{\mu^2}{m^2} + 6\right)\right\} \frac{-\sqrt{2}i}{m_{3A}^2} \langle \mathcal{O}(3_0^P)\rangle ,$$

$$f_T = \left\{1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left(\frac{8}{3}\ln\frac{\mu^2}{m^2} + \frac{88}{9}\right)\right\} \frac{-2}{m_T^2} \langle \mathcal{O}(3_0^P)\rangle ,$$

$$f_T^{\perp} = \left\{1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left(3\ln\frac{\mu^2}{m^2} + 10\right)\right\} \frac{2}{m_T^2} \langle \mathcal{O}(3_0^P)\rangle ,$$

Applications: γ*→η_Qγ, χ_{QJ}γ

$$i\mathcal{M}(\gamma^*(Q,\varepsilon_{\gamma^*}\to\eta_Q(p)\gamma(p',\varepsilon_\gamma))=\frac{i}{2}e^2Q_q^2\epsilon_\perp^{\mu\nu}\varepsilon_{\gamma^*\mu}\varepsilon^*_{\gamma\nu}f_{\eta_Q}\int_0^1 dyT_H^P(y;Q^2,\mu)\hat{\phi}_P(y;\mu)$$
$$=-2e^2Q_q^2\epsilon_\perp^{\mu\nu}\varepsilon_{\gamma^*\mu}\varepsilon^*_{\gamma\nu}\frac{\langle\mathcal{O}(^1S_0)\rangle}{m_{\eta_Q}}\left\{1+\frac{\alpha_s}{4\pi}C_F\left[(3-2\ln 2)L+\ln^2 2+3\ln 2-9-\frac{\pi^2}{3}\right]\right\},$$

$$i\mathcal{M}(\gamma^*(Q,\varepsilon_{\gamma^*}\to\chi_{Q1}(p,\varepsilon)\gamma(p',\varepsilon_\gamma))=\frac{i}{2}e^2Q_q^2\epsilon_\perp^{\mu\nu}\varepsilon_{\gamma^*\mu}\varepsilon^*_{\gamma\nu}f_{3A}\int_0^1 dyT_H^P(y;Q^2,\mu)\hat{\phi}^{\parallel}_{3A}(y;\mu)$$
$$=-2\sqrt{2}e^2Q_q^2\epsilon_\perp^{\mu\nu}\varepsilon_{\gamma^*\mu}\varepsilon^*_{\gamma\nu}\frac{\langle\mathcal{O}(^3P_0)\rangle}{m_{\chi_{Q1}}m}\left\{1+\frac{\alpha_s}{4\pi}C_F\left[(3-2\ln 2)L+\ln^2 2-\ln 2-7-\frac{\pi^2}{3}\right]\right\},\tag{108}$$

$$i\mathcal{M}(\gamma^*(Q,\varepsilon_{\gamma^*}\to\chi_{Q0}(p)\gamma(p',\varepsilon_\gamma))=-ie^2Q_q^2\varepsilon_{\gamma^*}\cdot\varepsilon^*_\gamma\frac{f_S}{2}\int_0^1 dyT_H^P(y;Q^2,\mu)\hat{\phi}_S(y;\mu)$$
$$=i4e^2Q_q^2\epsilon_\perp^{\mu\nu}\varepsilon_{\gamma^*\mu}\varepsilon^*_{\gamma\nu}\frac{\langle\mathcal{O}(^3P_0)\rangle}{\sqrt{3}m^2_{\chi_{Q0}}}\left\{1+\frac{\alpha_s}{4\pi}C_F\left[(1-2\ln 2)L+\ln^2 2+9\ln 2-\frac{\pi^2}{3}\right]\right\},\tag{109}$$

$$i\mathcal{M}(\gamma^*(Q,\varepsilon_{\gamma^*}\to\chi_{Q2}(p)\gamma(p',\varepsilon_\gamma))=-ie^2Q_q^2\varepsilon_{\gamma^*}\cdot\varepsilon^*_\gamma\frac{f_T}{\sqrt{6}}\int_0^1 dyT_H^P(y;Q^2,\mu)\hat{\phi}^{\parallel}_T(y;\mu)$$
$$=-i8e^2Q_q^2\epsilon_\perp^{\mu\nu}\varepsilon_{\gamma^*\mu}\varepsilon^*_{\gamma\nu}\frac{\langle\mathcal{O}(^3P_0)\rangle}{\sqrt{6}m^2_{\chi_{Q2}}}\left\{1+\frac{\alpha_s}{4\pi}C_F\left[(1-2\ln 2)L+\ln^2 2-3\ln 2-6-\frac{\pi^2}{3}\right]\right\},\tag{110}$$

Reproduce the asymptotic behavior of NRQCD factorization!

Chen & Sang PRD, 2010

Applications: Z→J/ψγ, $h_Q\gamma$

$$
\begin{aligned}
& i\mathcal{M}(Z(Q,\varepsilon_Z) \to J/\psi(\Upsilon)(p,\varepsilon)\gamma(p',\varepsilon_\gamma)) \\
&= -i\frac{g_A e Q_q}{4\cos\theta_W}\epsilon_\perp^{\mu\nu}\varepsilon_{Z\mu}\varepsilon^*_{\gamma\nu} f_{J/\psi(\Upsilon)}\int_0^1 dy T_H^P(y;Q^2,\mu)\hat{\phi}_V(y;\mu) \\
&= -i\frac{g_A e Q_q}{2\cos\theta_W}\epsilon_\perp^{\mu\nu}\varepsilon_{Z\mu}\varepsilon^*_{\gamma\nu}\frac{\langle\mathcal{O}(^1S_0)\rangle}{m_{J/\psi(\Upsilon)}}\left\{1+\frac{\alpha_s}{4\pi}C_F\left[(3-2\ln 2)L+\ln^2 2-\ln 2-9-\frac{\pi^2}{3}\right]\right\},
\end{aligned}
\tag{344}
$$

$$
\begin{aligned}
& i\mathcal{M}(Z(Q,,\varepsilon_Z) \to h_Q(p,\varepsilon)\gamma(p',\varepsilon_\gamma)) = \frac{g_A e Q_q}{8\cos\theta_W}\varepsilon_Z\cdot\varepsilon^*_\gamma f_{1A}\int_0^1 dy T_H^V(y;Q^2,\mu)\hat{\phi}^{\parallel}_{1A}(y;\mu) \\
&= -i\frac{g_A e Q_q}{2\cos\theta_W}\varepsilon_Z\cdot\varepsilon^*_\gamma\frac{\langle\mathcal{O}(^3P_0)\rangle}{m_{h_Q}m}\left\{1+\frac{\alpha_s}{4\pi}C_F\left[(1-2\ln 2)L+\ln^2 2-3\ln 2-4-\frac{\pi^2}{3}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
r[^3S_1] &\equiv \frac{\sigma^{\rm NLO}(e^+e^-\to Z^0\to J/\psi\gamma)}{\sigma^{\rm LO}(e^+e^-\to Z^0\to J/\psi\gamma)} \\
&= \frac{\alpha_s}{2\pi}C_F\left[(3-2\ln 2)\ln\frac{m_Z^2}{m_c^2}+\ln^2 2-\ln 2-5-\frac{\pi^2}{3}\right], \\
r[^1P_1] &\equiv \frac{\sigma^{\rm NLO}(e^+e^-\to Z^0\to h_c\gamma)}{\sigma^{\rm LO}(e^+e^-\to Z^0\to h_c\gamma)} \\
&= \frac{\alpha_s}{2\pi}C_F\left[(1-2\ln 2)\ln\frac{m_Z^2}{m_c^2}+\ln^2 2-3\ln 2-4-\frac{\pi^2}{3}\right].
\end{aligned}
$$

Reproduce the asymptotic behavior of NRQCD factorization!

Chen, Wu et al, PRD, 2014

Applications: h→J/ψγ

$$i\mathcal{M}(h(Q)\to J/\psi(p,\varepsilon_\psi)\gamma(p',\varepsilon_\gamma)) = -\frac{m_c e Q_q}{2v}\varepsilon^*_\psi\cdot\varepsilon^*_\gamma f^\perp_{J/\psi}\int_0^1 dy T_H^P(y;Q^2,\mu)\hat{\phi}_V(y;\mu)$$
$$= -i\frac{m_c e Q_q}{2v}\varepsilon^*_\psi\cdot\varepsilon^*_\gamma \frac{\langle\mathcal{O}(^1S_0)\rangle}{m_{J/\psi}}\left\{1-\frac{\alpha_s}{4\pi}C_F\left[4\ln 2\frac{-m_h^2-i\epsilon}{m_c^2}-2\ln^2 2-4\ln 2+7+\frac{2\pi^2}{3}\right]\right\}.$$

Shifman et al, NPB, 1981

$$M(h\to J/\Psi\gamma) = M_{\rm tr}(h\to J/\Psi\gamma)\left[1-\frac{\alpha_s(m_H^2)C_F}{2\pi}a(\kappa)\right], \quad (121)$$

$$a(\kappa) = 4-\frac{\pi^2}{12(1-\kappa)}-\frac{F(1-2\kappa)}{2(1-\kappa)}+\frac{\kappa-1}{1-2\kappa}+\frac{2\kappa(\kappa-2)}{(1-\kappa)^2}[\phi(\kappa)+F(1)-F(-1)]$$
$$+\left(4+\frac{4}{\kappa}+\frac{8}{1-\kappa}\right)\sqrt{\frac{\kappa}{1-\kappa}}\arctan\sqrt{\frac{\kappa}{1-\kappa}}+\left(\frac{4}{1-\kappa}+2+\frac{\kappa}{(2\kappa-1)^2}\right)\ln(2-2\kappa),$$

$$a(\kappa) = \frac{1}{2}\left[4\ln 2\ln\frac{-m_h^2-i\epsilon}{m_c^2}-2\ln^2 2-4\ln 2+\frac{2\pi^2}{3}+7\right]+\mathcal{O}(m_c^2/Q^2),$$

Reproduce the asymptotic behavior of NRQCD factorization!

Applications: Double parton fragmentation functions for quarkonia

In pQCD factorization approach, production of a heavy quarkonium H with momentum p is expanded by $1/p_T$[13, 14, 19, 20]

$$
\begin{aligned}
E_p \frac{d\sigma_{A+B\to H+X}}{d^3p}(p) \approx & \sum_f \int \frac{dz}{z^2} D_{f\to H}(z; m_Q) E_c \frac{d\hat{\sigma}_{A+B\to f(p_c)+X}}{d^3p_c}\left(p_c = \frac{1}{z}\hat{p}\right) \\
& + \sum_{[Q\bar{Q}(\kappa)]} \int \frac{dz}{z^2} \frac{d\zeta_1\, d\zeta_2}{4} \mathcal{D}_{[Q\bar{Q}(\kappa)]\to H}(z, \zeta_1, \zeta_2; m_Q) \\
& \times E_c \frac{d\hat{\sigma}_{A+B\to[Q\bar{Q}(\kappa)](p_c)+X}}{d^3p_c}(P_Q, P_{\bar{Q}}; P'_Q, P'_{\bar{Q}}),
\end{aligned} \tag{1}
$$

The FF for a perturbatively created $Q\bar{Q}$-pair of spin-color quantum number κ to fragment into a physical heavy quarkonium H is defined as [8]

$$
\begin{aligned}
\mathcal{D}_{[Q\bar{Q}(\kappa)]\to H}(z, \zeta_1, \zeta_2; m_Q, \mu_0) = & \int \frac{p^+ dy^-}{2\pi} \frac{p^+/z dy_1^-}{2\pi} \frac{p^+/z dy_2^-}{2\pi} e^{-i(p^+/z)y^-} e^{i(p^+/z)[(1-\zeta_2)/2]y_1^-} e^{-i(p^+/z)[(1-\zeta_1)/2]y_2^-} \\
& \times \mathcal{P}^{(s)}_{ij,kl}(p_c)\mathcal{C}^{[I]}_{ab,cd}\langle 0|\bar{\psi}_{c',k}(y_1^-)[\Phi^{(F)}_{\hat{n}}(y_1^-)]^\dagger_{c'c}[\Phi^{(F)}_{\hat{n}}(0)]_{dd'}\, \psi_{d',l}(0)|H(p)X\rangle \\
& \times \langle H(p)X|\bar{\psi}_{a',i}(y^-)[\Phi^{(F)}_{\hat{n}}(y^-)]^\dagger_{a'a}[\Phi^{(F)}_{\hat{n}}(y^- + y_2^-)]_{bb'}\psi_{b',j}(y^- + y_2^-)|0\rangle,
\end{aligned}
$$

Y.Q. Ma, J.W. Qiu, H.Zhang, **Phys.Rev. D89 (2014) 094029, 094030**

Double parton fragmentation functions for quarkonia -- color singlet part

- Can be expressed by the product of two LCDAs

$$\mathcal{D}_{[Q\bar{Q}(\kappa)]\to H}(z,\zeta_1,\zeta_2;m_Q,\mu_0) = \sum_{[Q\bar{Q}(n)]} \hat{d}_{[Q\bar{Q}(\kappa)]\to[Q\bar{Q}(n)]}(z,\zeta_1,\zeta_2;m_Q,\mu_0,\mu_\Lambda)\langle\mathcal{O}^H_{[Q\bar{Q}(n)]}(\mu_\Lambda)\rangle,$$

$$\hat{d}^{\,\mathrm{LO}}_{[Q\bar{Q}(a^{[1]})]\to Q\bar{Q}[({}^1S_0^{[1]})]}(z,\zeta_1,\zeta_2;m_Q) = \frac{1}{8\,m_Q}\delta(1-z)\delta(\zeta_1)\delta(\zeta_2).$$

$$\begin{aligned}\hat{d}^{\,\mathrm{NLO}}_{[Q\bar{Q}(a^{[1]})]\to Q\bar{Q}[({}^1P_1^{[8]})]}(z,\zeta_1,\zeta_2;m_Q,\mu_0,\mu_\Lambda) = \frac{\alpha_s z}{192\pi\,m_Q^3 N_c}\Big\{&4\,\Delta_0\,\delta(1-z)(\,\ln[\frac{\mu_\Lambda^2}{m_Q^2}]+2\ln 2+\frac{1}{3})\\ &+[\frac{\Delta_+^{[1]\prime\prime}}{(1-z)}+\Delta_+^{[1]\prime}+\Delta_+^{[1]}(1-z)](\,\ln[\frac{\mu_F^2}{m_Q^2}]-\frac{2}{3})\\ &+\frac{\Delta_+^{[1]\prime\prime}}{(1-z)}R_1(z)+\frac{\Delta_+^{[1]\prime}}{(1-z)}R_2(z)+\Delta_+^{[1]}\,R_3(z)\Big\},\end{aligned}$$

LCDAs for Bc mesons

- Two different flavors of heavy quark and anti-quark are inside of Bc mesons;
- Example 1: LCDA for pseudo-scalar Bc meson

$$
\begin{aligned}
\hat{\phi}_P(x;\mu) \;=\; & \delta(x-x_0) \\
& +\frac{\alpha_s}{4\pi}C_F\left\{2\left[\left(\ln\frac{\mu^2}{M^2(x_0-x)^2}-1\right)\left(\frac{x_0+\bar{x}}{x_0-x}\frac{x}{x_0}\theta(x_0-x)+\begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right)\right]_+\right. \\
& +\left[\frac{4x\bar{x}}{(x_0-x)^2}\right]_{++}+\left[4x_0\bar{x}_0\ln\frac{x_0}{\bar{x}_0}+2(2x_0-1)\right]\delta'(x-x_0) \\
& \left.+8\Delta\left[\frac{x}{x_0}\theta(x_0-x)+\begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right]_+\right\}, \qquad (97)
\end{aligned}
$$

LCDAs for Bc mesons

- Example 2: LCDA for P-wave Bc meson

$$
\begin{aligned}
\hat{\phi}^{\perp}_{3A}(x;\mu) = {} & \delta(x-x_0) + \frac{\alpha_s}{4\pi}C_F\Bigg\{2\Bigg[\Bigg(\ln\frac{\mu^2}{M^2(x_0-x)^2}-1\Bigg)\Bigg(\frac{x\theta(x_0-x)}{x_0(x_0-x)} + \begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\Bigg)\Bigg]_+ \\
& + \left[\frac{4x\bar{x}}{(x_0-x)^2}\right]_{++} - 2\left[\frac{\bar{x}_0}{x_0}\frac{x(2x_0-x)\theta(x_0-x)}{(x_0-x)^2} + \begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right]_{++} \\
& + \left[2x_0\bar{x}_0\ln\frac{x_0}{\bar{x}_0} + 2(2x_0-1)\right]\delta'(x-x_0)\Bigg\} \\
& - \frac{4x_0\bar{x}_0}{(2-d)(1-2x_0)}\Bigg(- \delta'(x-x_0)/2 \\
& + \frac{\alpha_s}{4\pi}C_F\Bigg\{-\Bigg[\ln\frac{\mu^2}{M^2(x_0-x)^2}\Bigg(\frac{x(2x_0-x)}{x_0^2(x_0-x)^2}\theta(x_0-x) - \begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\Bigg)\Bigg]_{++} \\
& -2\left[\frac{\bar{x}_0 x}{(x_0-x)^3}\theta(x_0-x) - \begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right]_{+++} - x_0\bar{x}_0\ln\frac{x_0}{\bar{x}_0}\delta''(x-x_0) \\
& -2\left[\frac{x^2}{x_0^2(x_0-x)^2}\theta(x_0-x) - \begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right]_{++} \\
& -\frac{(2x_0-1)}{2}\left[\frac{x(2x_0-x)}{x_0^2(x_0-x)^2}\theta(x_0-x) + \begin{pmatrix} x\leftrightarrow\bar{x} \\ x_0\leftrightarrow\bar{x}_0\end{pmatrix}\right]_{++} \\
& -2(d-2)\left[\frac{x\theta(x_0-x)}{x_0(x_0-x)} - \frac{\bar{x}\theta(x-x_0)}{\bar{x}_0(x-x_0)}\right]_{++} \\
& + \left[\frac{3}{2}\left(\ln\frac{\mu^2}{M^2} - 2x_0\ln x_0 - 2\bar{x}_0\ln\bar{x}_0\right) + 5 - (d-2)\right]\delta'(x-x_0) \\
& + \frac{1}{2}\left[-\ln\frac{\mu^2}{M^2} - (3-8x_0)\ln x_0 - (3-8\bar{x}_0)\ln\bar{x}_0 - 4 - \frac{8x_0\bar{x}_0}{1-2x_0}\ln\frac{x_0}{\bar{x}_0}\right]\delta'(x-x_0)\Bigg\}\Bigg),
\end{aligned}
$$

Under mb=mc limit, all LCDAs agree with ones for quarkonia!

Summary

- LCDAs are very important for hard-exclusive processes!
- Refactorization of LCDAs for quarkonia and Bc mesons bridges the collinear factorization to NRQCD factorization;
- Future works along this direction: relativistic corrections to LCDAs, 2-loop corrections ......
- Good excises for students!