# Wilson links for parton densities and resummation

Hsiang-nan Li (李湘楠)
Academia Sinica, Taipei

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# Outlines

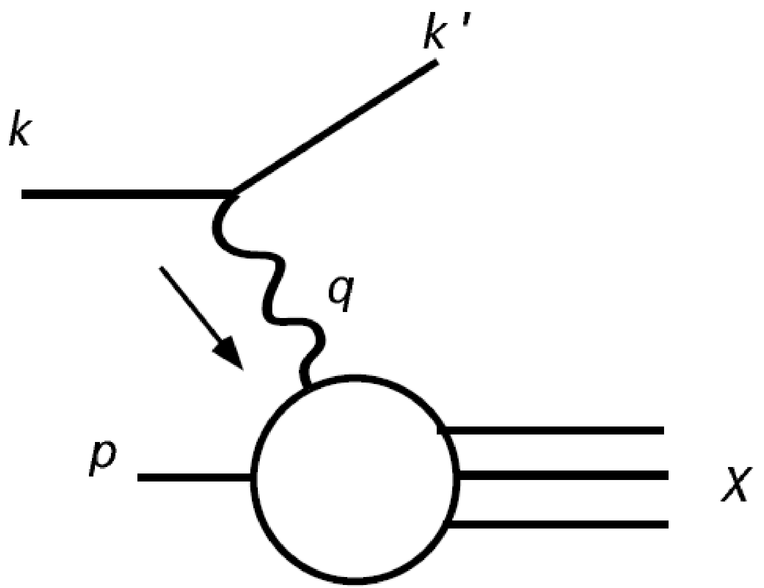
- Introduction
- Naïve TMD definition
- Collins' definition
- New definition with non-dipolar Wilson links
- Quasi-parton distribution function
- Resummation technique
- Summary

# Introduction

See also Zuo-Tang and Sayipjamal’s talks

# Deep inelastic scattering

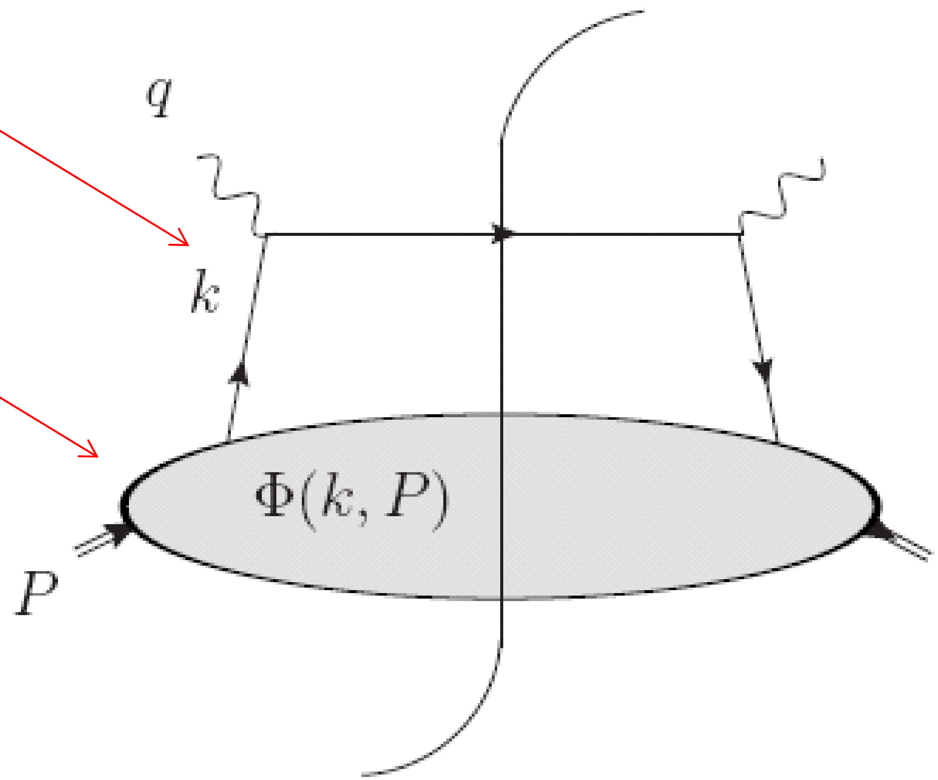
- Electron-proton DIS l(k)+N(p) -> l(k')+X
- Large momentum transfer $-q^2=(k-k')^2=Q^2$
- Calculation of cross section suffers IR divergence --- nonperturbative dynamics in the proton
- Origin if IR divergence?
- How to handle it?
- Factor out nonpert part from DIS, and leave it to other methods



# Factorization theorem

- Collinear divergence associated with proton
- Cross section = Hard (H) * Parton distribution function (PDF)
- H = short distance, LO
  PDF = long distance
- Collinear factorization

$$k = (xP^{+}, 0, 0_T)$$



# Collinear factorization

- Factorization of many processes investigated up to higher twists
- Hard kernels calculated to higher orders
- Parton distribution function (PDF) evolution from low to high scale derived (DGLAP equation)
- PDF database constructed (CTEQ)
- Logs from extreme kinematics resummed
- Soft, jet, fragmentation functions all studied

# $k_T$ factorization

- $k_T$ factorization applies to small x, or high energy region, especially to LHC physics
- Also to final-state spectra at low $q_T$, like direct photon and jet production
- Keep $k_T$ in hard kernel, $xP^+ \approx k_T,\, q_T \approx k_T$
- Parton $k = (xP^+, 0, k_T)$ enters hard kernel
- Parton $k_T$ is not integrated out in PDF

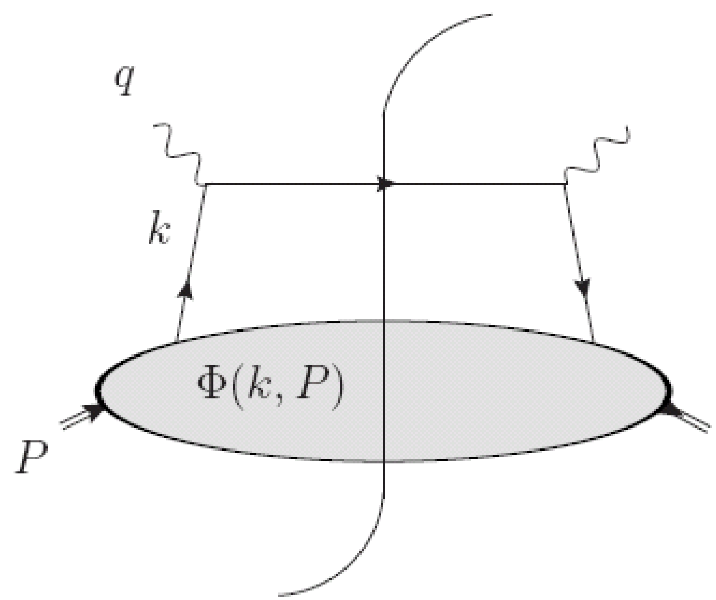
  $\Rightarrow$ $k_T$ dependent (TMD) parton density
- Many aspects of $k_T$ factorization not yet investigated in detail

# Factorization in $k_T$ space

Universal transverse-momentum-dependent (TMD) PDF $\Phi_{f/N}(\xi, k_T)$ describes probability of parton carrying momentum fraction and transverse momentum

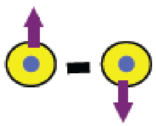
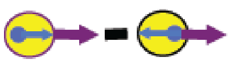
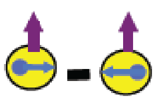
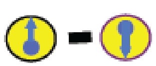
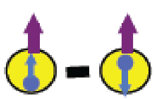
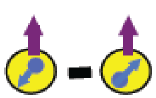
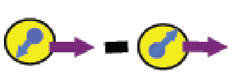
If neglecting $k_T$ in H, integration over $k_T$ can be worked out, giving

$$\int d^2 k_T \Phi_{f/N}(\xi, k_T) \Rightarrow \phi_{f/N}(\xi)$$



# 3-dimensional image of nucleon

At leading twist (twist 2)    $f, g, h$ :  quark un-, longitudinally, transversely polarized

| | | | | |
|---|---|---|---|---|
|  | $f_1(x,k_\perp)$ | | $q(x)$ | number density |
|  | $f_{1T}^\perp(x,k_\perp)$ | $\mathbf{0}$ | $\times$ | Sivers function |
|  | $g_{1L}(x,k_\perp)$ | | $\Delta q(x)$ | helicity distribution |
|  | $g_{1T}^\perp(x,k_\perp)$ | | $\times$ | Worm gear: trans-helicity |
|  | $h_1^\perp(x,k_\perp)$ | $\mathbf{0}$ | $\times$ | Boer-Mulders function |
|  | $h_{1T}(x,k_\perp)$ | | $\delta q(x)$ | transversity distribution |
|  | $h_{1T}^\perp(x,k_\perp)$ | | | pretzelocity |
|  | $h_{1L}^\perp(x,k_\perp)$ | | $\times$ | Worm gear: longi-transversity |
| | | if no gauge link | integrate over $k_\perp$ | |

# Naïve TMD definition

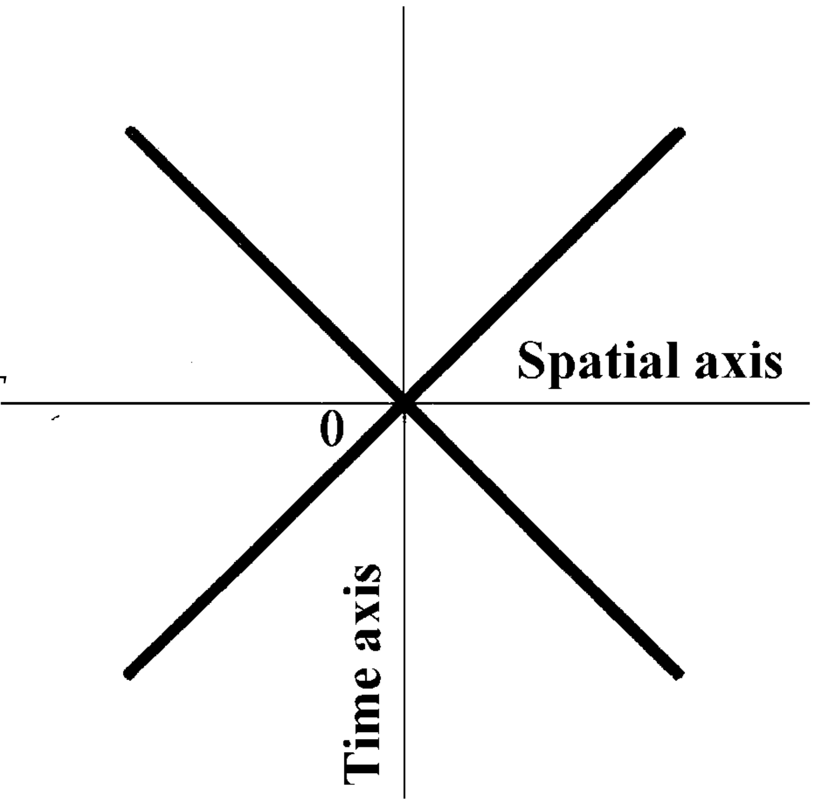
# Light-cone coordinates

- Analysis of infrared divergences simplified

$$l = (l^+, l^-, l_T)$$

$$l^\pm \equiv \frac{l^0 \pm l^3}{\sqrt{2}}$$

$$a \cdot b = a^+ b^- + a^- b^+ - a_T \cdot b_T$$

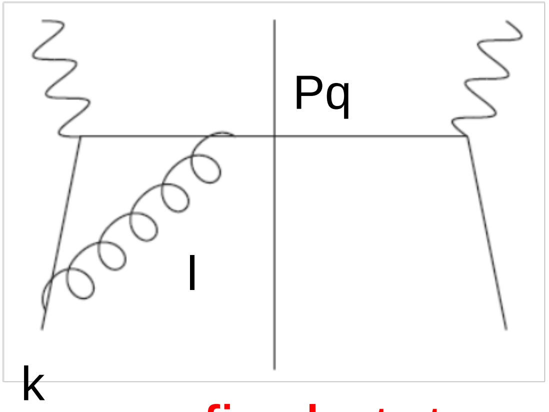



- As particle moves along light cone, only one large component is involve

# Eikonal approximation

$$\not{P}_q \gamma_\nu \frac{\not{P}_q + \not{l}}{(P_q + l)^2} \gamma^\mu \frac{\not{k} + \not{l}}{(k+l)^2} \gamma^\nu, \quad k \propto k^+, \quad P_q \propto P_q^-$$

$$\approx \not{P}_q \gamma^- \frac{\not{P}_q + \not{l}}{(P_q + l)^2} \gamma^\mu \frac{\not{k} + \not{l}}{(k+l)^2} \gamma^+, \quad l \propto l^+$$




final-state cut

$$\approx \not{P}_q \gamma^- \frac{\not{P}_q}{2P_q \cdot l} \gamma^\mu \frac{\not{k} + \not{l}}{(k+l)^2} \gamma^+$$

$$\approx \not{P}_q \frac{2P_q^- - \not{P}_q \gamma^-}{2P_q \cdot l} \gamma^\mu \frac{\not{k} + \not{l}}{(k+l)^2} \gamma^\nu, \quad \not{P}_q \not{P}_q = P_q^2 = 0$$

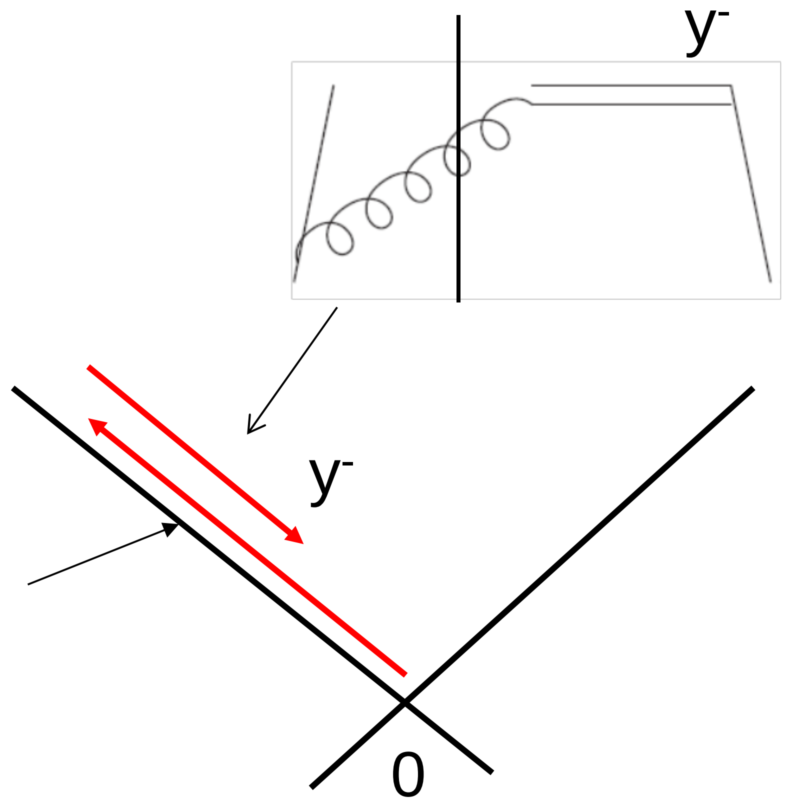
$$\approx \not{P}_q \gamma^\mu \frac{\not{k} + \not{l}}{(k+l)^2} \gamma^\nu \frac{n_{-\nu}}{n_- \cdot l}$$

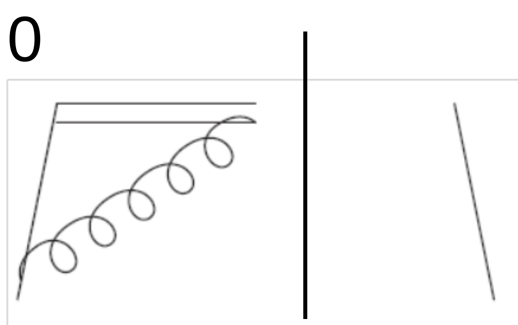
Feynman rules represented by Wilson lines

# Wilson lines

$$
\begin{aligned}
W(y^-,0) &= W(0)W^\dagger(y^-) \\
W(y^-) &= \mathcal{P}\exp\left[-ig\int_0^\infty d\lambda n_-\cdot A(y+\lambda n_-)\right]
\end{aligned}
$$

loop momentum flows through the hard kernel

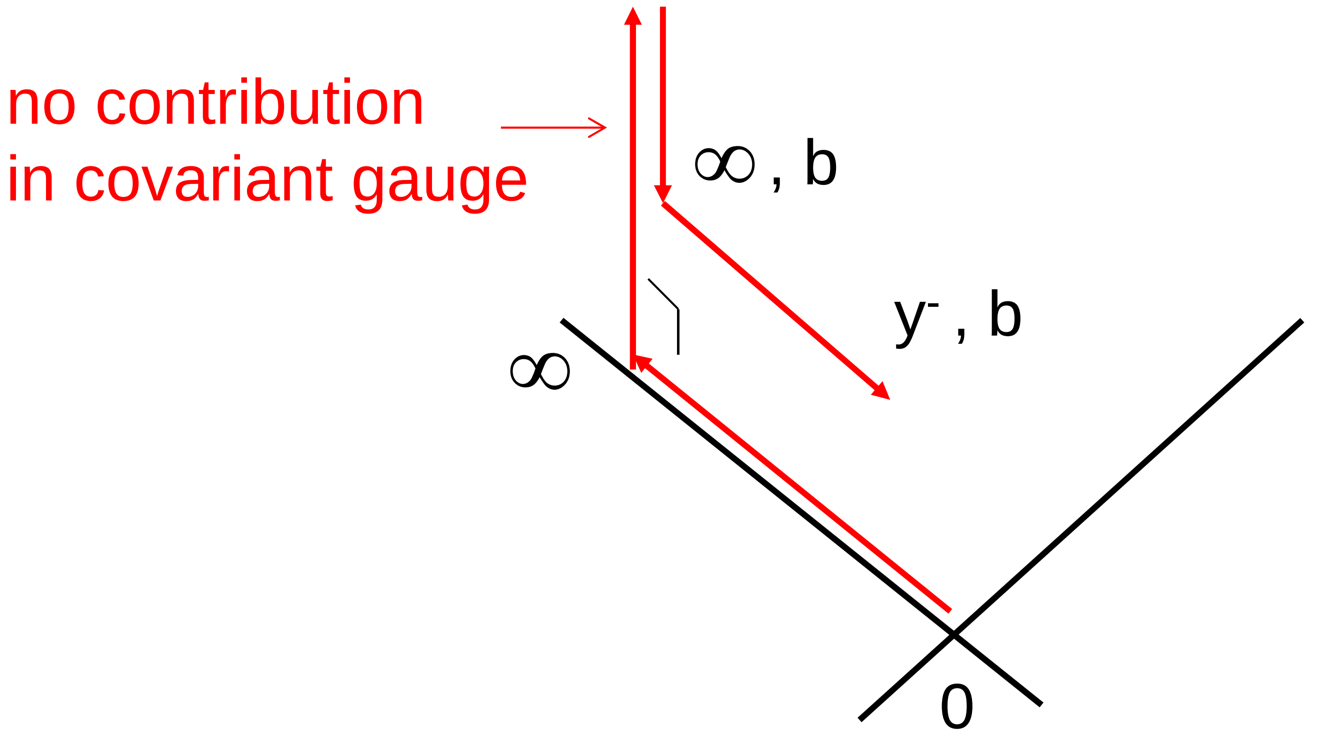



loop momentum does not flow through the hard kernel

# Transverse Wilson links

- Suppose factorization established. Quark fields nonlocal in transverse directions. Transverse Wilson links introduced



# Light-cone singularity

- Compute $H^{(1)} = G^{(1)} - \phi^{(1)} \otimes H^{(0)}$
- The pole $1/(n_- \cdot l) = 1/l^+$ from Wilson lines in $\phi^{(1)}$ gives the light-cone singularity.
- They cancel in collinear factorization

$$\phi^{(1)} \otimes H = \int \frac{dl^+}{l^+} [H(x) - H(x + l^+ / P^+)]$$

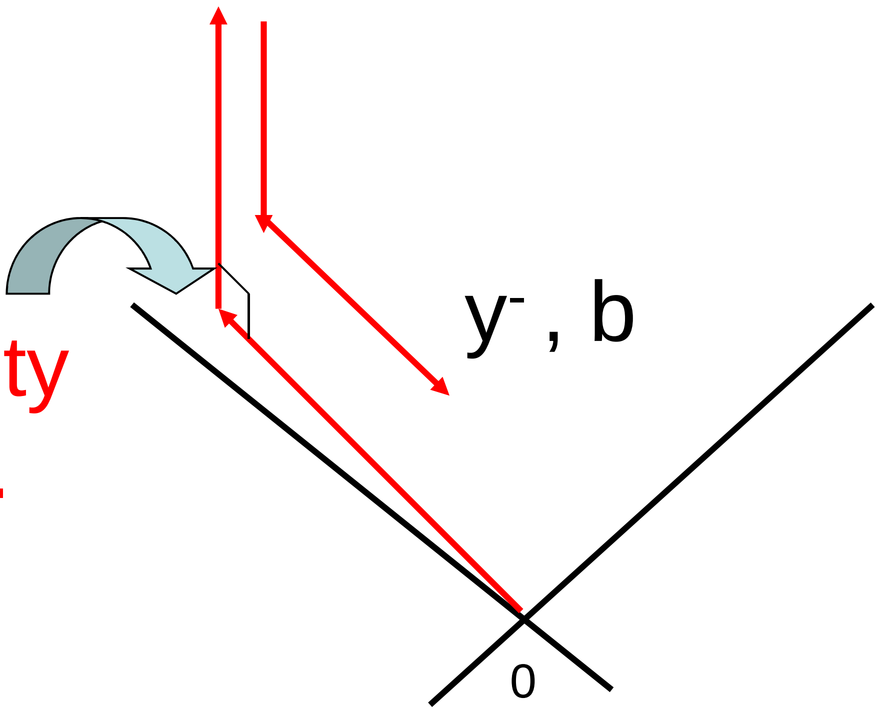
- The difference of $H^{(0)}$ removes singularity.
- They exist in $k_T$ factorization:

$$\int \frac{dl^+}{l^+} [H(x, k_T) - H(x + l^+ / P^+, k_T + l_T)]$$

- because $H(x, k_T) \neq H(x, k_T + l_T)$

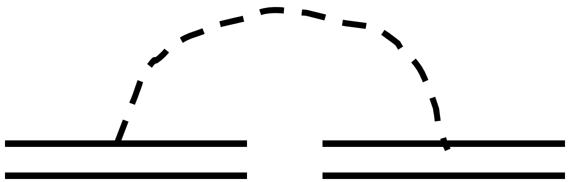
# Modification

- Naïve TMD is ill-defined
- Modified definition: $n_{-} \rightarrow n, n^{2} \neq 0$
- Collins 2003
- Light-cone singularity is regularized by $n^2$.



# New IR singularity

- Self-energy correction to Wilson links appears



- Proportional to $n^2$, vanishes originally as $n_-^2 = 0$
- Its Feynman integrand $\dfrac{1}{(n\cdot l + i\varepsilon)(n\cdot l - i\varepsilon)}$
- 1st pole $n\cdot l = 0$ leads to pinched singularity from 2nd eikonal propagator
- Off-light-cone Wilson links regularize light-cone singularity, but introduce new one

# Collins' definition

Foundations of perturbative QCD,
2011

# Collins' modification

- TMD with light-like Wilson links multiplied by

$$\lim_{\substack{y_1\to+\infty \\ y_u\to-\infty}} \sqrt{\frac{S(z_T;y_1,y_2)}{S(z_T;y_1,y_u)\,S(z_T;y_2,y_u)}}$$

(+,n)

(+,-) (n,-)

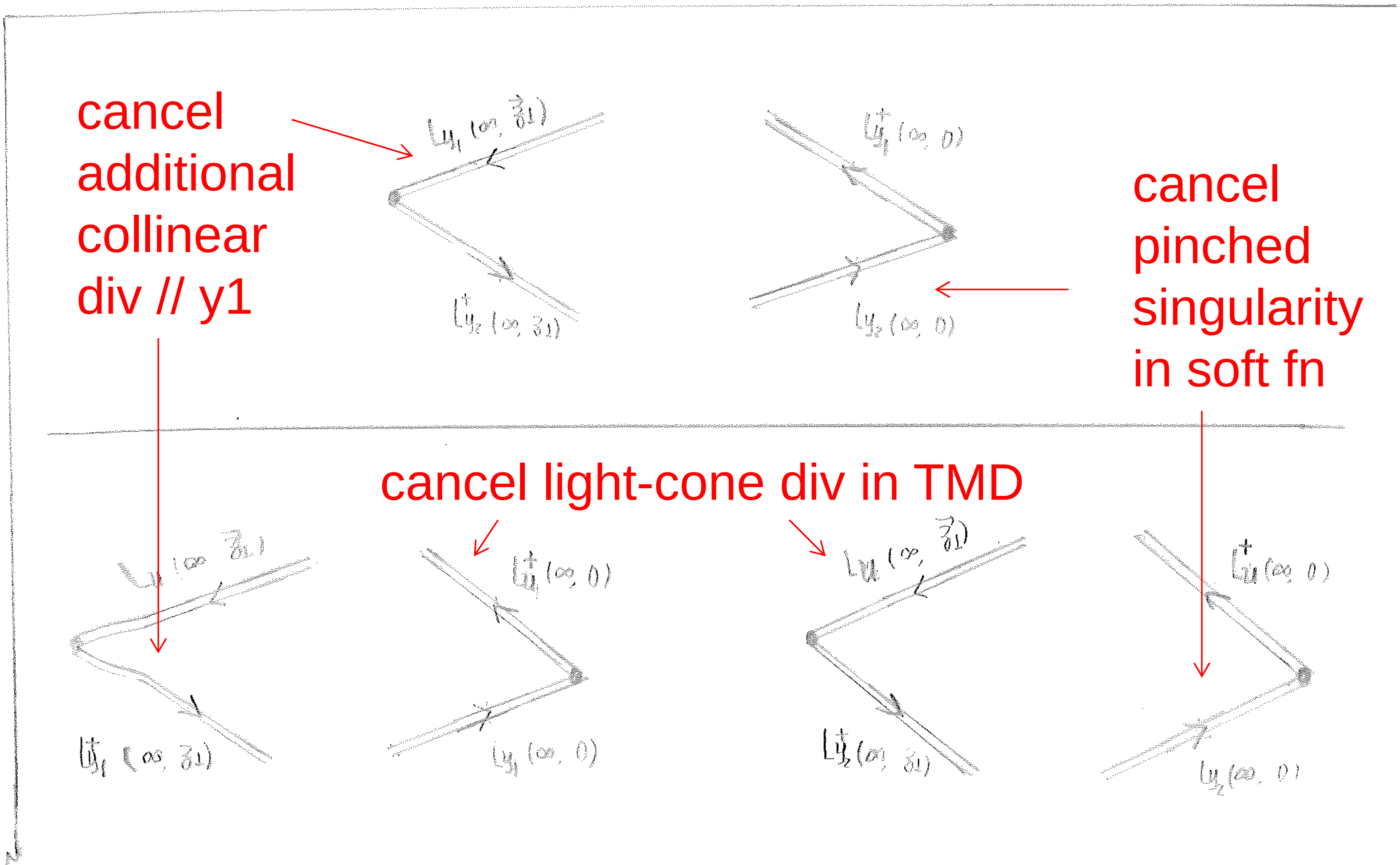
↑

Wilson-line rapidity

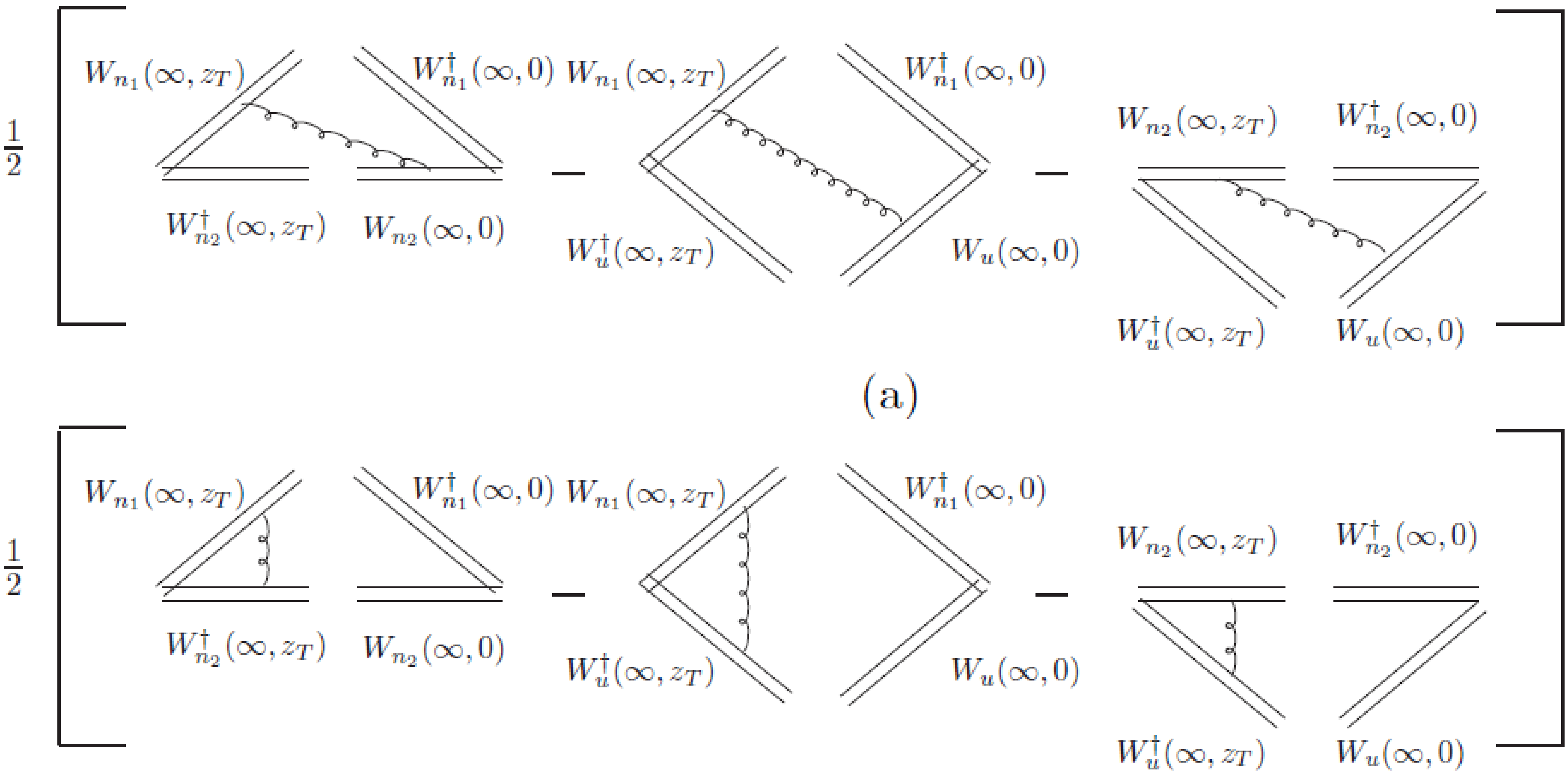
$$n_2 = (e^{y_2}, e^{-y_2}, \mathbf{0_T})$$

- u and n1 on light cone, n2 off light cone
- Off-light-cone Wilson links move into soft function
- Square root renders calculation difficult

# IR cancellations



# NLO diagrams

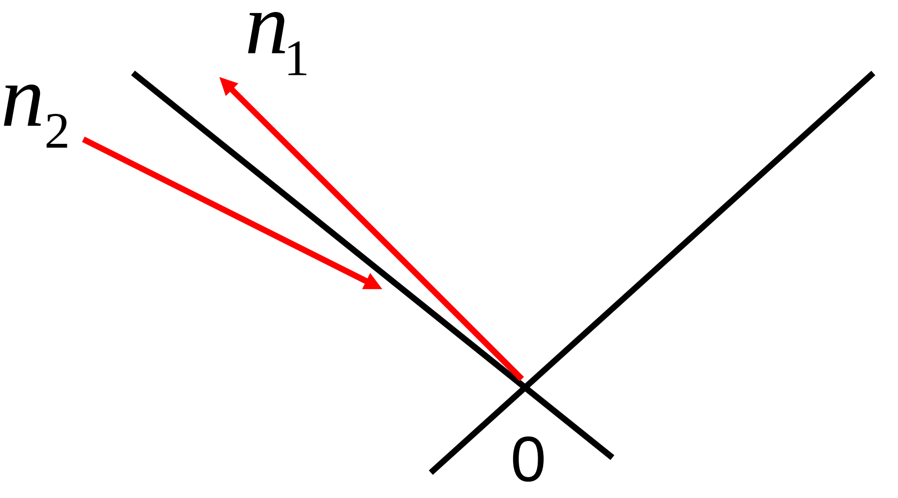


# New definition with non-dipolar Wilson links

# Our modification (Wang, Li 2014)

- Choose orthogonal gauge vectors for off-light-cone Wilson links
- Same light-cone limit as Collins' definition
- Pinched singularity disappears
- soft subtraction is not needed

$$\frac{n_1 \cdot n_2 = 0}{(n_1 \cdot l + i\varepsilon)(n_2 \cdot l - i\varepsilon)}$$



# Check IR behavior

- Take pion transition form factor as an example, whose hard kernel is simple
- Three definitions give the same collinear logarithm, the same as in QCD diagrams
- They all realize kT factorization at small x

$$\int_{-\infty}^{+\infty} dk'_+ \int_{-\infty}^{+\infty} d^{2-2\epsilon}k'_T \;\phi^{C(1)}(k'_+, k'_T, y_2)\, H^{(0)}(k'_+, k'_T)$$
$$= -\frac{\alpha_s C_F}{4\pi}\left[\ln\left(\frac{k_+}{p_+}\right)+2\right]\ln k_T^2\, H^{(0)}(k_+, k_T) + \cdots\,,$$

x

# Quasi-parton distribution function

See Jianhui’s talk

# Difficulty on lattice

- Ordinary light-cone PDF

$$q(x,\mu^2) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^- P^+} \langle P|\overline{\psi}(\xi^-)\gamma^+ \times \exp\left(-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)\right)\psi(0)|P\rangle$$

- Involve time dependence, not suitable for lattice calculation in Euclidean space
- Reason why only (few) moments (local objects) can be calculated by lattice

# Quasi PDF definition

- To facilitate direct lattice calculation of PDF, i.e., all-moment calculation, modify definition in large Pz limit (Ji, 2013)

$$\tilde{q}_n(x,\mu,P^z) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{izk^z} \langle P|\psi(w) W_n^\dagger(w) \gamma^z W_n(0) \psi(0)|P\rangle$$

$$W_n(w) = P\exp\left[-ig\int_0^\infty d\lambda\, T^a\, n\cdot A^a(\lambda n + w)\right]$$

- Then match Quasi PDF to light-cone PDF

$$\tilde{q}_n(x,\mu,P^z) = \int \frac{dy}{y} Z\left(\frac{x}{y}, \frac{\mu}{P^z}\right) q(y,\mu)$$

lattice data

calculable in perturbation

extracted

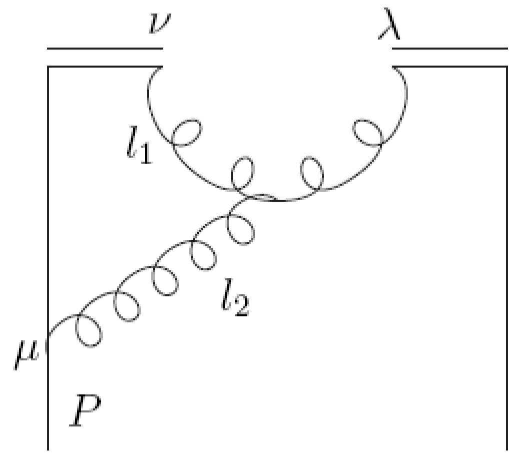
# Linear divergence

- Quasi PDF and light-cone PDF have the same collinear divergence as expected
- Rotation of Wilson links does not change collinear structure: $1/n\cdot l \approx 1/n^- l^+$
- But linear power divergence exists in quasi PDF from large zero component

$$\begin{aligned}\tilde{q}_n^{1c} &= -\frac{1}{2}g^2 C_F \int \frac{d^2 l_T}{(2\pi)^3}\frac{P^z}{l^0 (l^z)^2} \\ &= -\frac{\alpha_s}{2\pi}C_F \frac{\mu}{(1-x)^2 P^z}.\end{aligned}$$

# Breakdown of factorization

- This two-loop diagram violates factorization




leading-power collinear div $\alpha_s^2 \mu \ln m_g^2$

$$\int \frac{d^2 l_{1T}}{l_1^0} \frac{(2l_1 + l_2)^\mu g^{\nu\lambda} + (l_2 - l_1)^\lambda g^{\mu\nu} - (l_1 + 2l_2)^\nu g^{\mu\lambda}}{(l_1 + l_2)^2} n_\nu n_\lambda$$

$$\approx -\int \frac{d^2 l_{1T}}{l_1^0} \frac{g^{\mu +}}{l_2^+} - \int \frac{d^2 l_{1T}}{l_1^0} \frac{(l_2^z - l_1^z) n^\mu}{\sqrt{2}(l_1^0 - l_1^z) l_2^+} + \int \frac{d^2 l_{1T}}{l_1^0} \frac{(l_1^z + 2l_2^z) n^\mu}{\sqrt{2}(l_1^0 - l_1^z) l_2^+}$$

respect factorization

subleading-power collinear div, not power suppressed $\alpha_s^2 \ln \mu \ln m_g^2$

- Z becomes IR div at two loops!

# Modified quasi PDF

- Non-dipolar Wilson links also resolve linear divergence

$$\tilde{q}(x, \mu, P^z) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{izk^z} \langle P|\psi(w) W_{n_2}^{\dagger}(w) \gamma^z W_{n_1}(0)\psi(0)|P\rangle$$

$$n_1 = (0, 1, 0, 1) \text{ and } n_2 = (0, -1, 0, 1)$$

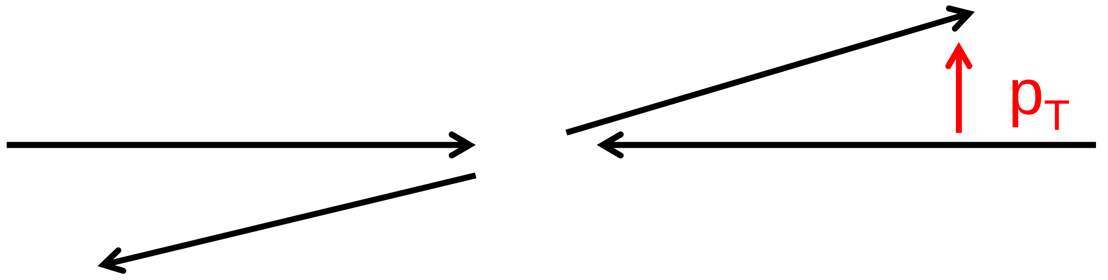
- $\tilde{q}^{1c} = 0$
- Subleading-power divergence is really power suppressed and neglected
- Factorization can be generalized to all orders

# Resummation technique

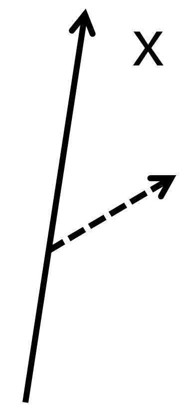
See Jian's talk

# Extreme kinematics

- QCD processes in extreme kinematic region, such as low $p_T$ and large x, generate double logs
- Limited phase space for real corrections
- Low $p_T$ jet, photon, W,… requires small $p_T$ real gluon emissions
- Top pair production requires large x



# Double logs




- Large x demands soft real gluon emission
- Cancellation between virtual and real corrections is not exact
- Large double logs are produced,

$$\alpha_S \ln^2(E/p_T), \alpha_S \ln(1-x)/(1-x)$$

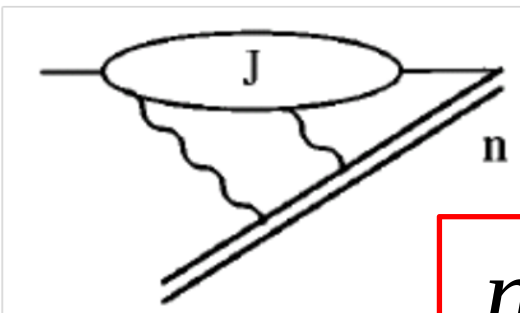
 E being beam energy

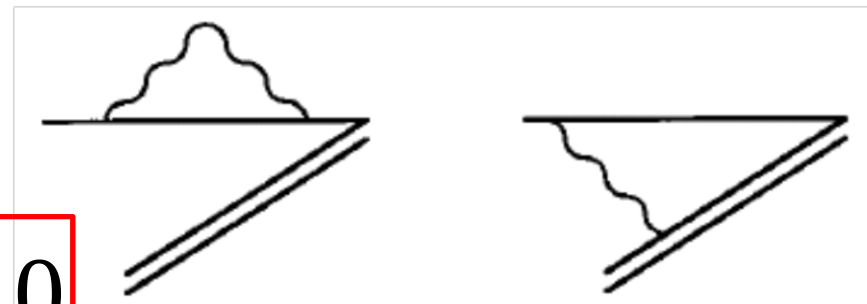
- Sum logs to all orders---resummation
- Resummation improves perturbation

# Jet function in covariant gauge

- Quark jet function at amplitude level

$$J(p,n)=\left\langle 0\left| P\exp\left[ig\int_0^\infty dz\; n\cdot A(nz)\right] q(0)\right| p\right\rangle$$



$$\boxed{n^2 \neq 0}$$

- NLO diagrams contain double logarithms
- Perform resummation in covariant gauge

# Key idea

- Derive differential equation $p^+ dJ/dp^+ = CJ$
- C contains only single logarithm
- Treat C by RG equation
- Solve differential equation, and solution resums double logarithms
- J depends on Lorentz invariants p.n, n^2
- Feynman rules for Wilson lines show scale invariance in n, $n_\nu / n \cdot l$
- J must depend on $(p \cdot n)^2 / n^2$

# Derivative with respect to n

- Derivative with respect to p can be replaced by derivative with respect to n

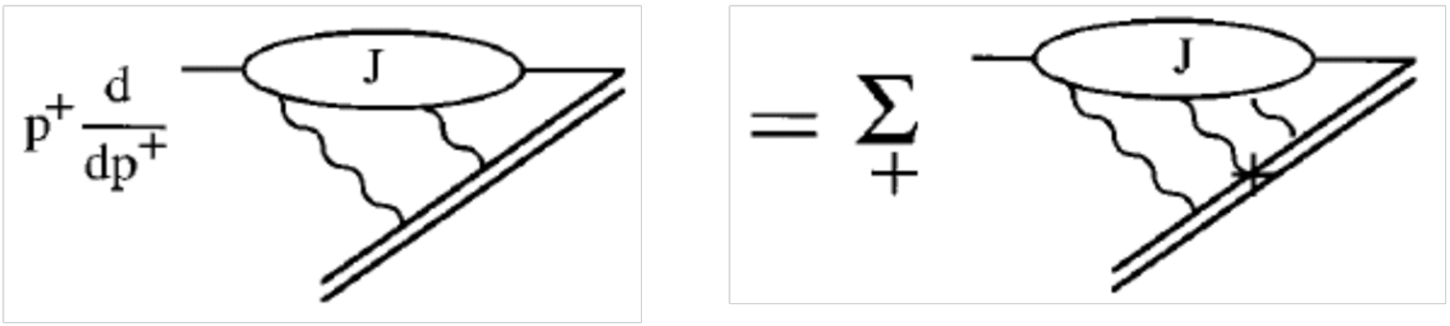
$$p^+ \frac{d}{dp^+} J = -\frac{n^2}{v \cdot n} v_\alpha \frac{d}{dn_\alpha} J$$

- Collinear dynamics independent of n
- Variation does not contain collinear dynamics, the reason why C contains only single logarithm
- Variation effect can be factorized

# Special vertex

- n appears only in Wilson lines

$$-\frac{n^2}{v\cdot n}v_\alpha\frac{d}{dn_\alpha}\frac{n_\mu}{n\cdot l}=\frac{n^2}{v\cdot n}\left(\frac{v\cdot l}{n\cdot l}n_\mu-v_\mu\right)\frac{1}{n\cdot l}\equiv\frac{\hat{n}_\mu}{n\cdot l}$$




- If gluon momentum l parallel to p, v.l vanishes.
- Contraction of v with J, dominated by collinear dynamics, also vanishes

# Soft factorization

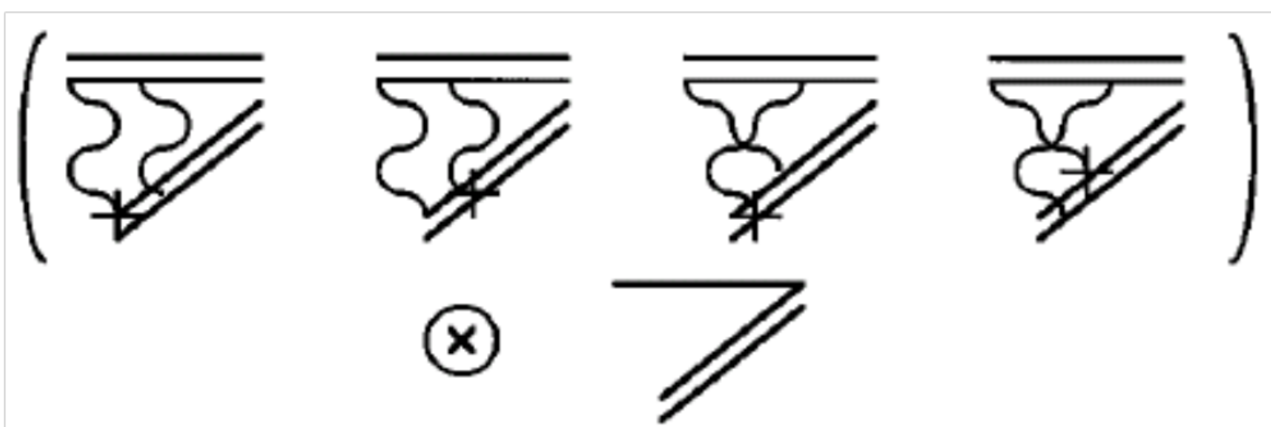
- Two-loop diagrams as example

- If l flowing through special vertex is soft, but another is not, only 1st diagram dominates
- Another gluon with finite momentum smears soft divergence in other three diagrams

# Soft kernel

- Factorizing the soft gluon with eikonal approximation


- If another gluon is also soft, we get two-loop soft kernel K



# Hard factorization

- If l flowing through special vertex is hard, only 2nd diagram dominates,
- It suppresses others, as another gluon is not hard
- Factorize hard kernel with Fierz transformation

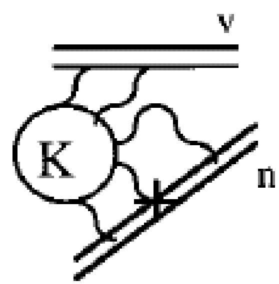
- If both gluons are hard, they contribute to two-loop hard kernel

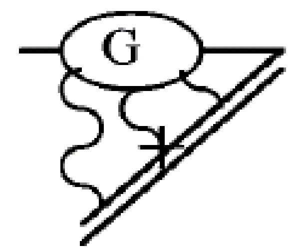
# Differential equation

- Extending the factorization to all orders

$$p^+ \frac{d}{dp^+} J = [K(m/\mu, \alpha_s(\mu)) + G(p^+ \nu/\mu, \alpha_s(\mu))] J$$

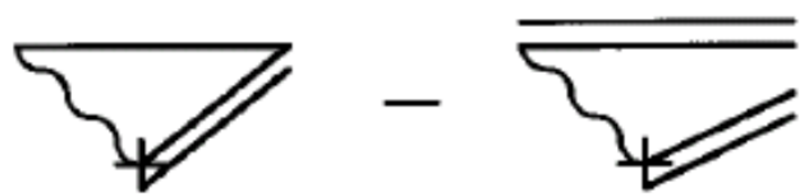
- Kernels K and G are described by




- At LO, K =

- G =



# Integrals for kernels

$$K = -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{\hat{n}_\mu}{n \cdot l} \frac{g^{\mu\nu}}{l^2 - m^2} \frac{v_\nu}{v \cdot l} - \delta K$$

$$= -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{n^2}{(n \cdot l)^2 (l^2 - m^2)} - \delta K$$

$$G = -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{\hat{n}_\mu}{n \cdot l} \frac{g^{\mu\nu}}{l^2} \left( \frac{\not{p} + \not{l}}{(p+l)^2} \gamma_\nu - \frac{v_\nu}{v \cdot l} \right)$$

$$- \delta G, \qquad ($$

- $\delta K$ , $\delta G$ additive counterterm
- Can derive kT resum, threshold resum, joint resum, and evolution equations…

# Summary

- $k_T$ factorization, more complicated than collinear factorization, has many difficulties
- TMD definition is one of them
- Rotation of Wilson links can be used to define TMD, quasi-PDF, derive resummation and evolution equations
- Non-dipolar Wilson links remove linear divergences in TMD and quasi-PDF, facilitate proof of kT factorization
- Should put modified quasi-PDF on lattice

# Back-up slides

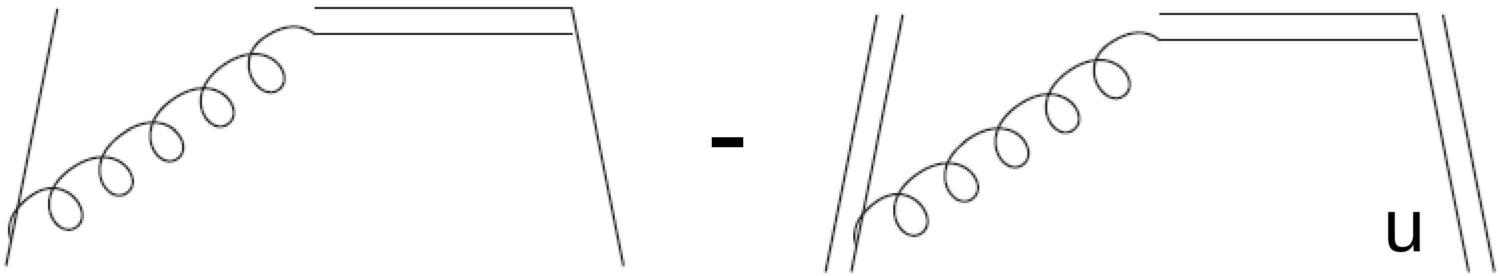
# Modification 2

- Soft subtraction

$$\frac{\langle N(P,\sigma)|\bar{q}_f(0,y^-,b)\frac{1}{2}\gamma^+W(y^-,b,0,0_T)q_f(0,0,0_T)|N(P,\sigma)\rangle}{\langle 0|W(y^-,b,0,0_T)W_u^\dagger(y^-,b,0,0_T)|0\rangle}$$

- Numerator and denominator have the same light-cone singularities.

parton line eikonalized in u



# Practical difficulty

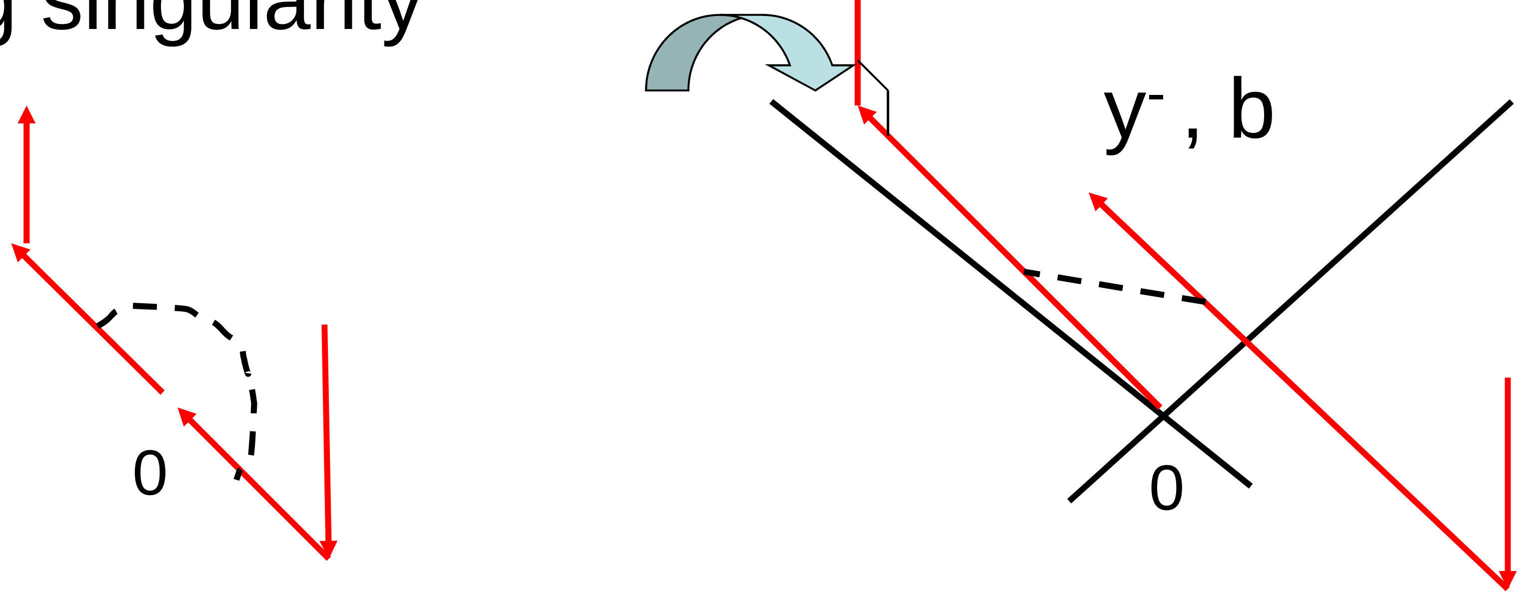
- Modification 2 is useless in practice
- Light-cone singularities in numerator and denominator still need to be regularized first, before showing their cancellation
- Regularization by means of off-light-cone Wilson line then brings the same pinched singularity
- How to regularize pinched singularity still arises

# Another modification

- Reverse one of off-light-cone Wilson links
- Pinched singularity reduces to log singularity

$$\frac{1}{(n\cdot l + i\varepsilon)^2}$$

- Introduce soft subtraction to remove this log singularity



# Check evolution

- Evolution equation in Wilson-line rapidity from Collins' definition

$$\frac{d}{dy_2}\ln\phi^C(k'_+,k'_T,y_2)=\frac{\alpha_s(\mu)\,C_F}{\pi}\left[\ln\left(\frac{2\,k'_+\bar{k}'_+}{\mu^2}\right)+2\,y_2\right]$$

factorization scale

- From our definitions

$\mu = p_+$

$$\frac{d}{dy_2}\ln\phi^L(k'_+,k'_T,y_2)=\frac{\alpha_s(\mu)\,C_F}{\pi}\left(\ln\frac{k'_+}{p_+}+1\right)$$

- They are equivalent at small x