Quasi Distribution and 2D QCD in Large N Limit

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PDF in 4d QCD

Light-cone parton distribution function

Ji, X.D. PRL78,610(1997), PRD55,7114(1997)
Radyushkin, A.V. PLB.380,417(1996), PLB385,333(1996)

- Experimental extraction. No problem.
- Lattice calculation. Very hard.

Quasi parton distribution function

Ji, X.D. et al. PRLs(2013), PRDs(2014, 2015)

- Lattice calculation. Directly.
- Extract light-cone PD via matching.

Why 2 dimensional QCD in large N limit ?

- 4d QCD is extremely complicate.
- 2d QCD in large N limit is simple but nontrivial.
- It is numerically solvable.

2d model of meson

Approach 1: 't Hooft's Lagrangian approach

't Hooft, Nucl. Phys. B75 (1974) 461-470.

Light cone coordinates :

$$x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$$

$$x_{\pm} = (x_0 \pm x_1)/\sqrt{2}$$

$$x_0 = x^0, x_1 = -x^1 \quad \Rightarrow \quad x_{\pm} = x^{\mp}$$

$$a \cdot b = a^+ b^- + a^- b^+$$

Light-cone gauge :

$$A^+ = 0 \quad \Rightarrow \quad F^{+-} = \partial^+ A^-, F^{-+} = -\partial^+ A^-.$$

Gamma matrices algebra :

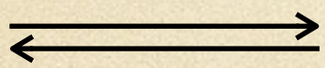
$$\gamma^+\gamma^- + \gamma^+\gamma^- = 2$$

$$\gamma^+\gamma^+ = \gamma^-\gamma^- = 0$$

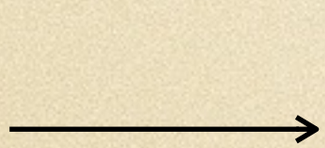
Lagrangian :

$$\mathcal{L}_{2d} = \frac{1}{2}\mathrm{Tr}\partial^+ A^- \partial^+ A^- + \bar{\psi}^i(i\not{\partial} - m_i + g\gamma^+ A^-)\psi^i \qquad (1)$$

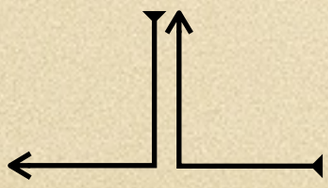
Feynman rules :



$$\frac{-i}{(k^+)^2}$$



$$\frac{i}{\gamma^- k^+ + \gamma^+ k^- - m_i} \qquad \frac{ik^+}{2k^- k^+ - m_i^2} \qquad (2)$$



$$ig\gamma^+ \qquad i2g$$

Large N limit :

$$N_c \to \infty, \quad g^2 N_c \to \text{Const.}$$

Advantages of 2d QCD in Large N limit :

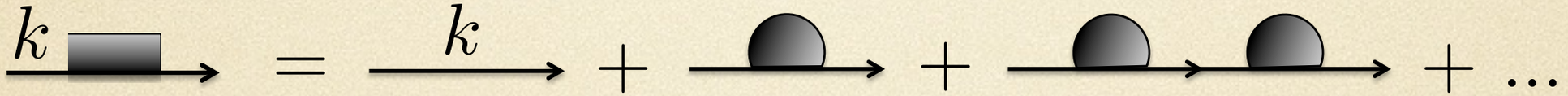
- No gluon self interaction.
- No gluon transverse degree of freedom.
- Only planar diagrams survive.
- No internal quark loop.
- Gluon has no propagating mode, no gluon radiation.

Irreducible self energy diagram :

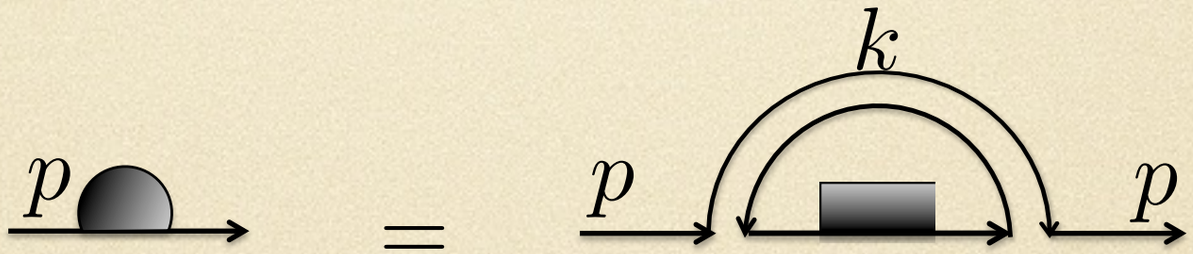
$$-i\Gamma(p) \quad =$$

Dressed propagator :

$$\frac{ik^+}{2k^+k^- - m_i^2 - k^+\Gamma(k)}$$



Bootstrap equation :



$$\Gamma(p) = \frac{-ig^2}{\pi^2}\int dk^+dk^- \frac{1}{(k^+)^2}\frac{k^+ + p^+}{(k^+ + p^+)(2k^- + 2p^- - \Gamma(k+p)) - m_i^2} \qquad (3)$$

$$\int dk^- \frac{1}{(k^+ + p^+)(2k^- - \Gamma(k^+ + p^+) - \frac{m_i^2}{k^+ + p^+} + i\epsilon)} = \frac{i\pi}{2|k^+ + p^+|}$$

$$\Downarrow$$

$$\Gamma(p^+) = \frac{g^2}{2\pi} \int dk^+ \frac{1}{(k^+)^2} \mathrm{sgn}(k^+ + p^+)$$

$$\Downarrow \text{IR divergent, cut-off } \lambda < |k^+| < \infty$$

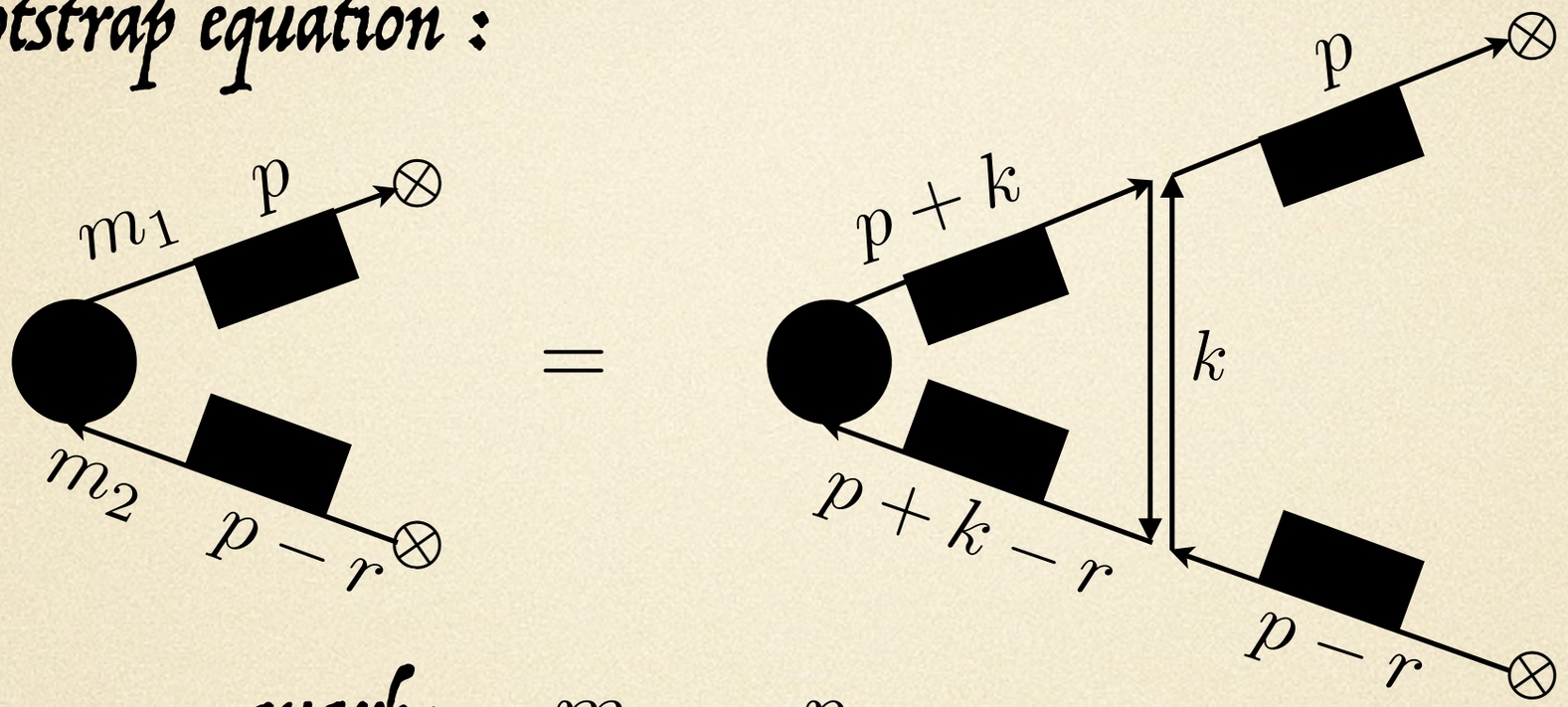
$$\Gamma(p^+) = \frac{g^2}{\pi} \left(\frac{\mathrm{sgn}(p^+)}{\lambda} - \frac{1}{p^+} \right)$$

$$\Downarrow$$

dressed propagator :

$$\longrightarrow = \frac{ik^+}{2k^+k^- + \frac{g^2}{\pi} - \frac{g^2}{\pi\lambda}|k^+| - m_i^2} \tag{4}$$

Another bootstrap equation :



quark: $m_1, \quad p$

anti-quark: $m_2, \quad r-p$

two-particles state: $\psi(p,r)$

$$M_i^2 = m_i^2 - \frac{g^2}{\pi}$$

$$\psi(p,r) = \frac{4g^2}{(2\pi)^2}\int dk^+ dk^- \psi(p+k,r)\frac{i}{(k^+)^2}\frac{p^+}{2p^+p^- - \frac{g^2}{\pi\lambda}|p^+| - M_1^2} * \tag{5}$$

$$* \frac{p^+ - r^+}{2(p^+ - r^+)(p^- - r^-) - \frac{g^2}{\pi\lambda}|p^+ - r^+| - M_2^2}$$

Defining: $$\psi(p^+, r) = \int \psi(p, r) dp^-$$

Integrating over p^- on both sides of (5)

$$\psi(p^+, r) = \int dp^- \frac{1}{(p^- - r^-) - \frac{M_2^2}{2(p^+ - r^+)} - (\frac{g^2}{2\pi\lambda} - i\epsilon)\mathrm{sgn}(p^+ - r^+)} \frac{1}{p^- - \frac{M_1^2}{2p^+} - (\frac{g^2}{2\pi\lambda} - i\epsilon)\mathrm{sgn}(p^+)} * \quad (6)$$
$$* \frac{ig^2}{(2\pi)^2} \int dk^+ \psi(p^+ + k^+, r) \frac{1}{(k^+)^2}$$

The first integral is non-zero only if :

$$\boxed{\mathrm{sgn}(p^+ - r^+) = -\mathrm{sgn}(p^+)}$$

$$\psi(p^+, r) = \frac{g^2}{2\pi} \frac{\theta(p^+)\theta(r^+ - p^+)}{-r^- + \frac{M_1^2}{2p^+} + \frac{M_2^2}{2(r^+ - p^+)} + \frac{g^2}{\pi\lambda}} \int dk^+ \frac{\psi(p^+ + k^+, r)}{(k^+)^2} \quad (7)$$

Hadamard regularization :

Hadamard, Lectures on Cauchy's problem in linear partial differential equations, Dover Phoenix editions, 1923.

$$\mathcal{H}\int_a^b \frac{f(x)}{(x-t)^2}dx = \lim_{\lambda\to 0^+}\left(\int_a^{t-\lambda}\frac{f(x)}{(x-t)^2}dx + \int_{t+\lambda}^b \frac{f(x)}{(x-t)^2}dx - \frac{2f(t)}{\lambda}\right) \quad (8)$$

$\Downarrow$

with cut-off

$$\int_a^b dk^+ \frac{\psi(p^+ + k^+, r)}{(k^+)^2} = \mathcal{H}\int_a^b dk^+ \frac{\psi(k^+ + p^+, r)}{(k^+)2} + \frac{2\psi(p^+, r)}{\lambda} \quad (9)$$

substituting (9) *into* (7)

$$\left(\frac{M_1^2}{2p^+} + \frac{M_2^2}{2(r^+ - p^+)}\right)\psi(p^+, r) - \frac{g^2}{2\pi}\mathcal{H}\int_{p^+}^{r^+ - p^+} dk^+ \frac{\psi(k^+ + p^+, r)}{(k^+)^2} = r^-\psi(p^+, r) \quad (10)$$

Defining new parameters :

$$\alpha_{1,2} = \frac{\pi M_{1,2}^2}{g^2}$$
$$\frac{p^+}{r^+} = x \tag{11}$$
$$2r^+r^- = \frac{g^2}{\pi}\mu^2$$

Equation (10) becomes 't Hooft equation:

$$\mu^2\phi(x) = \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right)\phi(x) - \mathcal{H}\int_0^1 \frac{\phi(y)}{(y-x)^2}dy \tag{12}$$

Approach 2: Hamiltonian method in axial gauge

Bars and Green, PRD.17.(1978)537-545

Kalashnikova, Nefediev, and Volodin, Phys. Atom. Nucl.63.(2000)1623-1628

Shifman, Advanced Topics in Quantum Field Theory, 2012

Axial gauge :

$$A_1(t,x) = 0$$

Gamma matrices convention :

$$\gamma^0 = \sigma_3, \gamma^1 = i\sigma_2, \gamma_5 = \gamma^0\gamma^1 = \sigma_1$$

Lagrangian :

$$\mathcal{L}_{2d} = \frac{1}{2}\mathrm{Tr}(\partial_1 A_0)^2 + \bar{\psi}^i(i\not{\partial} + g\gamma^0 A_0 - m_i)\psi^i \qquad (13)$$

Equation of motion :

$$(\partial_1)^2 A_0^a = g\psi^\dagger T^a \psi^i \tag{14}$$

Solution :

$$A_0^a(x) = \int dy|x-y| g\psi^\dagger(y) T^a \psi^i(y) \tag{15}$$

Hamiltonian

$$H = \int dx \psi^\dagger(-i\gamma_5\partial_1 + \gamma^0 m_i)\psi$$
$$+ \frac{g^2}{4}\int dxdy|x-y|\psi^\dagger(x)T^a\psi(x)\psi^\dagger(y)T^a\psi(y) \tag{16}$$

Dressing fermion field

$$\psi(t,x) = \int \frac{dk}{2\pi} e^{ikx}\left(b(k,t)u(k) + d^\dagger(-k,t)v(-k)\right) \tag{17}$$

$$b(k,t)|0\rangle = d(-k,t)|0\rangle = 0,$$
$$b(k,t)|0\rangle = |k\rangle,$$
$$d(-k,t)|0\rangle = |-k\rangle.$$

Commutation relations :

$$\{b(p,t), b^\dagger(q,t)\} = \{d(-p,t), d^\dagger(-q,t)\} = 2\pi\delta(p-q)$$
$$\{b(p,t), b(q,t)\} = \{d(-p,t), d(-q,t)\} = ... = 0$$

Spinors :

$$T(k) = -\frac{1}{2}\theta(k)\gamma^1 = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$
$$u(k) = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = T(k)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (18)$$
$$v(k) = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} = T(k)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Normal ordered Hamiltonian

$$H = H_0 + :H_2: + :H_4: \tag{19}$$

$$H_0 = \int dx \int \frac{dp}{2\pi} \mathrm{Tr}\left((\gamma_5 p + \gamma^0 m)\Lambda_-(p) + \frac{1}{4}\int dk \frac{\Lambda_+(k)\Lambda_-(k)}{(p-k)^2} \right)$$

$$:H_2: = \int dx : \psi^\dagger(x)(-i\gamma_5\partial_x + \gamma^0 m)\psi(x) :$$

$$- \int dx dy dk \frac{|x-y|}{4} : \psi^\dagger(x)\Big(\Lambda_+(k) - \Lambda_-(k)\Big)\psi(y) : e^{ik(x-y)}$$

$$:H_4: = -\frac{g^2}{4}\int dx dy : \psi^\dagger(x) T^a \psi(x) \psi^\dagger(y) T^a \psi(y) : |x-y|$$

$$\Lambda_\pm(p) = T(p)\frac{1 \pm \gamma^0}{2} T^\dagger(p)$$

Diagonalize Hamiltonian

$$:H_2:=\int\frac{dk}{2\pi}E(k)\Big(b^\dagger(k)b(k)+d^\dagger(-k)d(-k)\Big) \tag{20}$$

$$\int dx:\psi^\dagger(x)(-i\gamma_5\partial_x+\gamma^0 m)\psi(x):$$

$$=\int dp\Big((p\sin\theta_p+m\cos\theta_p)b^\dagger(p)b(p)+(p\cos\theta_p-m\sin\theta_p)b^\dagger(p)d^\dagger(-p)$$

$$+(p\cos\theta_p-m\sin\theta_p)d(-p)b(p)+(p\sin\theta_p+m\cos\theta_p)d^\dagger(-p)d(-p)\Big)$$

$$\int dxdydk\frac{|x-y|}{4}:\psi^\dagger(x)\Big(\Lambda_+(k)-\Lambda_-(k)\Big)\psi(y):e^{ik(x-y)}$$

$$=\int dp\int\frac{dk}{(p-k)^2}\left(\sin^2\left(\frac{\theta_p-\theta_k}{2}\right)b^\dagger(p)b(p)+\frac{\sin(\theta_p-\theta_k)}{2}b^\dagger(p)d^\dagger(-p)\right.$$

$$\left.+\frac{\sin(\theta_p-\theta_k)}{2}d(-p)b(p)-\cos^2\left(\frac{\theta_p-\theta_k}{2}\right)d^\dagger(-p)d(-p)\right)$$

Integral is divergent at the IR region, all infinite terms must be regularized, only finite terms are left.

$$\int \equiv \mathcal{H} \int$$

Gap equation for Bogolyubov-Valatin angle

$$p\cos\theta(p) - m\sin\theta(p) = \frac{1}{4}\int dk \frac{\sin[\theta(p)-\theta(k)]}{(p-k)^2} \tag{21}$$

Dispersion relation

$$E(p) = p\sin\theta(p) + m\cos\theta(p) + \frac{1}{4}\int dk \frac{\cos[\theta(p)-\theta(k)]}{(p-k)^2} \tag{22}$$

color singlet bilinear operators :

$$
\begin{aligned}
B(p,p') &= \frac{1}{\sqrt{N_C}} b_i^\dagger(p) b_i(p'), \\
D(p,p') &= \frac{1}{\sqrt{N_C}} d_i^\dagger(-p) d_i(-p'), \\
M(p,p') &= \frac{1}{\sqrt{N_C}} d_i(-p) b_i(p'), \\
M^\dagger(p,p') &= \frac{1}{\sqrt{N_C}} b_i^\dagger(-p) d_i^\dagger(p')
\end{aligned}
\tag{23}
$$

Commutation relations in Large N limit :

$$
\begin{aligned}
[M(p,p'), M^\dagger(q,q')] &= (2\pi)^2 \delta(p-q)\delta(p'-q'), \\
[B(p,p'), X(q,q')] &= 0, \\
[D(p,p'), X(q,q')] &= 0 \\
X(q,q') &\in \{B, D, M, M^\dagger\}
\end{aligned}
\tag{24}
$$

Ansatz :

Lenz, Thies, Levit and Yazaki, Ann.Phys.(N.Y.)208.(1991)1-89

$$B(p,p') = \frac{1}{\sqrt{N_C}} \int \frac{dq}{2\pi} M^\dagger(q,p) M(q,p'),$$
$$D(p,p') = \frac{1}{\sqrt{N_C}} \int \frac{dq}{2\pi} M^\dagger(p,q) M(p',q). \tag{25}$$

$$:H_2: + :H_4:$$
$$= \int \frac{dQdq}{(2\pi)^2} \Big(E(p) + E(Q-p) \Big) M^\dagger(p-Q,p) M(p-Q,p)$$
$$- \int \frac{dQdq}{(2\pi)^2} \frac{1}{4} \int \frac{dk}{(p-k)^2} \Big(2C(p,k,Q) M^\dagger(p-Q,p) M(k-Q,k) +$$
$$+ S(p,k,Q) \left[M(p,p-Q) M(k-Q,k) + M^\dagger(p,p-Q) M^\dagger(k-Q,k) \right] \Big)$$

$$C(p,k,Q) = \cos \frac{\theta(p) - \theta(k)}{2} \cos \frac{\theta(Q-p) - \theta(Q-k)}{2}$$
$$S(p,k,Q) = \sin \frac{\theta(p) - \theta(k)}{2} \sin \frac{\theta(Q-p) - \theta(Q-k)}{2}$$

Define operators :

$$
\begin{aligned}
m_n^\dagger(Q) &= \int \frac{dq}{2\pi}\Big(M^\dagger(q-Q,q)\phi_n^+(q,Q) + M(q,q-Q)\phi_n^-(q,Q)\Big) \\
m_n(Q) &= \int \frac{dq}{2\pi}\Big(M(q-Q,q)\phi_n^+(q,Q) + M^\dagger(q,q-Q)\phi_n^-(q,Q)\Big)
\end{aligned}
\tag{26}
$$

Normalization conditions :

$$
\begin{aligned}
\int \frac{dq}{2\pi}\Big(\phi_m^+(q,Q)\phi_n^+(q,Q) - \phi_m^-(q,Q)\phi_n^-(q,Q)\Big) &= \delta_{mn} \\
\int \frac{dq}{2\pi}\Big(\phi_n^+(q,Q)\phi_m^-(q,Q) - \phi_n^-(q,Q)\phi_m^+(q,Q)\Big) &= 0
\end{aligned}
\tag{27}
$$

Completeness conditions :

$$
\begin{aligned}
\sum_{n=0}^{\infty}\Big(\phi_n^+(q,Q)\phi_n^+(k,Q) - \phi_n^-(q,Q)\phi_n^-(k,Q)\Big) &= 2\pi\delta(q-k) \\
\sum_{n=0}^{\infty}\Big(\phi_n^+(q,Q)\phi_n^-(k,Q) - \phi_n^-(q,Q)\phi_n^+(k,Q)\Big) &= 0
\end{aligned}
\tag{28}
$$

Commutation relations :

$$[m_n(Q), m_m^\dagger(Q')] = 2\pi\delta(Q-Q')\delta_{nm}$$
$$[m_n(Q), m_m(Q')] = [m_n^\dagger(Q), m_m^\dagger(Q')] = 0$$

Diagonalize Hamiltonian :

$$:H_2: + :H_4: = \frac{1}{2}\sum_{n=0}^{\infty}\int\frac{dQ}{2\pi}Q_n^0(Q)m_n^\dagger(Q)m_n(Q)$$

Bars-Green equations :

$$\Big(E(p)+E(Q-p)-Q^0\Big)\phi^+(p,Q) = \frac{1}{2}\int\frac{dk}{(p-k)^2}\Big(C(p,k,Q)\phi^+(k,Q) - S(p,k,Q)\phi^-(k,Q)\Big)$$
$$\Big(E(p)+E(Q-p)+Q^0\Big)\phi^-(p,Q) = \frac{1}{2}\int\frac{dk}{(p-k)^2}\Big(C(p,k,Q)\phi^-(k,Q) - S(p,k,Q)\phi^+(k,Q)\Big)$$

(29)

Relation :

$$\phi(p,Q) = T(p)\Big(\frac{1+\gamma^0}{2}\gamma_5\phi^+(p,Q) + \frac{1-\gamma^0}{2}\gamma_5\phi^-(p,Q)\Big)T^\dagger(Q-p) \qquad (30)$$

Explanation :

Moving forward : $\phi^+(p,Q)$

Moving backward : $\phi^-(p,Q)$

Bars-Green claim :

Bars-Green equations are equivalent to 't Hooft equation

$$Q \to \infty \quad \Rightarrow \quad \phi^- \to 0 \text{ and } \phi^+ \to \phi$$

Static case :

Li, Wilets and Birse, J.Phys.G.13(1987)915-923

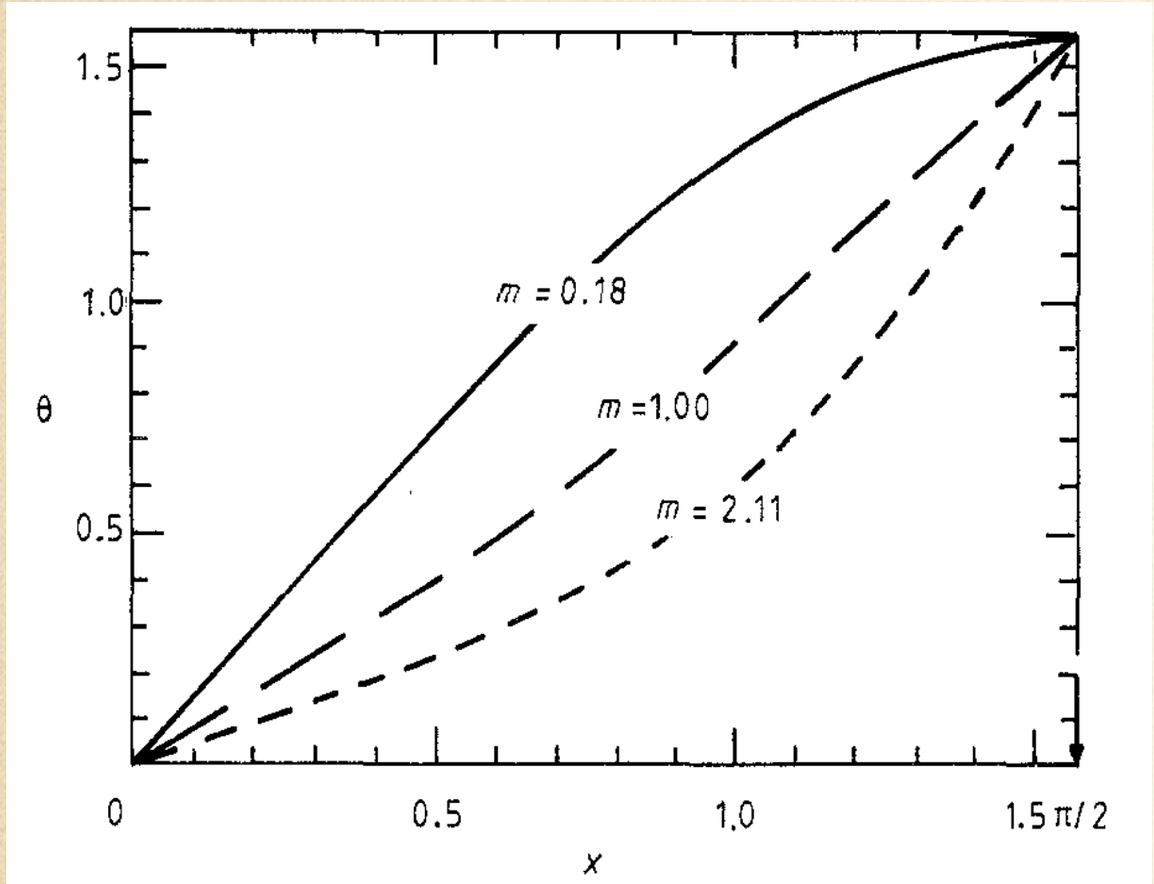
Bars-Green equations :

$$\Big(2E(p) - Q^0\Big)\phi^+(p) = \frac{1}{2}\int \frac{dk}{(p-k)^2}\Big(C(p,k)\phi^+(k) - S(p,k)\phi^-(k)\Big)$$

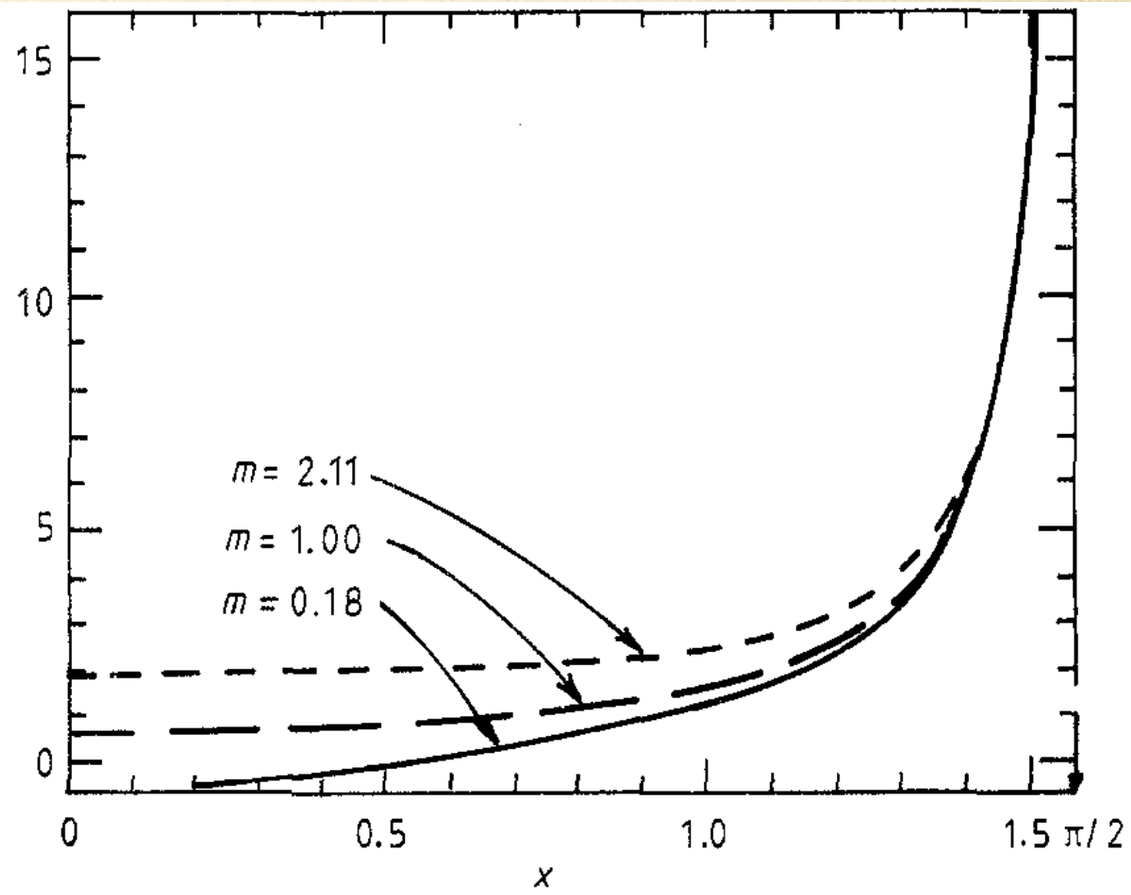
$$\Big(2E(p) + Q^0\Big)\phi^-(p) = \frac{1}{2}\int \frac{dk}{(p-k)^2}\Big(C(p,k)\phi^-(k) - S(p,k)\phi^-(k)\Big)$$

Numerical solution strategies :

1. Solving gap equation to get $\theta(p)$
2. Using dispersion equation to get $E(p)$
3. Finding a basis of functions.
4. Obtaining meson spectrum and related wave functions.



$\theta(p)$

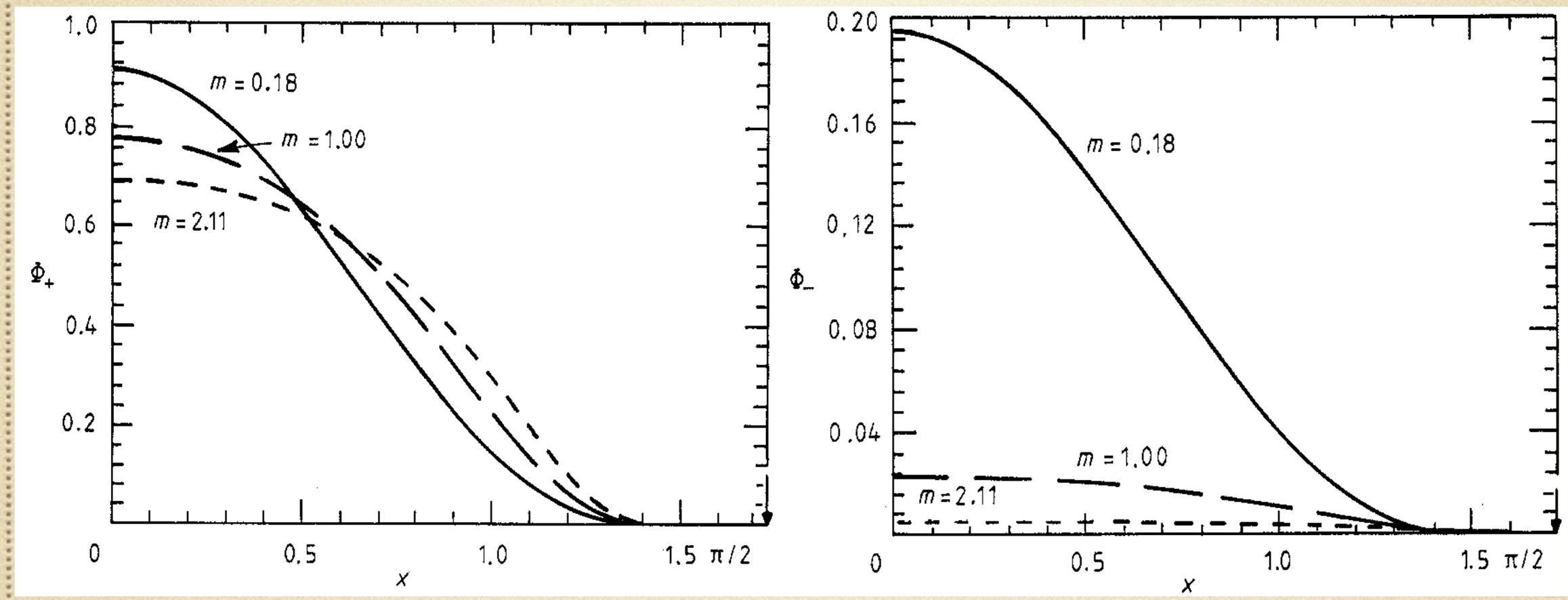


$E(p)$

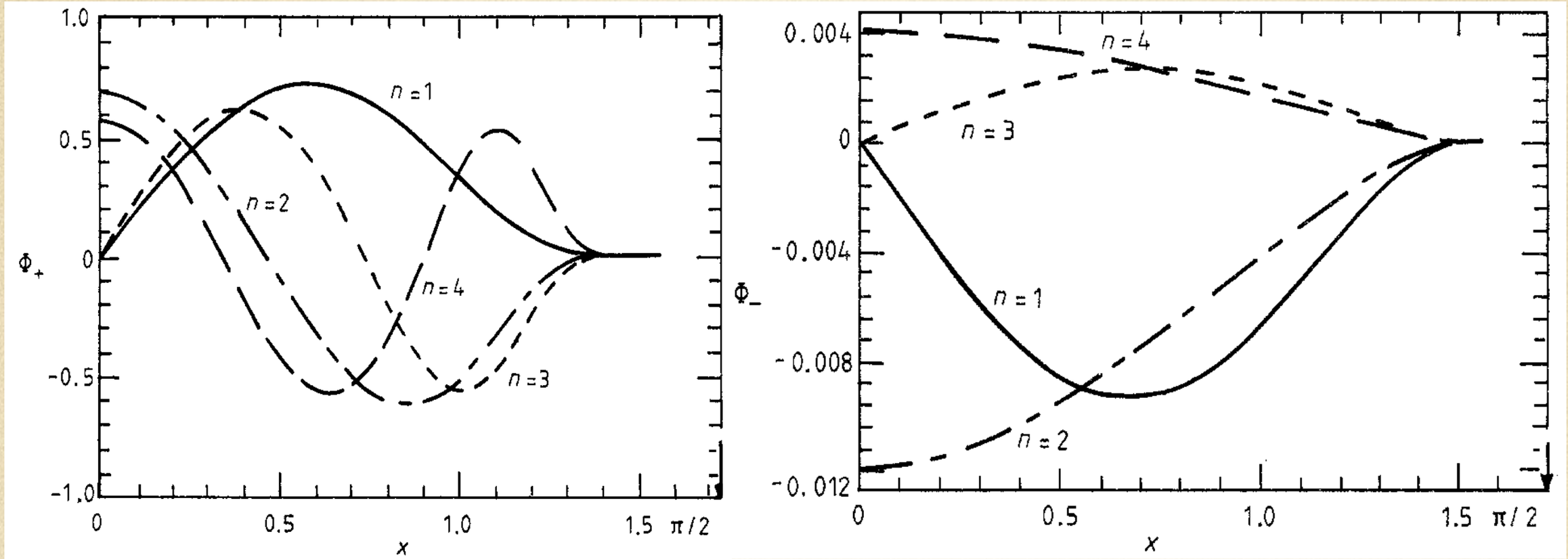
$$p = \tan(x)$$

n	$m=0.18$	$m=0.56$	$m=1.0$	$m=1.61$	$m=2.11$
0	0.88	1.78	2.70	3.92	4.91
1	2.73	3.39	4.16	5.26	6.17
2	4.00	4.54	5.21	6.21	7.06
3	5.00	5.47	6.09	7.02	7.84
4	5.86	6.28	6.85	7.73	8.51
5	6.61	7.00	7.54	8.38	9.13
6	7.30	7.66	8.17	8.98	9.71

Meson spectra with different quark masses m



functions of ground state of meson with different quark masses



functions of excited states of meson with quark mass m=0.18

Numerical results :

quark mass : $m \to \infty \Rightarrow \phi^- \to 0$

higher excited state : $n \to \infty \Rightarrow \phi^- \to 0$

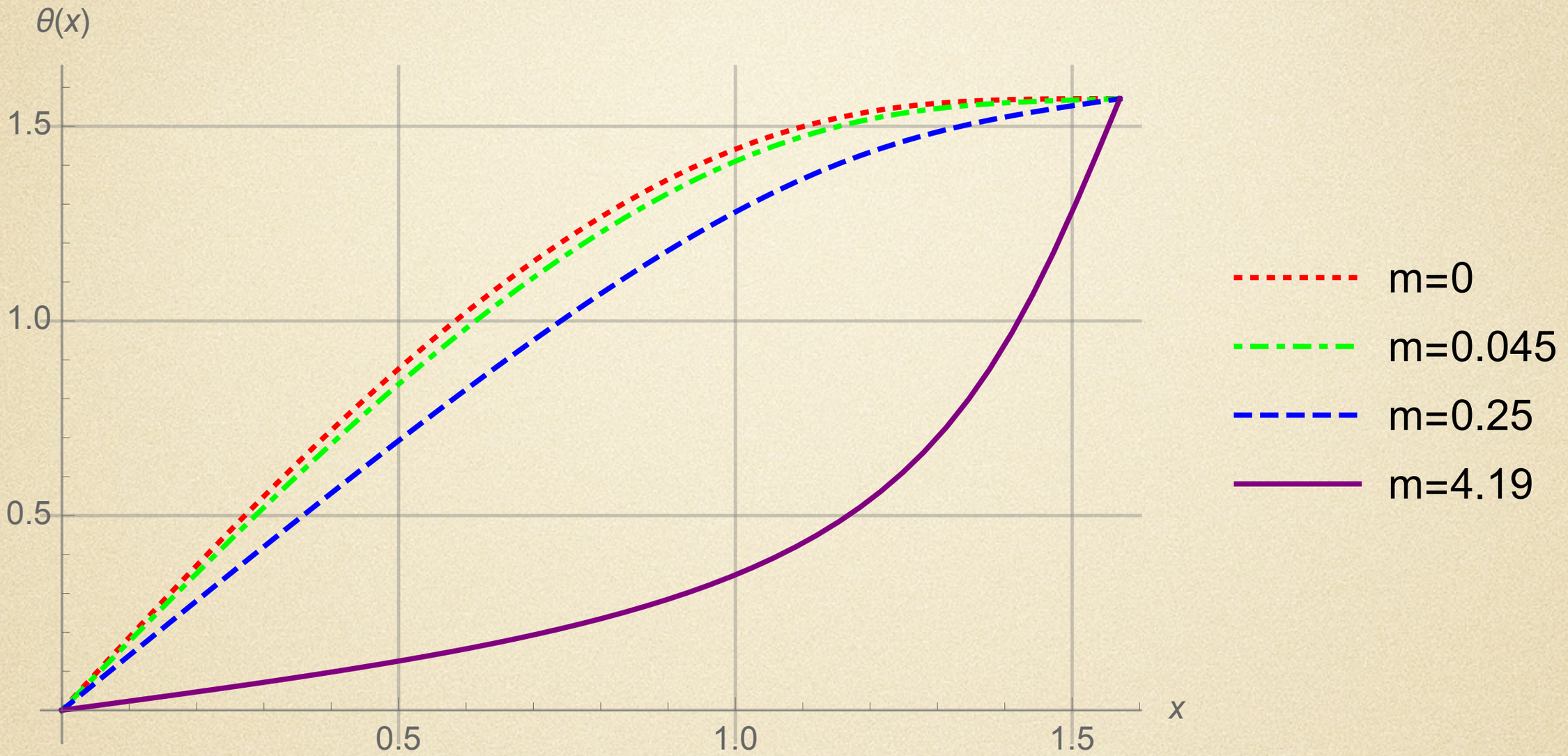
Ours results :

Jia, Liang, Li, and Xiong, in preparation

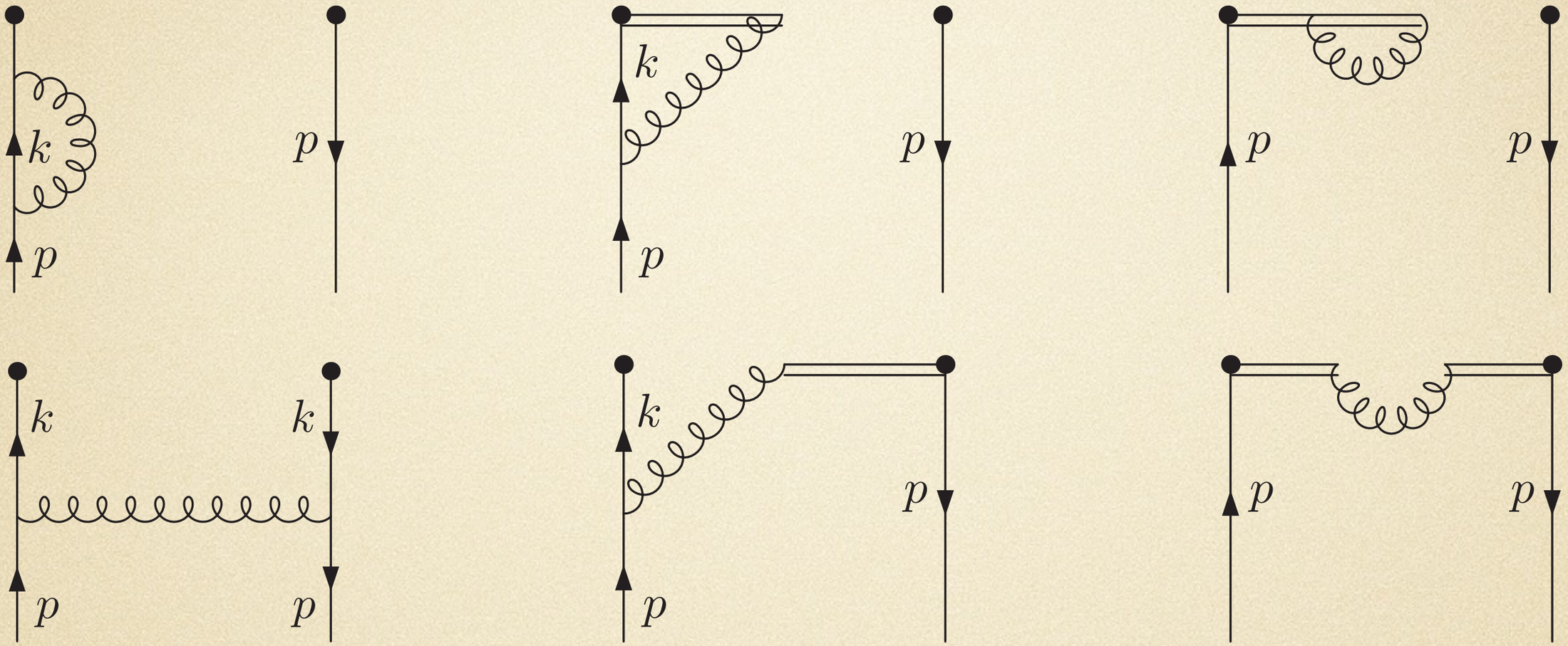
1. Solving gap equation to get $\theta(p)$ ✓
2. Using dispersion equation to get $E(p)$ ✓
3. Finding a basis of functions for moving mesons. ✓
4. Obtaining meson spectra and related wave functions. ✓

Results beyond LWB :

1. Finding a basis of functions for moving mesons.
2. Meson spectra and related wave functions of zero quark mass.
3. Verified Bars-Green's claim.



Bogolyubov-Valatin angle for different quarks



Relevant Feynman diagrams

X.D. Ji, J.H. Zhang,PRD92,(2015)034006

Taking Feynman gauge

light-cone PD in 2d QCD

$$q_a(x) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} f \frac{2^{2\epsilon+1}\pi^{\epsilon+1} m^{-2(\epsilon+1)}(1-x)^{-2\epsilon-3}\left(x\left((x\epsilon+2)+\epsilon\right)\csc(\pi\epsilon)\right.}{\Gamma(-\epsilon)}$$

$$z_a(x) = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{f}{2m^2}\delta(x-1)$$

$$q_b(x) = -\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} f \frac{x 2^{2\epsilon+1}\pi^{\epsilon+1} m^{-2(\epsilon+1)}(1-x)^{-2\epsilon-3}\csc(\pi\epsilon)}{\Gamma(-\epsilon)}$$

$$z_b(x) = -\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \frac{f}{m^2}\delta(x-1)$$

$$q_d(x) = 0$$

$$z_d(x) = 0$$

Quasi PDF in 2d QCD

Too complicated to be showed here.

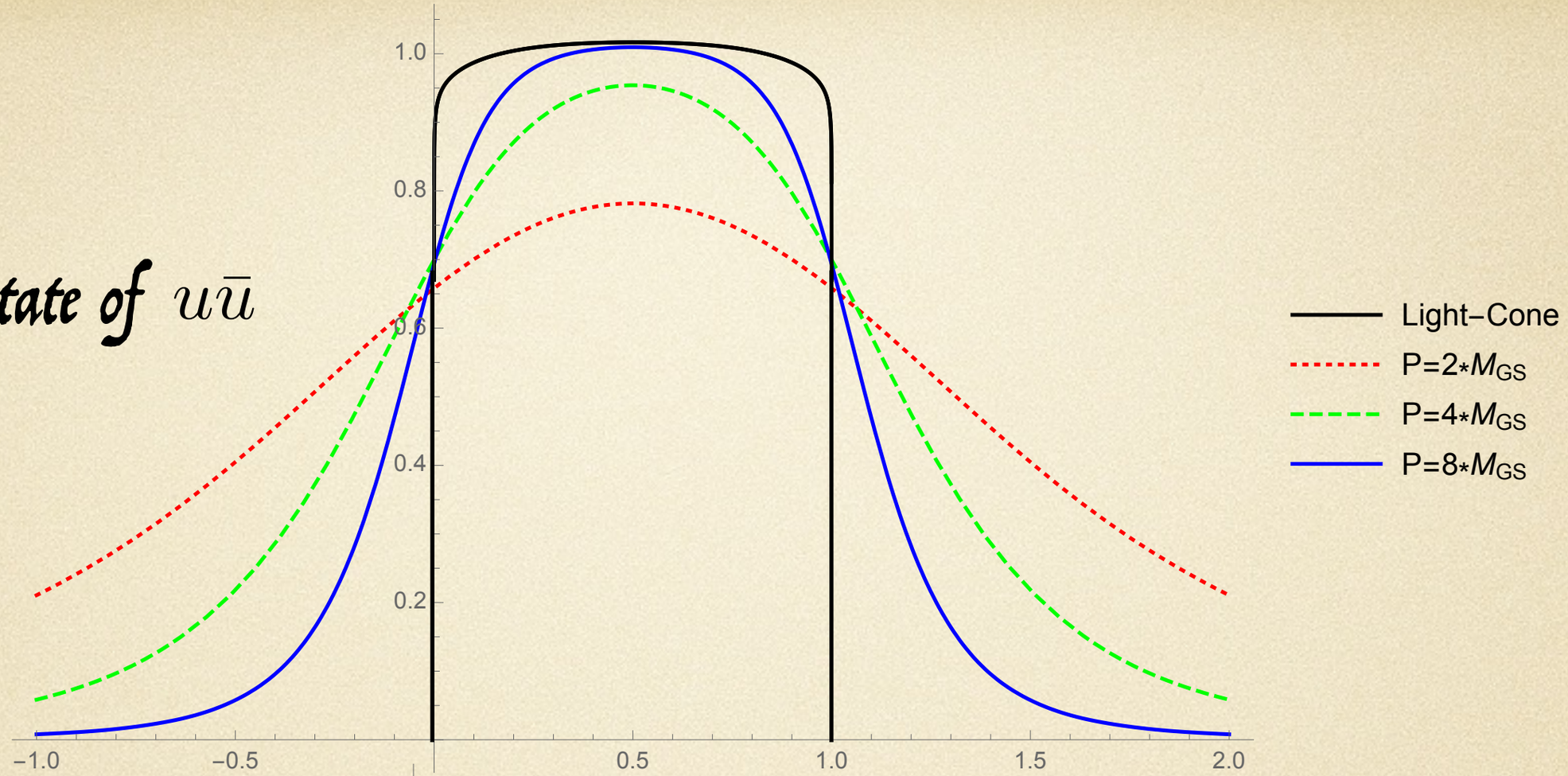
Sharing the same IR structure as light-cone PDF except some terms $\frac{m}{p^z}$

Matching condition

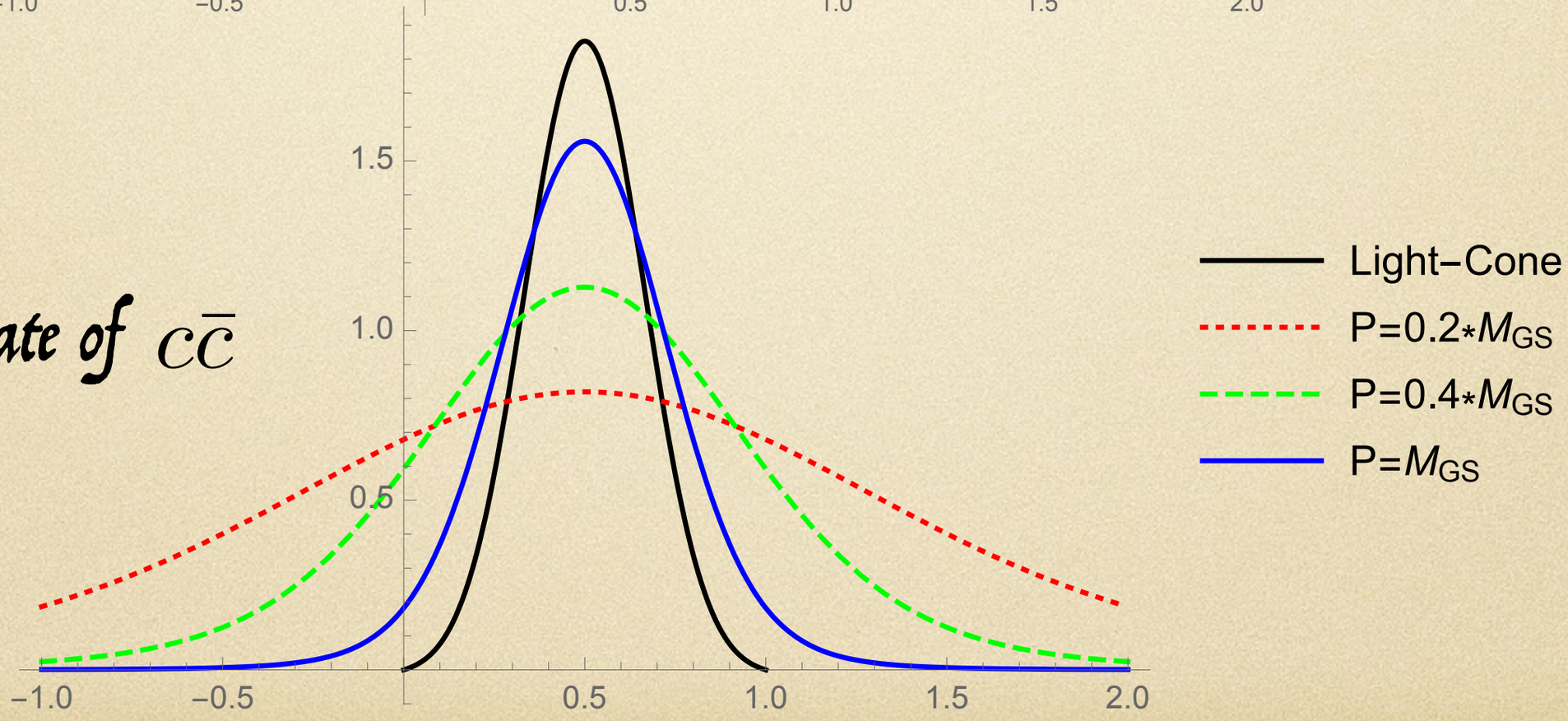
$$\tilde{q}\left(x, \frac{\mu}{p^z}\right) = \int \frac{dy}{|y|} Z\left(\frac{x}{y}, \frac{\mu}{p^z}\right) q(y)$$

$$Z\left(\frac{x}{y}\right)|_{p^z >> m} = \delta\left(\frac{x}{y} - 1\right)$$

Ground state of $u\bar{u}$



Ground state of $c\bar{c}$



Conclusions and outlooks :

- Equivalence of different approaches.
- Bars-Green's claim is right.
- Goldstone boson exists in 2d QCD at large N limit.
- Generalized Bars-Green equation for two different flavor quarks.
- Nonperturbative QCD variables in HQET will be calculated.
- ...

Thanks for your attention