Chandrasekhar-Kendall-Woltjer (CKW) State in Chiral Plasma

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with Hong Qin, Qun Wang, arXiv:1607.01126

What is CKW State?

$$\nabla \times \mathbf{B} = k\mathbf{B}$$

Eigen-state of the $\nabla\times$ operator.

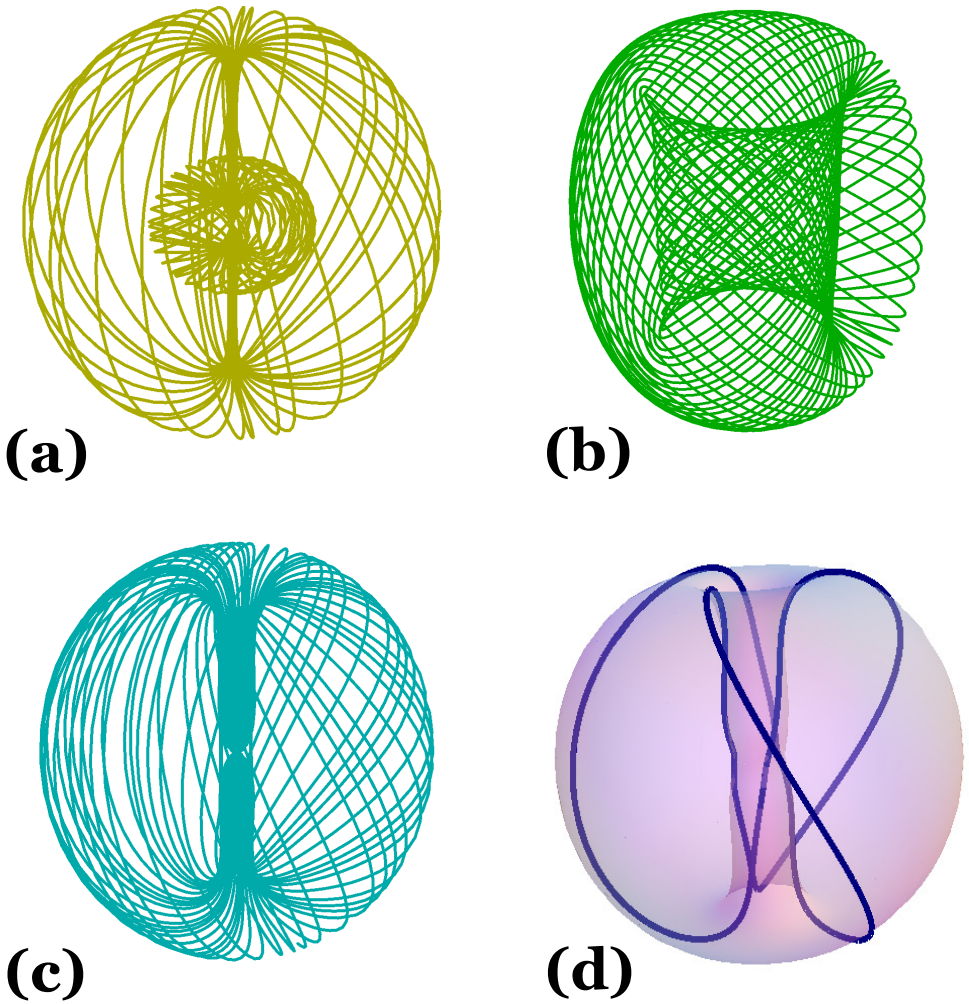
Solution by Chandrasekhar & Kendall (1957)

$$\mathbf{T}_{lm} = \nabla \times (\mathbf{r}\psi_{lm}), \quad \mathbf{S}_{lm} = \frac{1}{k}\nabla \times \mathbf{T}_{lm},$$

$\psi_{lm}(k; r, \theta, \phi) = \frac{i}{\sqrt{l(l+1)}} j_l(kr)\, Y_{lm}(\theta, \phi).$

$$\mathbf{W}^{\pm}_{lm} = \mathbf{T}_{lm} \pm \mathbf{S}_{lm},$$
$$\nabla \times \mathbf{W}^{\pm}_{lm} = \pm k\mathbf{W}^{\pm}_{lm}.$$

What is CKW State?



Magnetic line

Figure from:
M. N. Chernodub,
arXiv:1002.1473

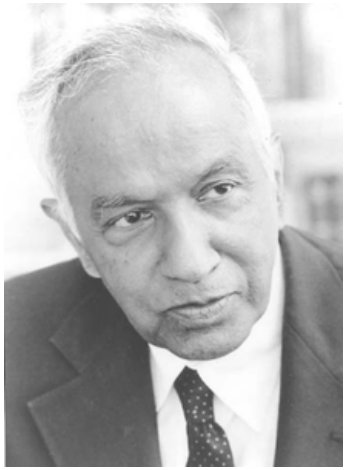
History on CKW State

Chandrasekhar-Kendall-Woltjer

- Lüst and Schlüter (1954) pointed that cosmic magnetic fields might often satisfy

$$\nabla \times \mathbf{B} = k\mathbf{B}$$

- Chandrasekhar, Kendall, Woltjer (1955s) studied it as force-free field
- Woltjer (1958): minimum-energy configuration for a fixed helicity
- J.B. Taylor (1974): natural-end configuration in classical plasma



S. Chandrasekhar (1910-1995)

Astrophysics, plasma physics, early universe...
How about QGP?

CKW & CME

CME

$$\mathbf{j} = \sigma_\chi \mathbf{B}$$

Ampere's law

$$\nabla \times \mathbf{B} = \sigma_\chi \mathbf{B}$$

- 2010, M. N. Chernodub, arXiv:1002.1473
- 2015, Y. Hirono, D. Kharzeev, Yi Yin, PhysRevD.92.125031

$$\begin{cases} \nabla \times \mathbf{B} & = \sigma \mathbf{E} + \sigma_\chi \mathbf{B}, \\ \nabla \times \mathbf{E} & = -\frac{\partial}{\partial t}\mathbf{B}. \end{cases} \Rightarrow \frac{\partial}{\partial t}\mathbf{B} = \eta \left(\nabla^2 \mathbf{B} + \sigma_\chi \nabla \times \mathbf{B}\right)$$

$\eta = \frac{1}{\sigma}$

<u>They identify the final state as the CKW state.</u>

$\sigma\mathbf{E}$ VS $\sigma_\chi\mathbf{B}$

$$\nabla \times \mathbf{B} = \sigma\mathbf{E} + \sigma_\chi\mathbf{B}$$

$\sigma\mathbf{E}$ VS $\sigma_\chi\mathbf{B}$

- $\sigma_\chi = 0$, no CKW;
- σ_χ strong enough $\Rightarrow$ CKW.

What does "*strong*" mean, quantitatively?

$\sigma\mathbf{E}$ VS $\sigma_\chi\mathbf{B}$

$$\nabla \times \mathbf{B} = \sigma\mathbf{E} + \sigma_\chi\mathbf{B}$$

$\sigma\mathbf{E}$ VS $\sigma_\chi\mathbf{B}$

- $\sigma_\chi = 0$, no CKW;
- σ_χ strong enough $\Rightarrow$ CKW.

What does "*strong*" mean, quantitatively?

Popular choices:

- const.
- $\sigma(t) \sim t^{-\alpha}$, $\sigma_\chi(t) \sim t^{-\beta}$
- $\sigma_\chi \propto H_{\mathsf{f}} = H_{\mathsf{total}} - H_{\mathsf{mag}}$

Hirono et al: $\sigma = \sigma_0$, $\sigma_\chi \propto H_{\mathsf{total}} - H_{\mathsf{mag}}$
Our work: **arbitrary function** of time $\sigma(t)$, $\sigma_\chi(t)$

Expansion of magnetic field

Expand $\mathbf{B}$ by CKW basis,

$$\mathbf{B}(\mathbf{x},t) = \sum_{l,m} \int_0^\infty \frac{dk}{\pi} k^2 \left[\alpha^+_{lm}(t,k)\mathbf{W}^+_{lm}(\mathbf{x};k) + \alpha^-_{lm}(t,k)\mathbf{W}^-_{lm}(\mathbf{x};k)\right]$$

Helicity

$$H = \int d^3x \mathbf{A}\cdot\mathbf{B} = \int_0^\infty dk\, k\left[g_+(t,k) - g_-(t,k)\right],$$

$g_\pm(k,t) = \frac{1}{\pi}\sum_{l,m}|\alpha^\pm_{lm}(k,t)|^2.$

Equation and solution

Equation:

$$\frac{\partial}{\partial t}\mathbf{B} = \eta\left(\nabla^2\mathbf{B} + \sigma_\chi \nabla\times\mathbf{B}\right)$$

$$\Downarrow$$

$$\frac{d}{dt}\alpha^{\pm}_{lm}(t,k) = \eta(-k^2 \pm \sigma_\chi k)\alpha^{\pm}_{lm}(t,k),$$

$$\frac{d}{dt}g_{\pm}(t,k) = 2\eta(-k^2 \pm \sigma_\chi k)g_{\pm}(t,k).$$

Solution:

$$\alpha^{\pm}_{lm}(t,k) = \alpha^{\pm}_{lm}(t_0,k)\,e^{-k^2\Lambda(t)\pm k\Theta(t)},$$

$$g_{\pm}(t,k) = g_{\pm}(t_0,k)\,e^{-2k^2\Lambda(t)\pm 2k\Theta(t)},$$

$\Lambda(t) = \int_{t_0}^{t}\eta(t)\,dt$, $\Theta(t) = \int_{t_0}^{t}\eta(t)\sigma_\chi(t)\,dt$.

CKW

$$\mathbf{B}(\mathbf{x},t)=\sum_{l,m}\int_0^\infty \frac{dk}{\pi}k^2\left[\alpha^+_{lm}(t,k)\mathbf{W}^+_{lm}(\mathbf{x};k)+\alpha^-_{lm}(t,k)\mathbf{W}^-_{lm}(\mathbf{x};k)\right]$$

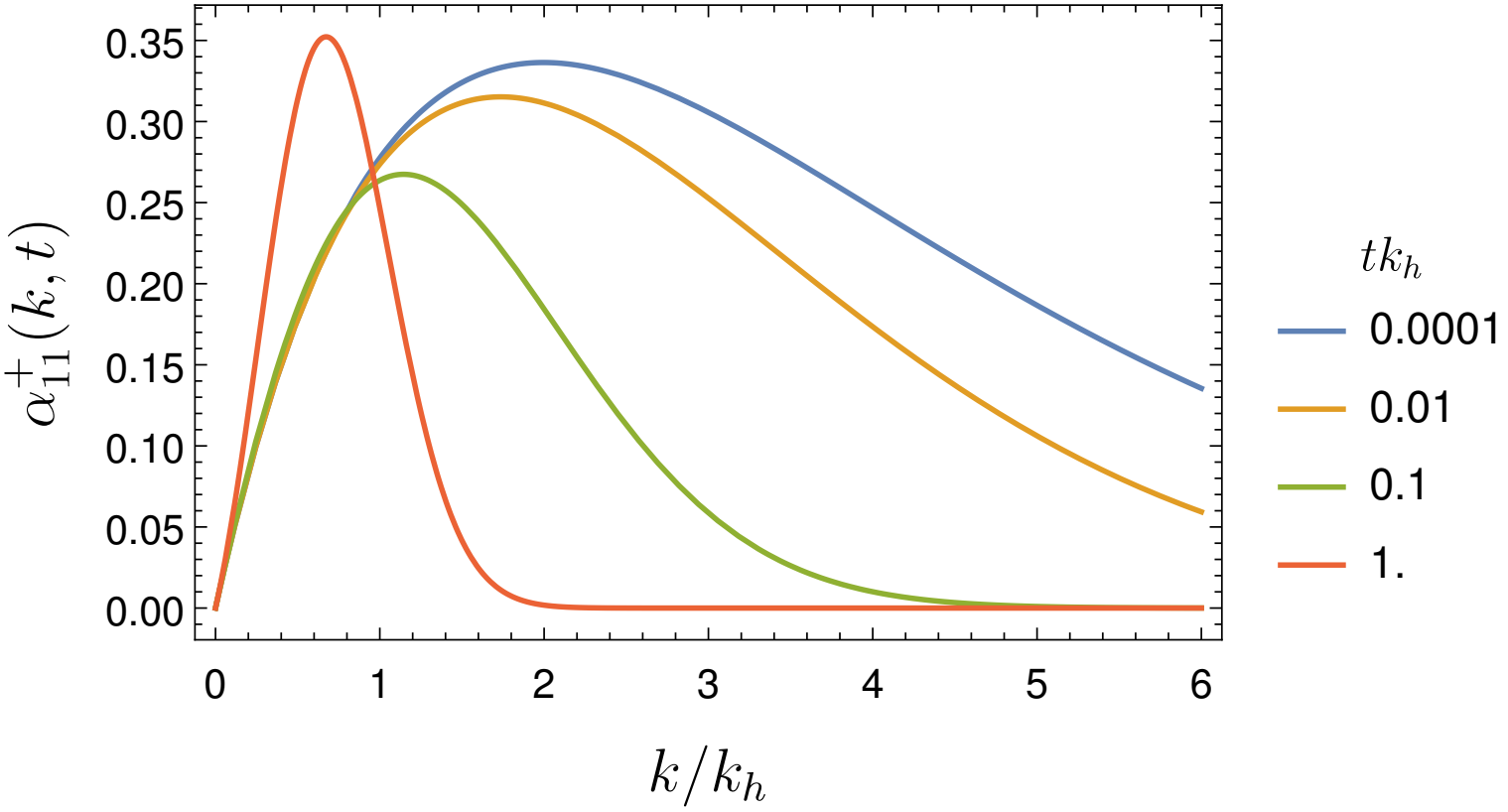
$$\nabla\times\mathbf{W}^\pm_{lm}(\mathbf{x};k)=\pm k\mathbf{W}^\pm_{lm}(\mathbf{x};k)$$

$\mathbf{B}$ is CKW state, if

$$\alpha^+_{lm}(t,k)\to\delta(k-k') \text{ and } \alpha^-_{lm}(t,k)\to 0;$$
$$\text{or } \alpha^-_{lm}(t,k)\to\delta(k-k') \text{ and } \alpha^+_{lm}(t,k)\to 0.$$

CKW

$g_+(t_0, k) \sim k^2 e^{-2kL}$, $\sigma_\chi = k_h \left(H_{\text{total}} - H_{\text{mag}}\right)$



$\alpha_{lm}^+(t, k) \to \delta(k - k')$

Measurement of CKW state

Our goal:

Given $\nabla \times \mathbf{B} = \mathbf{j}$ and $\mathbf{B}$.
Measure how nearly $\mathbf{B}$ is parallel with $\mathbf{j}$.

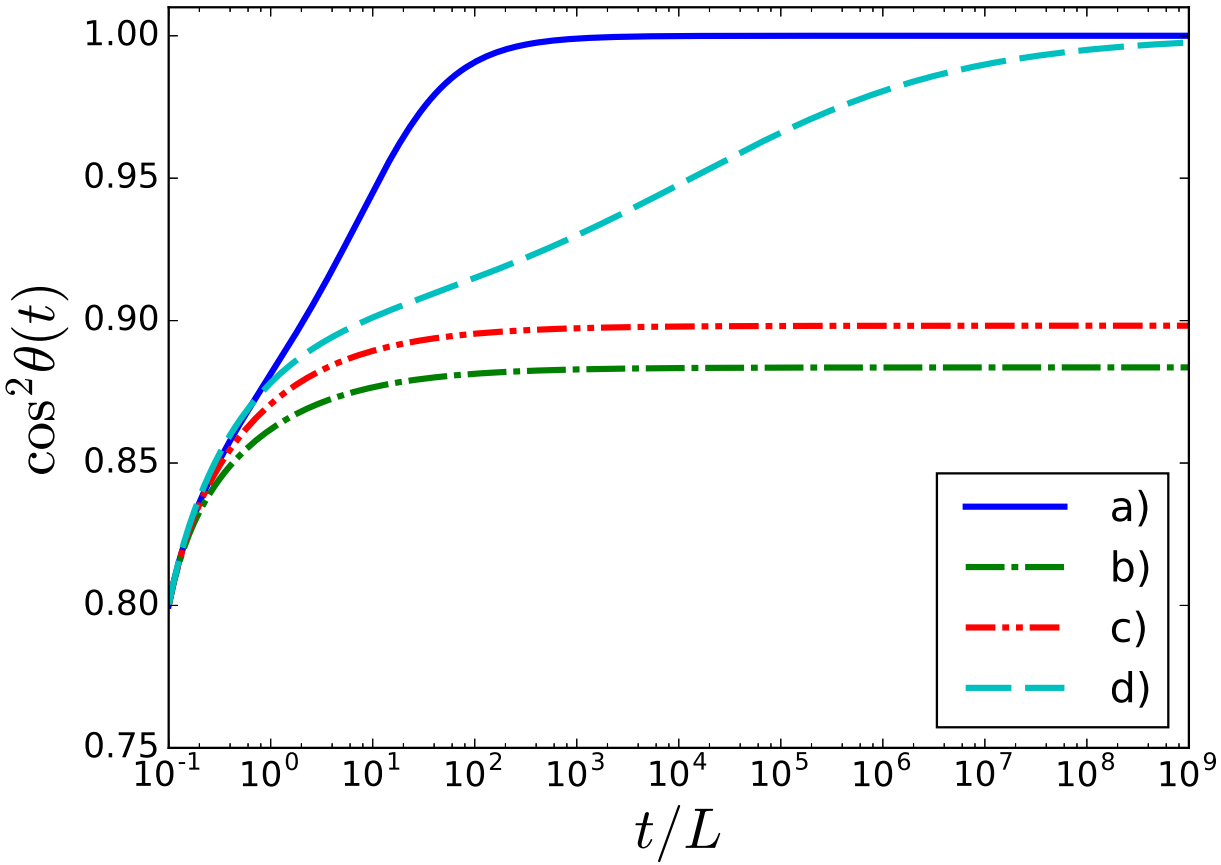
$$\cos^2\theta(t) \equiv \frac{\langle \mathbf{j}, \mathbf{B}\rangle^2}{\langle \mathbf{j}, \mathbf{j}\rangle\langle \mathbf{B}, \mathbf{B}\rangle} = \frac{\left(\int d^3x \mathbf{j}\cdot\mathbf{B}\right)^2}{\int d^3x \mathbf{j}\cdot\mathbf{j} \int d^3x \mathbf{B}\cdot\mathbf{B}}$$

$\cos^2\theta(t) \leq 1$,

$$\langle \mathbf{j}, \mathbf{B}\rangle^2 \leq \langle \mathbf{j}, \mathbf{j}\rangle \langle \mathbf{B}, \mathbf{B}\rangle$$

$\cos^2\theta(t) = 1$, iff CKW state is reached.

$\cos^2\theta(t)$



$$g_+(t_0, k) \sim k\, e^{-2Lk}$$

	σ	σ_χ
a)	const.	const.
b)	const.	0
c)	const.	$t^{-1/2}$
d)	$t^{-1/3}$	$t^{-1/2}$

Condition for CKW

We prove that:
For $g(t_0, k) = \sum_n c_n k^n e^{-2kL_1} e^{-k^2 L_2}$, condition for $\cos^2\theta(t) \to 1$ is:

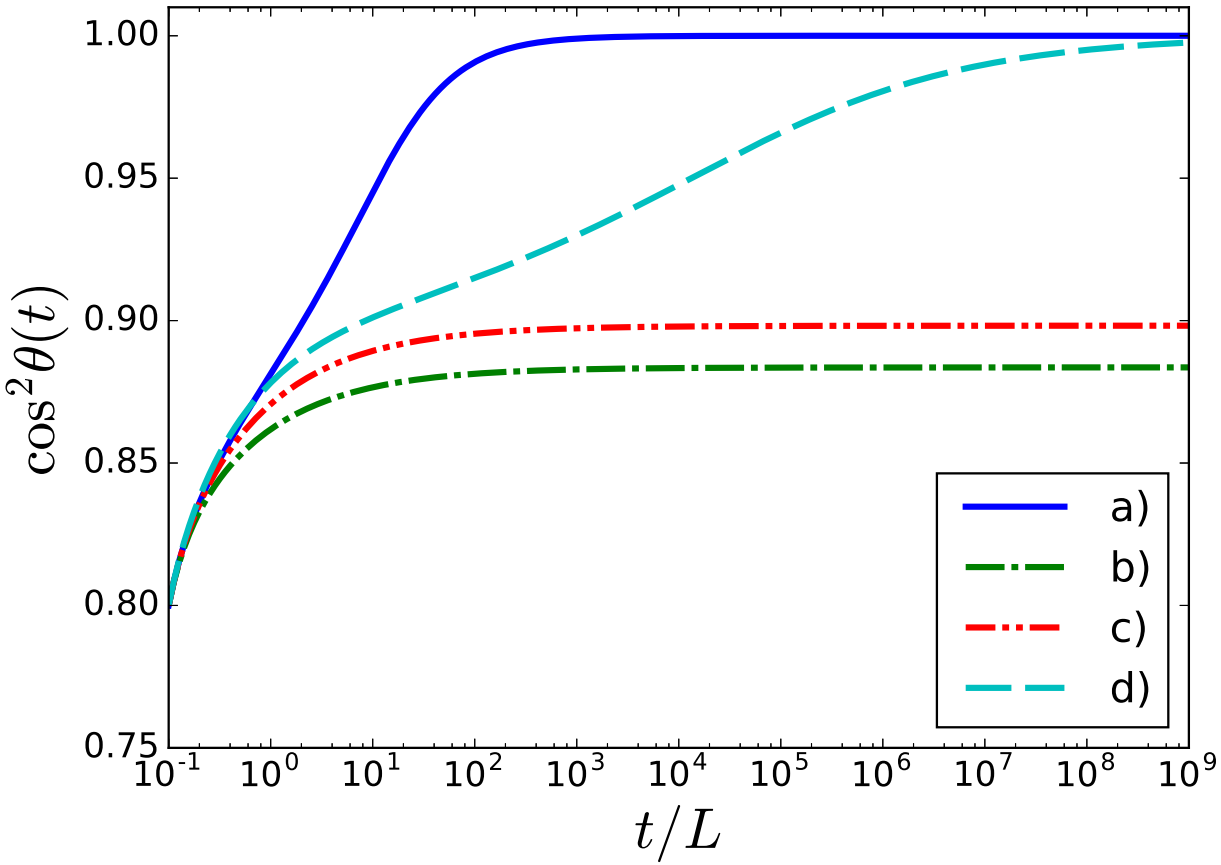
$$\lim_{t\to\infty} \frac{\int_{t_0}^{t} \eta(t)\sigma_\chi(t)\,dt}{\sqrt{\int_{t_0}^{t} \eta(t)\,dt}} = \infty.$$

$\int_{t_0}^{t} \eta(t)\sigma_\chi(t)\,dt$ grows faster in time than $\sqrt{\int_{t_0}^{t} \eta(t)\,dt}$

$\eta = \frac{1}{\sigma}$

$$\frac{\partial}{\partial t}\mathbf{B} = \eta\nabla^2\mathbf{B} + \eta\sigma_\chi\nabla\times\mathbf{B}$$

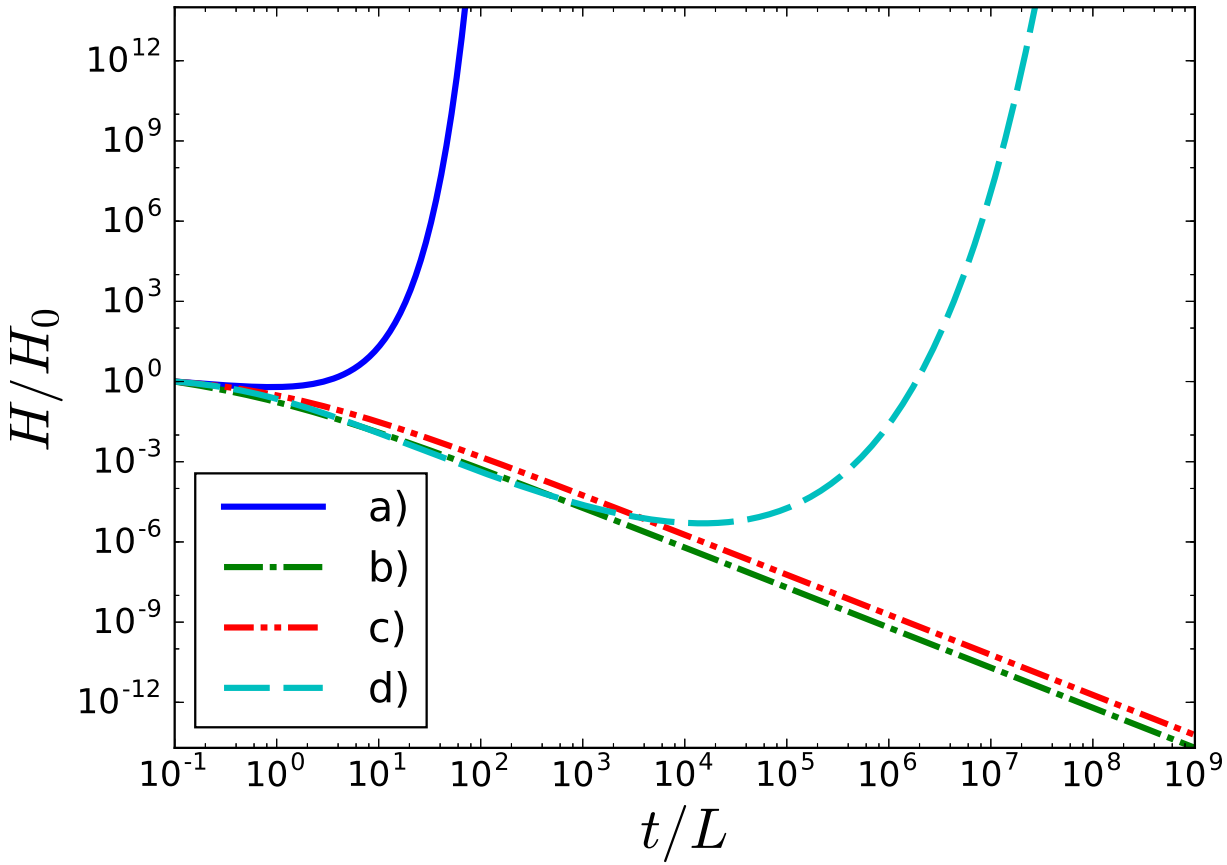
Condition for CKW



	σ	σ_χ
a)	const.	const.
b)	const.	0
c)	const.	$t^{-1/2}$
d)	$t^{-1/3}$	$t^{-1/2}$

$\sqrt{\int \eta dt}$	$\int \eta \sigma_\chi dt$
$t^{1/2}$	t
$t^{1/2}$	0
$t^{1/2}$	$t^{1/2}$
$t^{2/3}$	$t^{5/6}$

Helicity divergence



$$g_+(t_0, k) \sim k\, e^{-2Lk}$$

	σ	σ_χ
a)	const.	const.
b)	const.	0
c)	const.	$t^{-1/2}$
d)	$t^{-1/3}$	$t^{-1/2}$

Why?

$\mathbf{B}$ induces $\mathbf{B}$ itself!

How to avoid this divergence,

meanwhile realize the CKW state?

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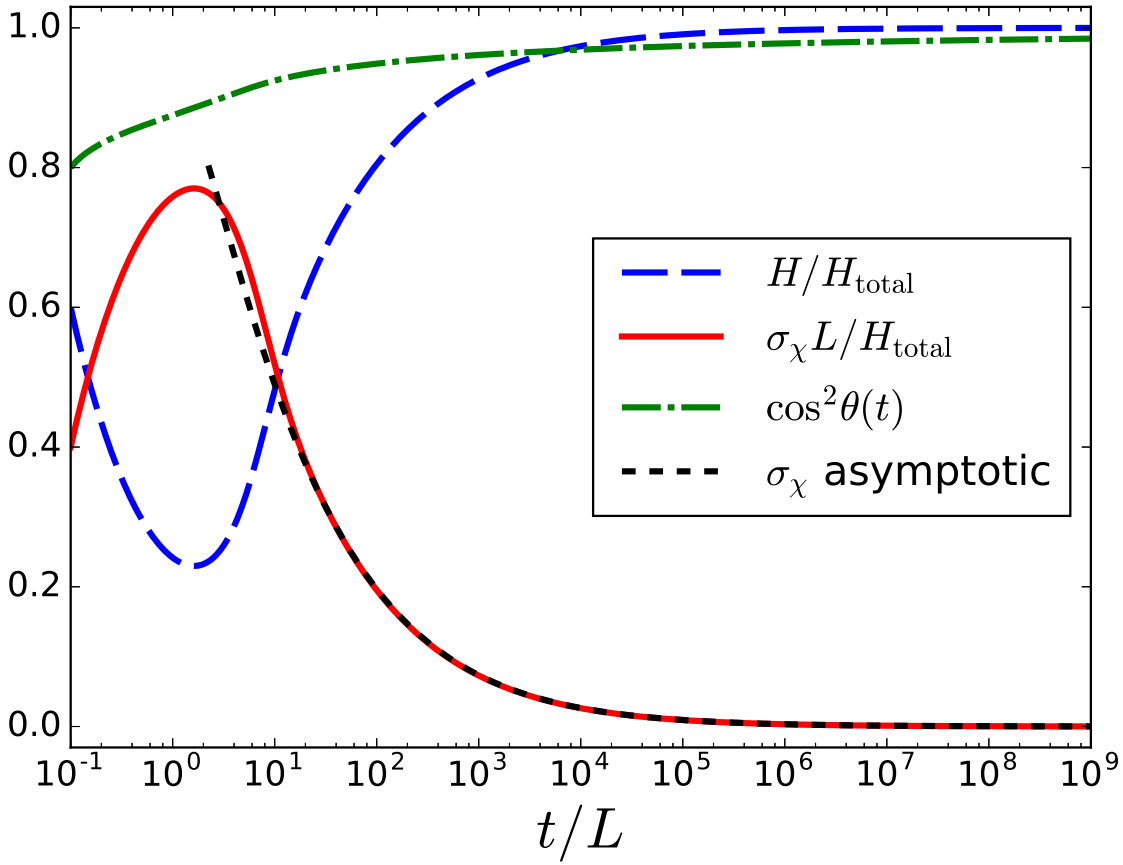

Negative feedback

$$\sigma_\chi(t) \propto H_{\rm total} - H(t)$$

$$\sigma_\chi \Rightarrow H \nearrow \Rightarrow \sigma_\chi \searrow$$

Manuel et al, PhysRevD.92.074018

Hirono et al, PhysRevD.92.125031

How to avoid this divergence,

meanwhile realize the CKW state?

For arbitrary $\sigma(t)$ and initial spectrum $g_+(t_0, k) \sim k^r e^{-2kC_1 - k^2 C_2^2}$,

$$\Theta(t) \sim \sqrt{\Lambda(t) \mathrm{plog}\left[\Lambda(t)^{(r+2)/(r+1)}\right]},$$

$\mathrm{plog}()$ is inverse function of $y = xe^x$,

$\Lambda(t) = \int_{t_0}^{t} \eta(t)\, dt$, $\Theta(t) = \int_{t_0}^{t} \eta(t)\sigma_\chi(t)\, dt$.

<u>$\Theta(t)$ grows faster than $\sqrt{\Lambda(t)} \Rightarrow$ CKW state.</u>

Summary

We study the time evolution of the magnetic field with CME.

- We define a measure $\cos^2\theta(t)$ for the CKW state.
  $\cos^2\theta(t) \to 1$ is more *intuitive* than $\alpha_{lm}^{+/-}(t,k) \to \delta(k-k')$.
- We find condition that the CKW state can be reached:
  (i) the presence of σ_χ;
  (ii) $\int_{t_0}^{t} \eta\sigma_\chi dt$ grows faster in time than $\sqrt{\int_{t_0}^{t} \eta dt}$.
  $\sigma(t)$, $\sigma_\chi(t)$ are arbitrary function of time.
- We discuss system in which the CKW state can be reached, avoiding unphysical divergence.

Thank you!

backup: $\mathbf{W}^{\pm}_{lm}$

$\mathbf{W}^{\pm}_{lm}$ are CKW states

$$\nabla \times \mathbf{W}^{\pm}_{lm}(\mathbf{x}; k) = \pm k \mathbf{W}^{\pm}_{lm}(\mathbf{x}; k)$$

Orthogonality

$$\int d^3x \mathbf{W}^{s_1 *}_{l_1 m_1}(\mathbf{x}; k) \cdot \mathbf{W}^{s_2}_{l_2 m_2}(\mathbf{x}; k') = \frac{\pi}{k^2} \delta(k - k') \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{s_1 s_2}$$

backup: CKW VS Fourier

CKW basis: scalar coefficient, vector base

$$\mathbf{B} = \sum_{l,m} \int_0^\infty \frac{dk}{\pi} k^2 \left[\alpha^+_{lm}(k,t)\mathbf{W}^+_{lm}(\mathbf{x};k) + \alpha^-_{lm}(k,t)\mathbf{W}^-_{lm}(\mathbf{x};k)\right]$$

$$H = \int_0^\infty dk\, k \left[g_+(t,k) - g_-(t,k)\right]$$

$$\frac{d}{dt} g_\pm(t,k) = 2\eta(-k^2 \pm \sigma_\chi k) g_\pm(t,k)$$

Fourier basis: vector coefficient, scalar base

$$\mathbf{B} = \sum_k \mathbf{B}_k \exp(i\mathbf{k}\cdot\mathbf{x})$$

$$H = V \sum_k \frac{i\mathbf{B}_k \times \mathbf{B}^*_k}{k^2} \cdot \mathbf{k}$$

backup: Helicity

$$\nabla \times \mathbf{W}^{\pm}_{lm}(\mathbf{x}; k) = \pm k \mathbf{W}^{\pm}_{lm}(\mathbf{x}; k)$$

$\pm$ are positive/negative magnetic helicity

$$\mathbf{A}(\mathbf{x}, t) = \sum_{l,m} \int_0^\infty \frac{dk}{\pi} k \left[\alpha^+_{lm}(t, k) \mathbf{W}^+_{lm}(\mathbf{x}; k) - \alpha^-_{lm}(t, k) \mathbf{W}^-_{lm}(\mathbf{x}; k)\right]$$

$$\mathbf{B}(\mathbf{x}, t) = \sum_{l,m} \int_0^\infty \frac{dk}{\pi} k^2 \left[\alpha^+_{lm}(t, k) \mathbf{W}^+_{lm}(\mathbf{x}; k) + \alpha^-_{lm}(t, k) \mathbf{W}^-_{lm}(\mathbf{x}; k)\right]$$

$$H = \int d^3x \mathbf{A} \cdot \mathbf{B} = \int_0^\infty dk\, k \left[g_+(t, k) - g_-(t, k)\right],$$

$g_\pm(k, t) = \frac{1}{\pi} \sum_{l,m} |\alpha^\pm_{lm}(k, t)|^2$.