Lectures on Evaluating Feynman Integrals

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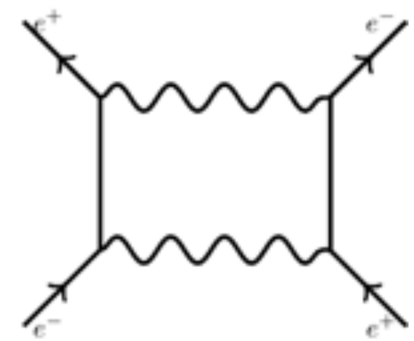
Plan

- Introduction
- Sector decomposition
- Mellin-Barnes method
- Differential equation method

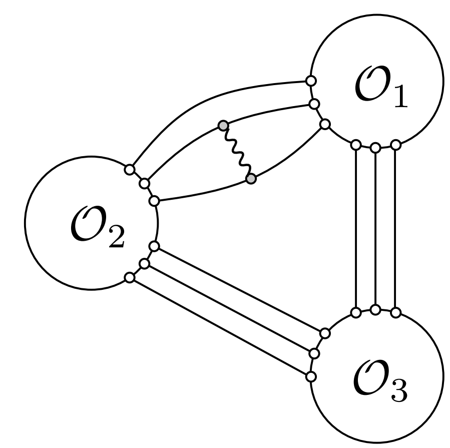
Introduction

Physical quantities in QFT

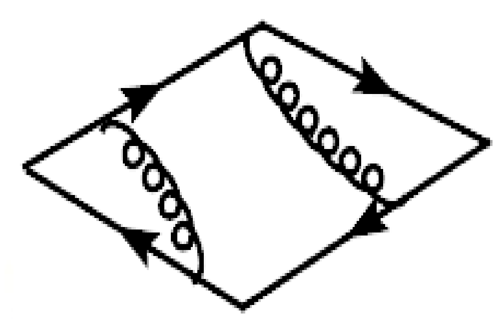
Scattering amplitudes



Correlation functions



Wilson loops



Scattering amplitudes

$$\mathcal{A} = \sum \text{Feynman diagrams}$$

$$\text{Integrand} = \sum c_i \times I_i$$

$$\downarrow$$

PV, IBP

Unitarity

Comment on Unitarity method

Obtain integrand based on physical singularities, without using Feynman diagrams

$$\text{Integrand} = \sum c_i \times I_i$$

Advantages (compared to Feynman diagram method):

- more compact expression
- better UV behaviour
- structure and symmetries made obvious
- way to find 'nice' basis

Scattering amplitudes

$$\text{Integrand} = \sum c_i \times I_i$$ = functions

↓

computing integrals

Feynman parameters

Parametric representation

Schwinger parametrization

$$\frac{1}{A^a} = \frac{1}{\Gamma(a)} \int_0^\infty dx x^{a-1}\, e^{-xA}$$

Feynman parametrization

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx_1 dx_2\, \frac{\delta(1-x_1-x_2)\, x_1^{a-1} x_2^{b-1}}{(x_1 A + x_2 B)^{a+b}}$$

One-loop case

$$G_n^{(1)} = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2+m_1^2)^{a_1}[-(k+p_1)^2+m_2^2]^{a_2}\cdots[-(k+p_1+\ldots+p_{n-1})^2+m_n^2]^{a_n}}$$

$$\Big\downarrow$$

Feynman/Schwinger parametrization
\+ Wick rotation to Euclidean space
\+ Gaussian integral over the loop momentum k

$$G_n^{(1)} = \frac{\Gamma(a-D/2)}{\prod_{i=1}^n \Gamma(a_i)} \int_0^\infty \Big[\prod_{i=1}^n dx_i x_i^{a_i-1}\Big] \delta(1-\sum_{i=1}^n c_i x_i) \frac{U^{a-D}}{(V+U\sum_{i=1}^n m_i^2 x_i)^{a-D/2}}$$

$a=\sum_{i=1}^n a_i,\, U=\sum_{i=1}^n x_i$

$V=\sum_{i<j} x_i x_j[-(p_i+p_{i+1}+..+p_{j-1})^2]$

Higher loop cases

$$G^{(L)} = \int \Big[\prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}}\Big] \frac{1}{(-q_1^2+m_1^2)^{\nu_1}[-q_2^2+m_2^2]^{\nu_2}\cdots[-q_N^2+m_N^2]^{\nu_N}}$$

$\downarrow$ similar procedure

$$\boxed{G^{(L)} = \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^{N}\Gamma(\nu_i)} \int_0^\infty \Big[\prod_{i=1}^{N} dx_i x_i^{\nu_i-1}\Big]\,\delta(1-\sum_{i=1}^{N} x_i)\,\frac{U^{\nu-(L+1)D/2}}{F^{\nu-LD/2}}}$$

$$U(\vec{x}) = \sum_{T\in T_1}\prod_{i\notin T_1} x_i\,,$$

$$V(\vec{x}) = \sum_{T\in T_2}\prod_{i\notin T_2} x_i(-s_T)\,,$$

$$F(\vec{x}) = V + U\sum_{i=1}^{n} m_i^2 x_i\,.$$

$$\nu \;=\; \textstyle\sum_{i=1}^{N}\nu_i$$

U, F are homogeneous polynomial in x, with degree L and L+1 respectively.

At one-loop, one can set U=1.
In massless case, F = V.

Divergences and regularization

Dimensional regularization: $D = 4 - 2\epsilon$

UV divergences $\epsilon_{\rm UV} > 0$

IR divergences $\epsilon_{\rm IR} < 0$

$$G^{(L)} = \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int_0^\infty \left[\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1}\right] \delta(1 - \sum_{i=1}^{N} x_i) \frac{U^{\nu-(L+1)D/2}}{F^{\nu - LD/2}}$$

Divergences and regularization

Dimensional regularization: $D = 4 - 2\epsilon$

UV divergences $\epsilon_{\text{UV}} > 0$

IR divergences $\epsilon_{\text{IR}} < 0$

overall UV div.

$$G^{(L)} = \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int_0^\infty \left[\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right] \delta(1 - \sum_{i=1}^{N} x_i) \frac{U^{\nu-(L+1)D/2}}{F^{\nu - LD/2}}$$

Divergences and regularization

Dimensional regularization: $D = 4 - 2\epsilon$

UV divergences $\epsilon_{\mathrm{UV}} > 0$

IR divergences $\epsilon_{\mathrm{IR}} < 0$

UV sub-div.

$$G^{(L)} = \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int_0^\infty \left[\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right] \delta(1 - \sum_{i=1}^{N} x_i) \frac{U^{\nu-(L+1)D/2}}{F^{\nu - LD/2}}$$

Divergences and regularization

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IR div.

Divergences and regularization

Comment 1: degree of divergences

Both logarithm and quadratic UV divergences ~ $\dfrac{1}{\epsilon}$

$$\Gamma(2-\tfrac{D}{2}) \qquad \Gamma(1-\tfrac{D}{2})$$

Do we lose the information about degree of divergences?

Divergences and regularization

Comment 1: degree of divergences

Both logarithm and quadratic UV divergences ~ $\frac{1}{\epsilon}$

$$\Gamma(2-\tfrac{D}{2}) \qquad \Gamma(1-\tfrac{D}{2})$$

Do we lose the information about degree of divergences?

No. Dimensional regularization translates the degree of divergence into the analytic properties of regulated amplitudes in D dimensions.

Divergences and regularization

Comment 2: integral containing both UV and IR divergences

Typical examples are scaleless integrals

$$\int_0^\infty \frac{dx}{x^{1+\epsilon}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} + \int_1^\infty \frac{dx}{x^{1+\epsilon}} = -\frac{1}{\epsilon} + \frac{1}{\epsilon} = 0$$

Divergences and regularization

Comment 2: integral containing both UV and IR divergences

Typical examples are scaleless integrals

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$$= -\frac{1}{\epsilon_{\mathrm{IR}}}\bigg|_{\epsilon_{\mathrm{IR}}<0} + \frac{1}{\epsilon_{\mathrm{UV}}}\bigg|_{\epsilon_{\mathrm{UV}}>0} \stackrel{?}{=} 0$$

Divergences and regularization

Comment 2: integral containing both UV and IR divergences

Typical examples are scaleless integrals

$$\int_0^\infty \frac{dx}{x^{1+\epsilon}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} + \int_1^\infty \frac{dx}{x^{1+\epsilon}} = -\frac{1}{\epsilon} + \frac{1}{\epsilon} = 0$$

$$= -\frac{1}{\epsilon_{\rm IR}}\bigg|_{\epsilon_{\rm IR}<0} + \frac{1}{\epsilon_{\rm UV}}\bigg|_{\epsilon_{\rm UV}>0} \quad = 0$$

In practice, we set: $\epsilon_{\rm IR} = \epsilon_{\rm UV} = \epsilon$

We can do this, since we know that IR and UV divergences must vanish separately in physical observables.

Counting master integrals

$$\text{Integrand} = \sum c_i \times I_i$$

$$\downarrow$$

IBP

Can we know the number of master integrals without doing IBP?

Counting master integrals

[Lee, Pomeransky 2013]
see also [Baikov 2005]

# of basis integrals $\sim$ **# of proper critical points**

$$\boxed{\frac{\partial G}{\partial x_i} = 0 \quad (i = 1, \ldots, m) \qquad \text{and} \qquad G \neq 0.}$$

$$G(\vec{x}) = U(\vec{x}) + F(\vec{x})$$

This can be counted using algebraic techniques:

$$I = \left\langle \frac{\partial G}{\partial \alpha_1}, \ldots, \frac{\partial G}{\partial \alpha_m}, \alpha_0 G - 1 \right\rangle$$ → Gröbner basis → number of irreducible monomials

For further technical point, see: [Lee, Pomeransky 2013] [Boels, Kniehl, Yang 2015]

Sector decomposition method

General picture

$$\int_0^1 dx \int_0^1 dy (x+y)^{-2+\epsilon}$$

Goal: to separate the divergences

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$$\int_0^1 dx \int_0^1 dy (x+y)^{-2+\epsilon}$$

Goal: to separate the divergences

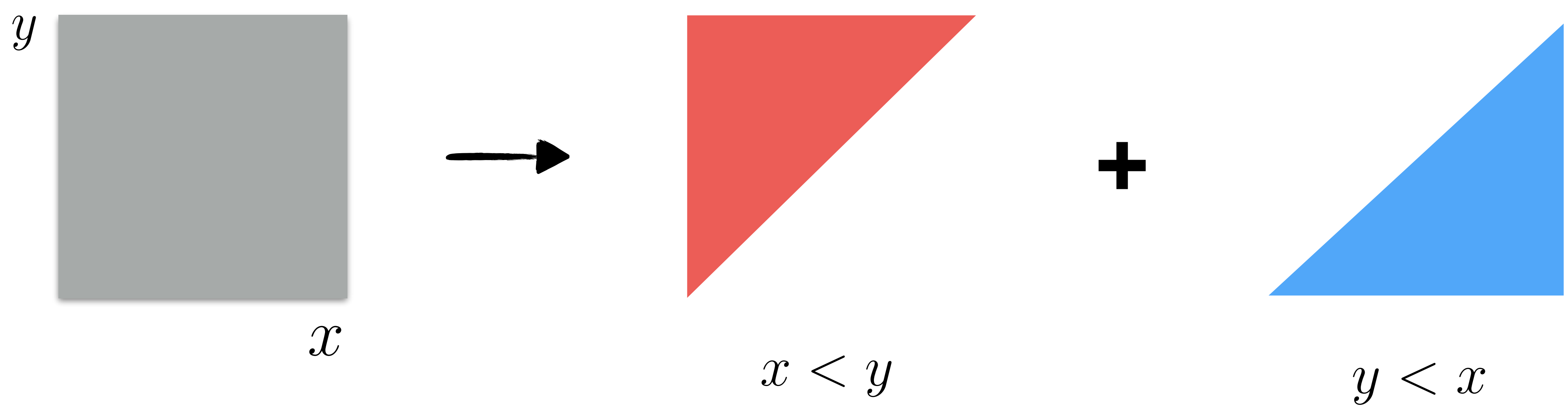
divergent part x finite part

can be trivially integrated out

can be expanded in ϵ at integrand level

Sector decomposition

Basic idea:



Sector decomposition

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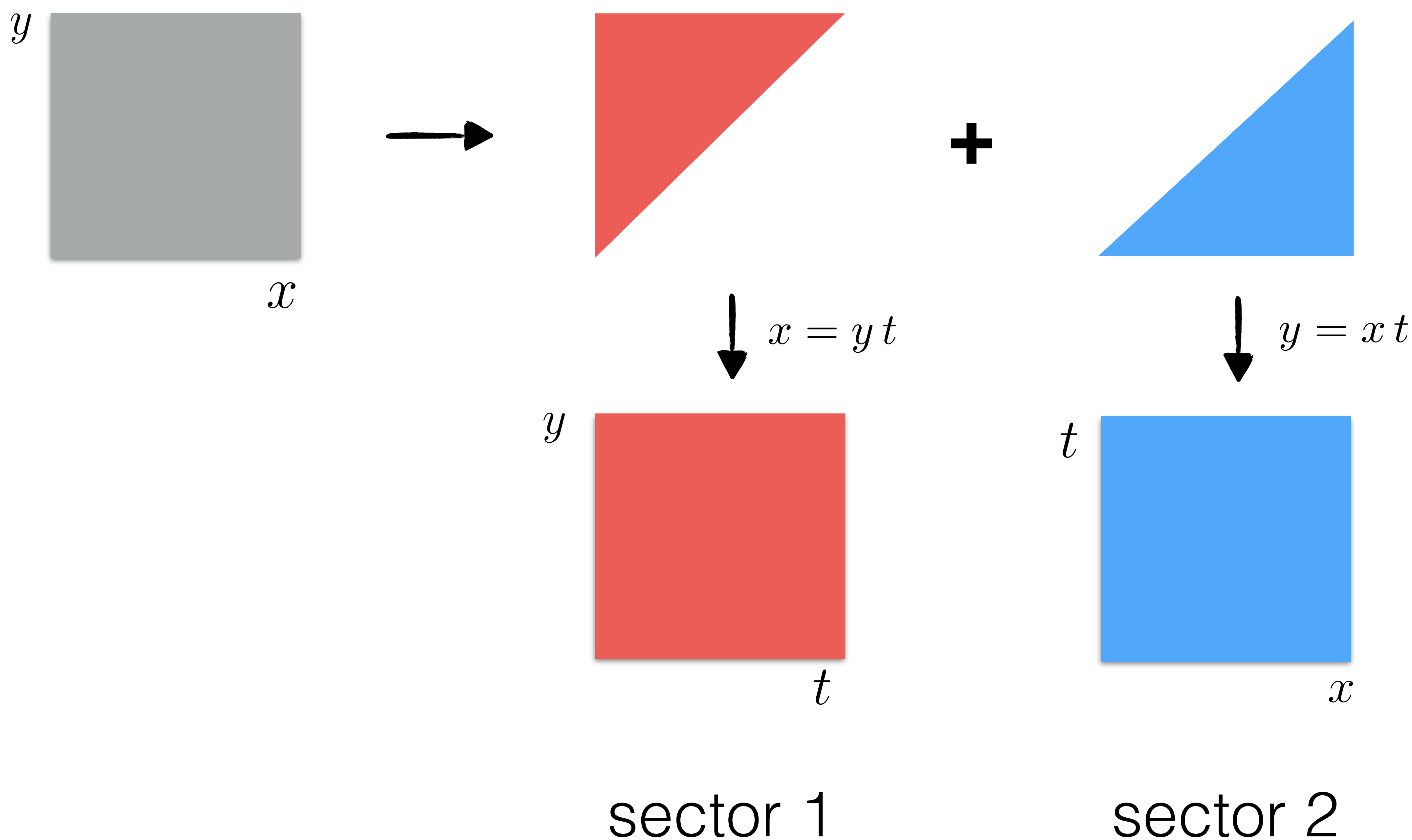

Sector decomposition

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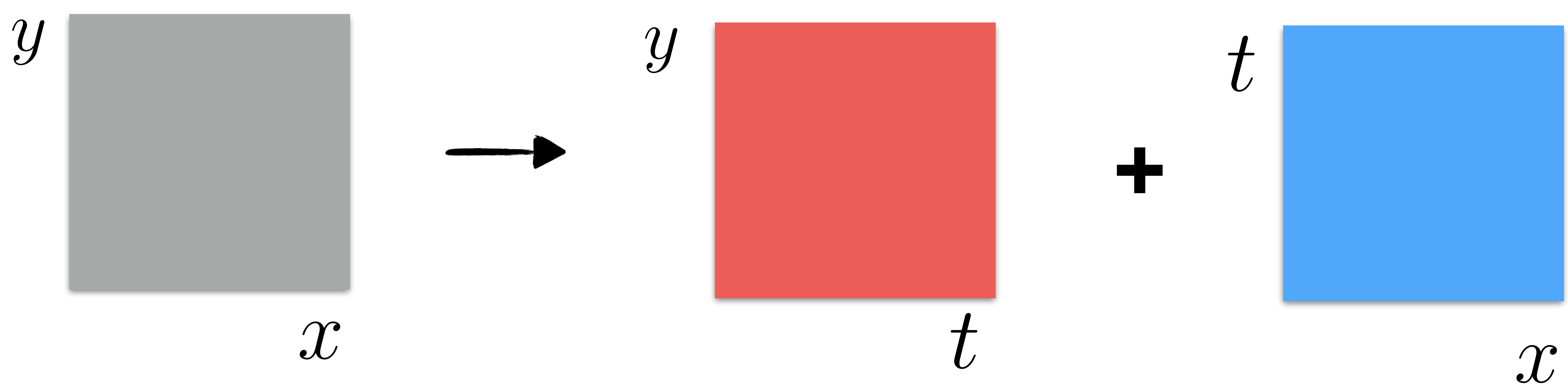

$$\int_0^1 dx \int_0^1 dy (x+y)^{-2+\epsilon} = \int_0^1 dt (1+t)^{-2+\epsilon} \int_0^1 dy y^{-1+\epsilon} + \int_0^1 dx x^{-1+\epsilon} \int_0^1 dt (1+t)^{-2+\epsilon}$$

History

- 1966 K. Hepp (BP**H**Z)

"Proof of the Bogoliubov-Parasiuk Theorem on Renormalization"

- 2000 T. Binoth, G. Heinrich

"An automatized algorithm to compute infrared divergent multi-loop integrals"

- 2007 C. Bogner, S. Weinzierl -> sector_decomposition

"Resolution of singularities for multi-loop integrals"

- 2008 A. Smirnov, M.N. Tentyukov, et.al -> FIESTA

"Feynman Integral Evaluation by a Sector decomposiTion Approach (FIESTA)"

- 2010 J. Carter, G. Heinrich, et.al -> SecDec

"SecDec: A general program for sector decomposition"

Sector decomposition

Basic procedure: [T. Binoth, G. Heinrich 2000]

- Primary sectors
- Subsectors → choice of strategies: S, X, KU,…
- Epsilon expansion

↓

Final form:

$$G = \Gamma(\nu - LD/2) \sum_{i=1}^{N} \frac{1}{\Gamma(\nu_i)} \sum_{k=1}^{\alpha(i)} G_{ik} \qquad G_{ik} = \sum_{m=-r}^{2L} \frac{C_{ik,m}}{\epsilon^m} + \mathcal{O}(\epsilon^{r+1})$$

finite integrals

Reference: G. Heinrich 0803.4177

See talks of Jianxiong Wang and Renyou Zhang for further details

Mathematical facts

Can arbitrary kind of number / function appear in analytic expressions for Feynman integrals?

Mathematical facts

Theorem [Bogner, Weinzierl 2009]:

In the case where all scalar product $p_i \cdot p_j$ are negative or zero, all internal masses positive, and all ratios of invariants algebraic, the coefficients of the Laurent expansion of a Feynman integral are periods.

A number is a period if it can be written as integrals of an algebraic function with algebraic coefficients over a domain defined by polynomial inequalities with algebraic coefficients.

periods: $\text{algebraic number}, \; \pi, \; \log z, \; \mathrm{Li}_2(z), \; \ldots$

not periods: $e, \; \gamma_E, \; 1/\pi, \; \log \pi, \; \ldots$

$$\mathrm{Li}_2(z) = \int_{0 \le t_2 \le t_1 \le z} \frac{dt_1 \, dt_2}{t_1 \, (1 - t_2)}$$

Mathematical facts

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In the case where all scalar product $p_i \cdot p_j$ are negative or zero, all internal masses positive, and all ratios of invariants algebraic, the coefficients of the Laurent expansion of a Feynman integral are periods.

$$G^{(L)} = \int \Big[\prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}}\Big] \frac{1}{(-q_1^2+m_1^2)^{\nu_1}[-q_2^2+m_2^2]^{\nu_2}\cdots[-q_N^2+m_N^2]^{\nu_N}}$$

$$= \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^{N}\Gamma(\nu_i)} \int_0^\infty \Big[\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1}\Big] \delta(1 - \sum_{i=1}^{N} x_i) \frac{U^{\nu-(L+1)D/2}}{F^{\nu - LD/2}}$$

$$\Gamma(1+L\epsilon) = \exp\Big(-L\gamma_E\epsilon + \sum_{k=2}^{\infty}\frac{(-L)^k}{k}\epsilon^k\zeta_k\Big)$$

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$$= \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^{N}\Gamma(\nu_i)} \int_0^\infty \Big[\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1}\Big] \delta(1 - \sum_{i=1}^{N} x_i) \frac{U^{\nu-(L+1)D/2}}{F^{\nu - LD/2}} \quad \times \quad e^{L\epsilon\gamma_E}$$

$$\Gamma(1+L\epsilon) = \exp\Big(- L\gamma_E\epsilon + \sum_{k=2}^{\infty} \frac{(-L)^k}{k}\epsilon^k\zeta_k\Big)$$

or $\quad g_{\mathrm{YM}}^2 \to g_{\mathrm{YM}}^2 e^{\epsilon\gamma_E}$

Mellin-Barnes method

Mellin-Barnes integral

Robert Hjalmar Mellin
(1854 – 1933)

Ernest William Barnes
(1874 – 1953)

History

- 1974 M. Bergère, Y-M. Lam

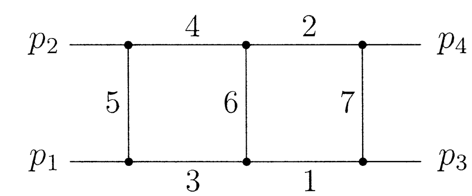
"Asymptotic expansion of Feynman amplitudes"

- 1975 N. Usyukina

"On a representation for the three-point function"

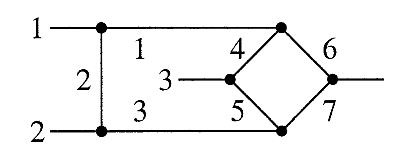
- 1991 E. Boos, A. Davydychev

"A method of evaluating massive Feynman integrals"



- 1999 V. Smirnov

"Analytical result for dimensionally regularized massless on-shell double box"



- 1999 B. Tausk

"Non-planar massless two-loop Feynman diagrams with four on-shell legs"

- 2005 M. Czakon -> MB.m

"Automatized analytic continuation of Mellin-Barnes integrals"

- 2007 J. Gluza, K. Kajda T. Riemann -> AMBRE.m

"AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals"

Mellin-Barnes integral

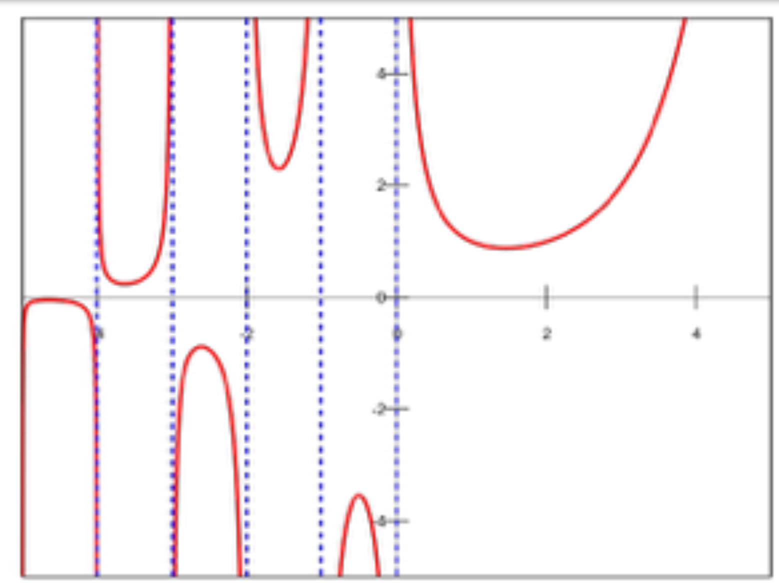
Basic equation:

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$

What is the contour?

Gamma function has simple poles at value of non-positive integer numbers.

analytic structure of Gamma function



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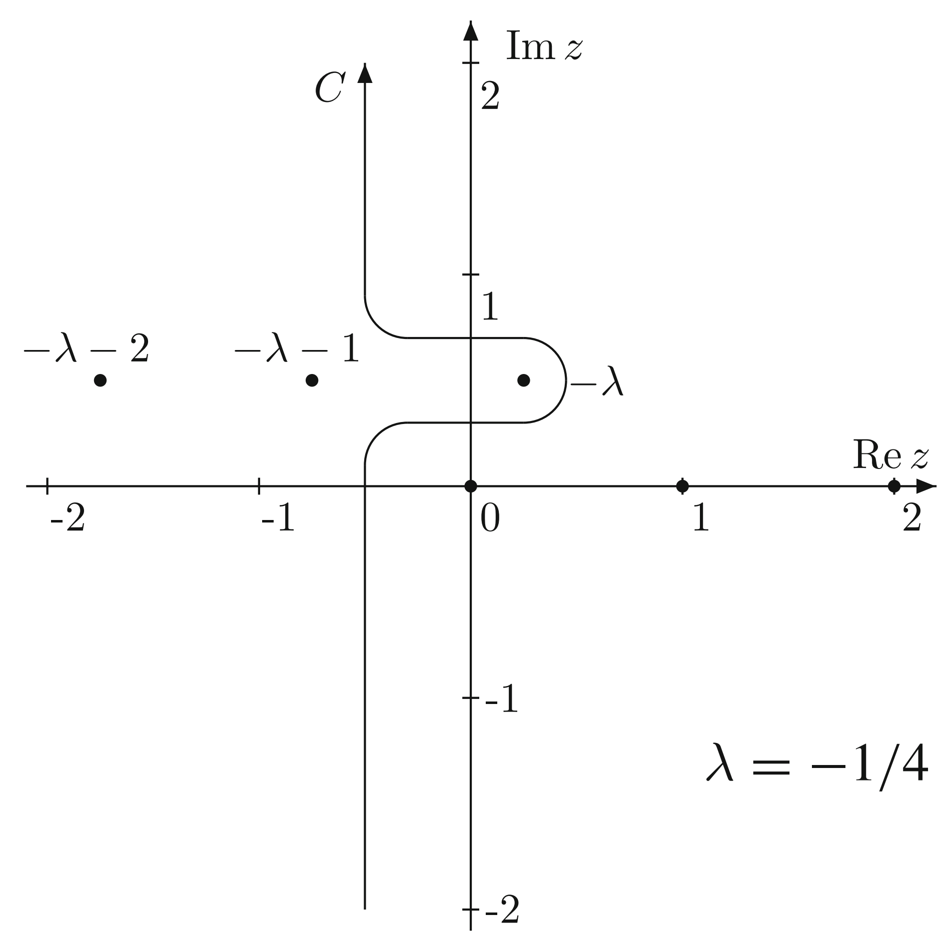
Contour is chosen such that

poles of $\Gamma(\cdots + z)$ are to the left of it

poles of $\Gamma(\cdots - z)$ are to the right of it



$\lambda = -1/4 - \mathrm{i}/2$

A simple proof

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$

Taylor expansion of LHS:

$$\frac{1}{(A+B)^\lambda} = \frac{1}{A^\lambda}\frac{1}{(1+\tilde{B})^\lambda} = \frac{1}{A^\lambda}\sum_{n=0}^{\infty}(-1)^n\frac{\lambda(\lambda+1)\dots(\lambda+n-1)}{n!}\tilde{B}^n\,, \qquad \tilde{B} = \frac{B}{A}$$

Compute RHS by residue theorem:

$$\frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\tilde{B}^z = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}2\pi i\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+n)}{(-1)^n n!}\tilde{B}^n$$

note also: $\Gamma(\lambda+n) = \lambda(\lambda+1)\dots(\lambda+n-1)\Gamma(\lambda)$

Generalization

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$

$$\downarrow$$

$$\frac{1}{(A_1+A_2+\cdots A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{+i\infty} dz_1 \cdots \int_{-i\infty}^{+i\infty} dz_{n-1} \prod_{i=1}^{n-1} A_i^{z_i}$$

$$\times A_n^{-\lambda-\sum_{i=1}^{n-1} z_i} \Gamma(\lambda + \sum_{i=1}^{n-1} z_i) \prod_{i=1}^{n-1} \Gamma(-z_i) \, .$$

Massive propagator

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$

Apply to massive propagator:

$$\frac{1}{(k^2-m^2)^\beta} = \frac{1}{(k^2)^\beta}\ \frac{1}{\Gamma(\beta)}\ \frac{1}{2\pi \mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\ \left(-\frac{m^2}{k^2}\right)^s\ \Gamma(-s)\ \Gamma(\beta+s)$$

MB representation

Parametric form:

$$G_n^{(1)} = \frac{\Gamma(\nu)}{\prod_{i=1}^{n}\Gamma(\nu_i)} \int_0^1 \left[\prod_{i=1}^{n} dx_i x_i^{\nu_i-1}\right] \delta(1-\sum_{i=1}^{n} x_i) \frac{1}{(V+U\sum_{i=1}^{n} m_i^2 x_i)^{\nu-D/2}}$$

$$\downarrow$$

Parameters x_i can be integrated out trivially after MB transformation:

$$\int_0^1 \left[\prod_{i=1}^{n} dx_i x_i^{\nu_i-1}\right] \delta(1-\sum_{i=1}^{n} x_i) = \frac{\Gamma(\nu_1)\ldots\Gamma(\nu_n)}{\Gamma(\nu_1+\ldots+\nu_n)}$$

General MB form:

$$\downarrow$$

$$\frac{1}{(2\pi i)^n}\int_{-i\infty}^{+i\infty}\prod_{l=1}^{n} dz_l \prod_{k} y_k^{d_k} \times \frac{\sum_i \Gamma(a_i + b_i\epsilon + \sum_j c_{ij} z_j)}{\sum_{i'} \Gamma(a'_{i'} + b'_{i'}\epsilon + \sum_{j'} c'_{i'j'} z_{j'})}$$

AMBRE.m [J. Gluza, K. Kajda T. Riemann]

Resolve singularities

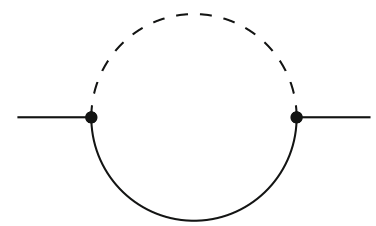
As in sector decomposition method, we want to separate the divergences:

divergent part x finite part

↓ obtained by resolving the singularities

↓ can be expanded in ϵ at integrand level

Resolve singularities

Example:  $$\int \frac{\mathrm{d}^d k}{(k^2 - m^2)(q-k)^2}$$

$$\Gamma(\varepsilon + z)\Gamma(-z)\Gamma(1-\varepsilon-z) \xrightarrow{\varepsilon = 0} \Gamma(z)\Gamma(-z)\Gamma(1-z)$$

There is no contour can be chosen such that all argument of Gamma functions are positive in the limit of $\varepsilon = 0$.

Evidence of divergence.

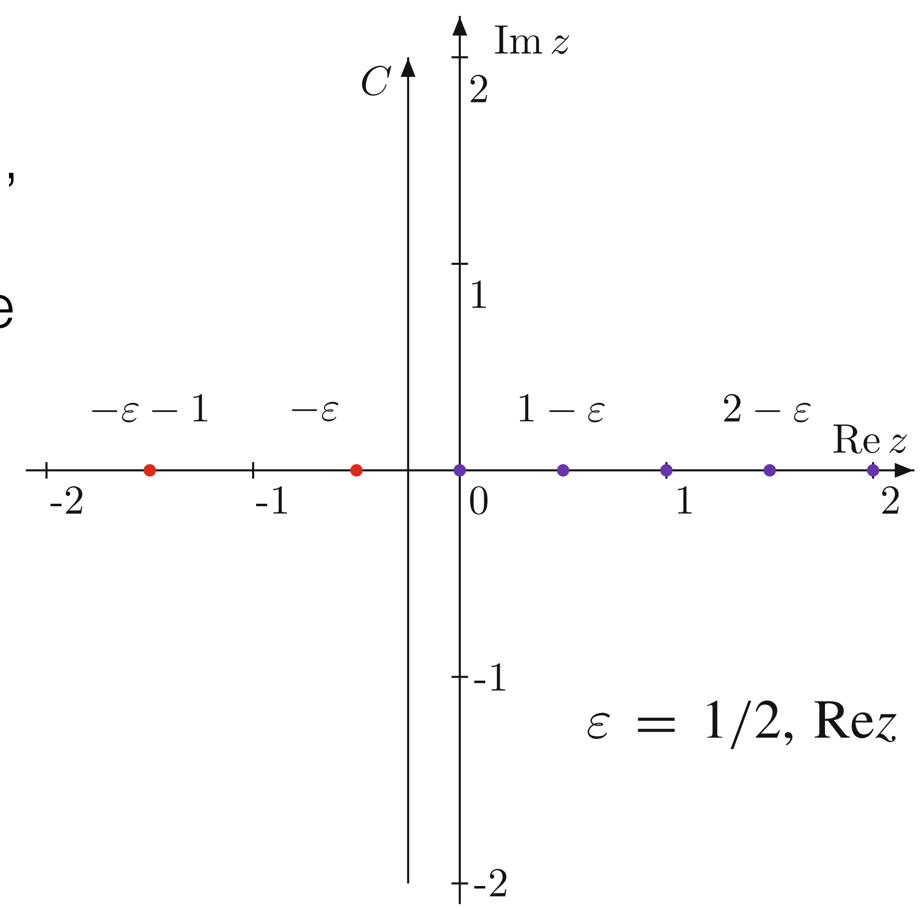
Resolve singularities

Strategy A: MBresolve.m [A. Smirnov, V. Smirnov]

Deform the integration contours, and then shift them past the poles of the Gamma functions, which results in residue integrals.

Strategy B: MB.m [M. Czakon] $\Gamma(\varepsilon+z)\Gamma(-z)\Gamma(1-\varepsilon-z)$

Choose an initial value of ε and values of the real parts of the integration variables z's, such that the real parts of all the arguments of the gamma functions in the numerator are positive.
Then one tends ε to zero and whenever the real part of the argument of some gamma function vanishes one crosses this pole and adds a corresponding residue which has one integration less.



$\varepsilon = 1/2$, $\mathrm{Re} z = -1/4$

Practical strategy

- Obtain MB representation

  AMBRE.m [J. Gluza, K. Kajda T. Riemann]

- Resolve eps singularities

  MB.m [M. Czakon] MBresolve.m [A. Smirnov, V. Smirnov]

- Perform epsilon expansion

  MB.m [M. Czakon]

- Evaluate the finite integrals numerically

  MB.m [M. Czakon]

Reference: V. Smirnov's books

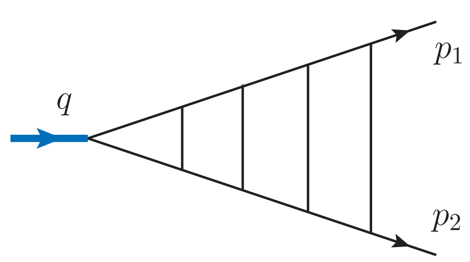
Various codes are collected at webpage:

http://mbtools.hepforge.org/

Mellin-Barnes method

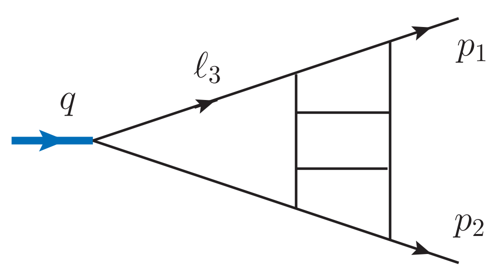
Advantages:

- sometimes possible to get analytic results
- in many cases are much faster and with better precision than sector decomposition method



MB: ~ 200 s

FIESTA: > 10 h



MB: < 10 h

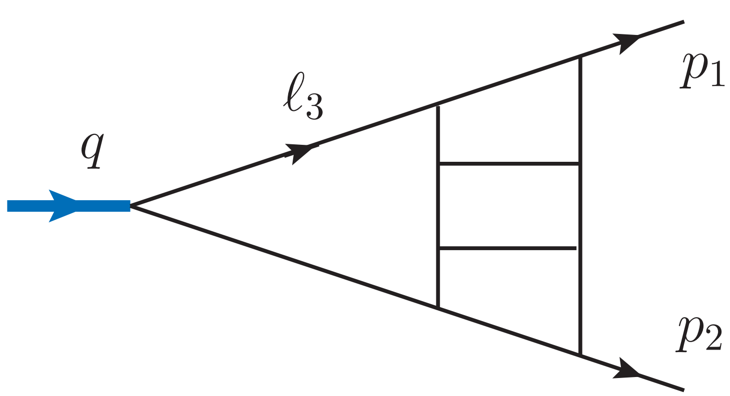
11-dim MB rep.

FIESTA: ?

Disadvantages:

- so far not work for general non-planar integral, at least not in a systematic way

Mellin-Barnes method



MB: < 10 h

11-dim MB rep.

FIESTA: ?

Differential equation method can solve it analytically !

Differential equation method

History

- 1991 A. Kotikov

"Differential equations method: New technique for massive Feynman diagrams calculation"

- 1997 E. Remiddi

"Differential equations for Feynman graph amplitudes"

- 1999 T. Gehrmann, E. Remiddi

"Differential Equations for Two-Loop Four-Point Functions"

- 2013 J. Henn

"Multiloop integrals in dimensional regularization made simple"

Differential equation

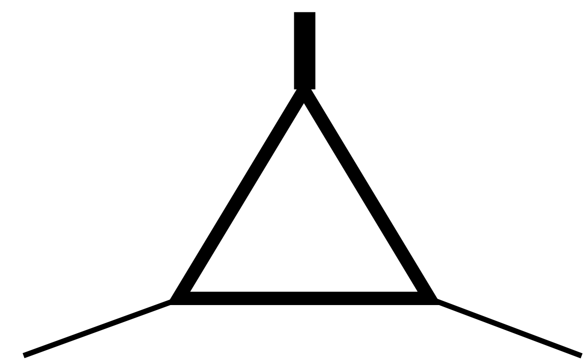
Differentiation + IBP guarantee us a system of first order differential equations for master integrals:

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\,\vec{f}(x,\epsilon)$$

$\vec{f}(x,\epsilon)$ are the set of master integrals, and x's are the Mandelstam variables or masses.

$A(x,\epsilon)$ is an $N \times N$ matrix, and is rational in x and ϵ.

Example



Example: $G(a_1, a_2, a_3)$

$$:= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(-k^2+m^2)^{a_1}(-(k+p_1)^2+m^2))^{a_2}(-(k+p_2)^2+m^2)^{a_3}}$$

Master integrals: $\mathbf{g} := \{G(0,1,0),\, G(0,1,1),\, G(1,1,1)\}$

$$\partial_s \mathbf{g}(s, m^2; \epsilon) = \mathcal{A}(s, m^2; \epsilon)\, \mathbf{g}(s, m^2; \epsilon)$$

$$\mathcal{A}(s, m^2; \epsilon) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{2(\epsilon-1)}{s(4m^2-s)} & -\frac{2m^2-s\epsilon}{s(4m^2-s)} & 0 \\ \frac{\epsilon-1}{sm^2(4m^2-s)} & \frac{2\epsilon-1}{s(4m^2-s)} & -\frac{1}{s} \end{pmatrix}$$

Differential equation

Key new idea: [Henn, 2013] Reference: Henn 1412.2296

choose an optimal basis of integrals that would lead to a system of differential equations in a canonical form.

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\,\vec{f}(x,\epsilon)$$

$$\downarrow$$

$$\boxed{\partial_x \vec{f}(x,\epsilon) = \epsilon\,A(x)\,\vec{f}(x,\epsilon)}$$

canonical form

Differential equation

Key new idea: [Henn, 2013]

canonical form

$$\boxed{\partial_x \vec{f}(x,\epsilon) = \epsilon\, A(x)\, \vec{f}(x,\epsilon)}$$

Once the canonical form is obtained, it is almost trivial to solve the basis integrals iteratively:

$$\vec{f}(x,\epsilon) = \sum_{k\geq 0} \epsilon^k\, \vec{f}^{(k)}(x)$$

or

$$\vec{f}(x,\epsilon) = \mathbb{P}\exp\left[\epsilon \int_\gamma A(x)\right] \vec{f}_0(\epsilon)$$

Example (continued)

Choose a different basis:

$$
\begin{aligned}
\vec{f} &= \{f_1,\, f_2,\, f_3\}\,, \\
f_1 &= \frac{\epsilon}{\Gamma(1+\epsilon)} m^{2\epsilon}\, G(0,2,0)\,, \\
f_2 &= \frac{\epsilon}{\Gamma(1+\epsilon)} m^{2\epsilon} s \sqrt{1-\frac{4m^2}{s}}\, G(0,1,2)\,, \\
f_3 &= \frac{\epsilon^2}{\Gamma(1+\epsilon)} m^{2\epsilon} s\, G(1,1,1)\,,
\end{aligned}
$$

$$\longrightarrow \qquad \partial_x \vec{f}(x,\epsilon) = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) \vec{f}(x,\epsilon)$$

$$-\frac{m^2}{s} = \frac{x}{(1-x)^2} \qquad a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example (continued)

Boundary condition: $\vec{f}(x=1,\epsilon) = \{1,\,0,\,0\}$

↓

Solution: $\vec{f}(x,\epsilon) = \sum_{k\geq 0} \epsilon^k\,\vec{f}^{(k)}(x)$

$$
\begin{aligned}
\vec{f}^{(0)} &= \{1,\,0\,0\}\,,\\
\vec{f}^{(1)} &= \{0,\,\log x,\,0\}\,,\\
\vec{f}^{(2)} &= \{0,\,\frac{\log^2 x - 4\log x\log(1+x) - 4\mathrm{Li}_2(-x)}{2} - \frac{\pi^2}{6},\,-\frac{\log^2 x}{2}\}\,,\\
\vec{f}^{(3)} &= \{0,\,\Big(\frac{1}{6}\log x - \log(1+x)\Big)\big(\log^2 x - \pi^2\big) + 2\log^2(1+x)\big[\log x - \log(-x)\big]\\
&\qquad -2\mathrm{Li}_3(-x) - 4\mathrm{Li}_3(1+x) + 2\zeta_3,\\
&\qquad -\frac{1}{6}\log x\big[\log^2 x + 12\mathrm{Li}_2(-x) - \pi^2\big] + 4\mathrm{Li}_3(-x) + 3\zeta_3\}\,.
\end{aligned}
$$

Differential operator

For example for massless box:

$$p_i^2 = 0\,, \;\; \sum_{i=1}^{4} p_i = 0$$

$$s = (p_1 + p_2)^2\,, \quad t = (p_2 + p_3)^2$$

We want to operate at integrand level:

$$\frac{\partial}{\partial s}\mathcal{I}(p_1, p_2, p_3, k, m) =?$$

Differential operator

For example for massless box:

$$p_i^2 = 0\,, \ \sum_{i=1}^{4} p_i = 0$$

Ansatz:

$$D_s = \frac{\partial}{\partial s} = (\alpha p_1^\mu + \beta p_2^\mu + \gamma p_3^\mu)\frac{\partial}{\partial p_1^\mu}$$

$$D_s p_1^2 = 0\,,$$
$$D_s(p_1+p_2+p_3)^2 = 0\,,$$
$$D_s(p_1+p_2)^2 = 1\,.$$

$$\longrightarrow \quad D_s = \frac{1}{2}\left[\frac{2s+t}{s(s+t)}p_1^\mu + \frac{1}{s}p_2^\mu + \frac{1}{s+t}p_3^\mu\right]\frac{\partial}{\partial p_1^\mu}$$

Major question

Once the canonical form is found, the solution is obtained for free.

How to find the canonical form?
Or in other words, how to find the canonical basis?

Finding canonical form

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\, \vec{f}(x,\epsilon)$$

↓ remove higher order poles

$$\partial_x \vec{f}(x,\epsilon) = \left[\sum_k \frac{a_k(\epsilon)}{x - x_k} + p(x,\epsilon)\right] \vec{f}(x,\epsilon)$$

subtlety of choosing proper 'letters' x_k.

↓ remove $p(x,\epsilon)$

$$\partial_x \vec{f}(x,\epsilon) = \left[\sum_k \frac{a_k(\epsilon)}{x - x_k}\right] \vec{f}(x,\epsilon)$$

↓ let $a_k = \mathcal{O}(\epsilon)$

$$\partial_x \vec{f}(x,\epsilon) = \epsilon\left[\sum_k \frac{\tilde{p}_k(\epsilon)}{x - x_k}\right] \vec{f}(x,\epsilon) \quad \text{or} \quad \partial_x \vec{f}(x,\epsilon) = \epsilon\left[\sum_k \frac{A_k}{x - x_k}\right] \vec{f}(x,\epsilon)$$

These procedure is relatively well-understood for single scale integrals.

Why only simple poles

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\,\vec{f}(x,\epsilon)$$

$$\partial_x \vec{f}(x,\epsilon) = \left[\sum_k \frac{a_k(\epsilon)}{x - x_k} + p(x,\epsilon)\right] \vec{f}(x,\epsilon)$$

Fuchsian system of differential equations

Consider a simple case:

$$\partial_x g(x,\epsilon) = \frac{a}{x^N}\, g(x,\epsilon)$$

$$N = 1: \qquad g(x,\epsilon) \sim x^a\,,$$

$$N = 2: \qquad g(x,\epsilon) \sim e^{-a/x}$$

$\rightarrow$ essential singularity, not allowed in physical amplitudes

Major question

How to find the canonical form?
Or in other words, how to find the canonical basis?

One of the most efficient strategies is to find the basis by doing **unitarity cuts!**

Finding canonical basis

The solution of canonical form has uniform transcendentality (UT).

$$\vec{f}(x,\epsilon) = \mathbb{P}\exp\left[\epsilon \int_\gamma A(x)\right] \vec{f}_0(\epsilon)$$

↓

The basis are UT basis, which take d-log form, and are related to leading singularity via unitarity cuts.

N=4 SYM can play important role in constructing canonical basis which apply to general theories including QCD!

Applications

- Massless three-loop four-point integrals
- Two-loop four-point with two off-shell legs
- Massless two-loop five-point
- Sudakov form factor at four-loop

Thank you!