New algorithm to construct and triangulate convex polyhedral cone for Geometric sector decomposition

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Outline

1. 简介
2. Geometric method for sector decomposition
3. 我们的新解法

- Unitary Cut, On shell Helicity Amplitude, IBP，Symbol，Sector Decomposition, Mellin-Barnes(MB), Differential Equation(DE)
- 关键一步 Master interal 解析分离发散, 剩下有限部分 (解析结果，数值积分）
- 2010 年 T.kaneko at al 在 setor decomposition 框架下把解析分离发散问题非常漂亮的化为多维空间中的多面锥的构建和三角化问题，并利用数学家的的解决方法，最后一个三圈 box, 要用 53cpu 小时解决。
- 我们研究了这个问题，提出了新的多维空间中的多面锥的构建和三角化的解决方法，把同一个问题提高到了 3 cpu 分钟之内解决。
- 而这各种各样的三圈图在物理过程的高阶修正计算中会遇到很多，所以这个研究对于提高高阶修正计算的速度很有用（两方面：1：解析分离发散，2：有限部分的数值计算或解析计算）。

- T. kaneko 的文章—研究多项式的性质化为研究多维空间中的多面锥的构建和三角化的问题，
- 数学家的解法（Algorithms in Combinatorial Geometry 422 页）

Diagram	A	B	C	S	X	H	This method	Exponential S.D.
Bubble	2	2	2	2*	2		2	2
Triangle	3	3	3	3*	3		3	3
Box	12	12	12	12	12		12	8
Tbubble	58	48	48	48*	48		48	36
Double box, $p_i^2 = 0$	775	586	586	362	293	282	266	106
Double box, $p_4^2 \neq 0$	543*	245*	245*	230*	192*	197	186	100
Duble box, $p_i^2 = 0$ nonplanar	1138	698	698	441*	395		360	120
D420	8898	564	564	180	F		168	100
3 loop vertex (A8)	4617*	1196*	1196*	871*	750*	684	684	240
Triple box	M	114 256	114 256	22 657	10 155		6568	856

			Double box	Triple Box
T. Kaneko, et al	2010	unpublished	29 s	53 h
Our		FDC-Sec-Dec	1 s	3 minus

		Double box		Triple box	
Package	Strategy	No. of Sectors	Time(s)	No. of Sectors	Time(s)
FIESTA4	S	362	1.4	23783	831
	B	586	0.5	121195	127
	X	282	0.3	10259	44
	KU	266	13.7	6822	7472
	KU0	326	4.5	10556	6487
	KU2	266	61.9	-	-
SecDec3.0	X	320	1.4	11384	700
	G1	270	4.2	7871	574
Method proposers		266	29.2	6568	123280
Our result		266	0.6	6568	120

Comparision with FIESTA4 [[?]], SecDec3.0 [[?]] and method proposers [[?]]. Strategies KU, KU0, KU2, G1 and G2 are all based on the geometric method, while G2 is different in the strategy of primary sectors.

decomposition only, without the integration of finite coefficients

Geometric method for sector decomposition

Sector Decomposition is a method used to separate divergences in loop integral.

With α presentation of a propagator

$$\frac{1}{D_l^{a_l}} = -i\int_0^\infty \mathrm{d}\alpha_l \exp(iD_l\alpha_l a_l),$$

An h-loop integral with N propagators can be expressed as

$$G = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \cdots \mathrm{d}^d k_h}{D_1^{a_1} D_2^{a_2} \cdots D_N^{a_N}} = \int \mathrm{d}^d k \int \mathrm{d}^N \alpha \exp\left(i \sum_{l=1}^{N} D_l \alpha_l a_l \right)$$

After integration on loop momenta, it becomes

$$G = C \int_0^\infty \mathrm{d}^N \alpha \prod_l \alpha_l^{a_l - 1} U^{-d/2} \mathrm{e}^{-iF/U} \tag{1}$$

where U and F are homogeneous polynomials of α_i with the homogeneity degrees h and $h+1$, and C is a constant.

Let $\eta = \sum \alpha_l$, insert $\delta(\eta - \sum \alpha_l)$ into the integral, and make the transformation $\alpha_l = \eta \alpha_l'$. After the integration over η, the integral becomes

$$G = C' \int_0^1 \mathrm{d}^N \alpha \delta \left(1 - \sum \alpha_l\right) \prod_l \alpha_l^{a_l - 1} \frac{U^{a-(h+1)d/2}}{F^{a-hd/2}} \qquad (2)$$

with $a = \sum a_l$.

One can always reach Eq.(2) with usual loop integral techniques. And this is where sector decomposition starts. In this integral, only the integration over α_i is remained, and the interval is now limited to $[0, 1]$ due to the delta function.

And this is how sector decomposition works on it:

- separate the integration domain into N sectors $\Delta_{k,k=1,2,\ldots,N}$, where Δ_k is defined by $\alpha_i \leq \alpha_k, i \neq k$.
- do the transformation $\alpha_i' = \alpha_i/\alpha_k, i \neq k$ in Δ_k, and integrate over α_k with the delta function
- now, the integral in the integration domain Δ_k (labelled with G_k) becomes

$$G_k = C' \int_0^1 \mathrm{d}^{N-1} \alpha \prod_l \alpha_l^{a_l - 1} \frac{U_k^{a-(h+1)d/2}}{F_k^{a-hd/2}} \qquad (3)$$

where U_k and F_k are obtained by setting α_k to 1 in U and F.

- Usually these Δ_k are called primary sectors. But they are not sufficient since the divergences are still hidden inside.
- Further decomposition is needed.
- Here we introduce the geometric method [[**?**]], which can separate the divergence after one more decomposition (free from infinite recursion).
- For convenience, we rewrite G_k into

$$G_k = C' \int_0^1 \mathrm{d}^{N-1}\alpha \alpha^v U_k^\beta F_k^\gamma \tag{4}$$

with $v = \{a_1 - 1, a_2 - 1, \ldots, a_{N-1} - 1\}$, $\alpha^v = \prod_l \alpha_l^{a_l - 1}$,
$\beta = a - (h+1)d/2$ and $\gamma = -(a - hd/2)$

This is how the geometric method works:

- A monomial in $U_k(F_k)$ has the form $c_b\alpha^b = c_b\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_{N-1}^{b_{N-1}}$, and can be characterized by a vector $b = \{b_1, b_2, \ldots, b_{N-1}\}$, thus we have two vector sets Z^U and Z^F
- do transformation $\alpha_i = \mathrm{e}^{-y_i}$, the Jacobian is $\mathrm{e}^{-\sum_i y_i}$, the integral becomes

$$G_k = C' \int_0^\infty \mathrm{d}^{N-1} y \mathrm{e}^{-v\cdot y} U_k^\beta F_k^\gamma \tag{5}$$

- now $U_k = \sum_{b\in Z^U} c_b \mathrm{e}^{-b\cdot y}$, $F_k = \sum_{b'\in Z^F} c_{b'}\mathrm{e}^{-b'\cdot y}$
- suppose $\mathrm{e}^{-b(b')\cdot y}$ is maximal in $U_k(F_K)$, label this domain as $\Delta_{bb'}$.
- extract these two term from U_k and F_k, and the integral can be further decomposed into

$$\begin{aligned} G_k &= \sum_{b\in Z^U}\sum_{b'\in Z^F}\int_{\Delta_{bb'}} \mathrm{d}^{N-1} y \mathrm{e}^{-(v+b\beta+b'\gamma)\cdot y} \times \left[c_b + \sum_{d\in Z^U/\{b\}} c_d \mathrm{e}^{-(d-b)\cdot y}\right]^\beta \\ &\quad \times \left[c_{b'} + \sum_{d'\in Z^F/\{b'\}} c_{d'}\mathrm{e}^{-(d'-b')\cdot y}\right]^\gamma \end{aligned} \tag{6}$$

where $Z^U/\{b\}$ denotes subset of Z^U obtained by removing b from Z^U

- We have supposed that in the domain $\Delta_{bb'}$, $\mathrm{e}^{-b(b')\cdot y}$ is maximal in $U_k(F_K)$ for certain $b(b')$.
- $\mathrm{e}^{-(d-b)\cdot y} \leqslant 1$ and $\mathrm{e}^{-(d'-b')\cdot y} \leqslant 1$ for all $d(d')$
- $(d-b)\cdot y \geq 0$, $(d'-b')\cdot y \geq 0 \rightarrow$ restrictions on $\Delta_{bb'}$
- Last two terms in Eq.(6) is always finite when $\alpha_i = 0$
- All physical poles (if exist) have been factorized into first term
- Now we can expand last two term with respect to $\epsilon = (D-4)/2$ to required order and obtained a Laurent series of the whole integral in terms of ϵ, whose coefficients are expressed by finite integrals
- The question is how to obtain the domain $\Delta_{bb'}$

- Define $Z_b^{U(F)} \equiv \{v - b | v \in Z^{U(F)} / \{b\}\}$, $S \equiv Z_b^U \cup Z_{b'}^F \cup R^{N-1}$, where R^{N-1} is the standard basis of $(N-1)$-dimensional Euclidean space.
- Let $C(S)$ be the vector space generated by S, $C(S) \equiv \{\sum_i r_i v_i | r_i \geq 0, v_i \in S\}$, then $C(S)$ is just a convex polyhedral cone in $(N-1)$-dimensional Euclidean space.
- The dual cone of C(S) is defined by $C(S)^V \equiv \{y | v \cdot y \geq 0, \forall v \in C(S)\}$, $C(S)^V$ is also a convex polyhedral cone in $(N-1)$-dimensional Euclidean space.
- $\Delta_{bb'} = \{y | v \cdot y \geq 0, \forall v \in S\} = C(S)^V$. R^{N-1} is also in S because of the fact $y_i \geq 0$.
- Let V be the set of all edges of $C(S)^V$, then $C(S)^V$ can also be written as $\{\sum_i u_i v_i | u_i \geq 0, v_i \in V\}$
- If $C(S)^V$ is simplicial (has only $N-1$ edges), change the integration variables from y_i into u_i, $\Delta_{bb'}$ is obtained.
- If not, triangulate $C(S)^V$ (split it into several simplicial sub-cones), and change integration variables in each sub-cones (more sub-sectors).
- This is why this method is called 'geometric'

我们的新解法：

T. Kaneko: 为了直接应用数学家的解法，**d维空间**中的多面锥的构建和三角化问题复杂化为多面体问题（给多面锥加了一个d-1维截面）。V. A. Smirnov， G. Heinrich 等人几乎沿用了Kaneko的数学家的解法。

我们直接解决多面锥问题（数学家没有研究),并且这个物理问题（高圈图费曼参数积分发散项解析分离）中，多面锥的棱矢量空间是高度简并的，在解决方案中提高速度的关键是尽量利用其高度简并性。如果用我们的方案去解决一个高度非简并的多面锥问题，结果会跟他们的解法一样慢，甚至比他们的慢。（**得到的经验是：已经有的解法很可能对于你的问题不是最优方案，在复杂情况下可能比最优方案慢很多**）

从多项式得到的矢量空间 [V^a] 对偶到积分变量空间 [V^y],需要求解不等式组：

$$V_i^a \bullet V_j^y = 0, \quad \forall V_i^a \subset [V^a], \quad \forall V_i^y \subset [V^y]$$

我们可以证明生成对偶**d维**空间的问题与生成**d维空间**多面锥的**d-1维**外表面问题等价。这个问题如果是高度非简并的：需要遍历下面数目的d-1维面

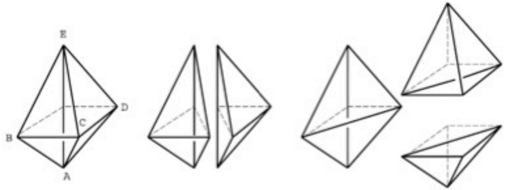
$$C_{d+n}^{d-1} \quad = \quad 308万， \quad 当 d = 9(triplet-box), \quad n = 29(大于1各种数)$$

在后面会给出生成**d维空间**多面锥的**d-1维**外表面的我们的方法(寻找次数会大大的小于遍历数)

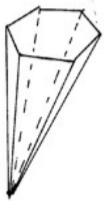
三角化多面锥前，对于组成多面锥的棱矢量进行了线性无关子空间的分解，对于每一个子空间分别进行多面锥的三角化拆分，然后子空间之间直积成多个分块.

对于不可分子空间构成的多面锥的三角化拆分就需要生成它的所有**d-1维**外表面

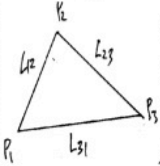
多维空间中的多面锥的构建和三角化Computional Geometry (algebraic Geometry, 代数几何)



(a) (b) (c)



三维空间多面锥



二维空间三角形

$L_{12}: a_{12}x + b_{12}y + c_{12} = 0$

$L_{23}: a_{23}x + b_{23}y + c_{23} = 0$

$L_{31}: a_{31}x + b_{31}y + c_{31} = 0$

有一点 $P_a = (x_a, y_a)$

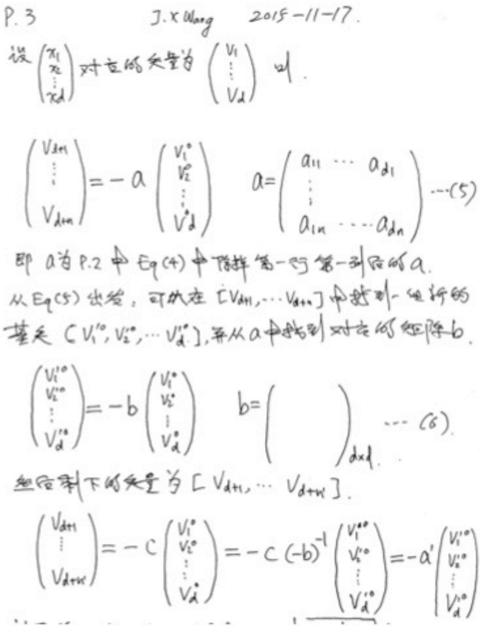
判断在这个三角形的关系

d维空间中，d-1维超面A，（A上点的坐标）

A: $V_A \cdot X + C = 0$, $V_A = (V_1, V_2, \cdots V_d)$, $X = (x_1, \cdots x_d)$

任一点 $P_a = (x_1^a, x_2^a, \cdots x_d^a)$ 与A的关系.

计算 $V_A \cdot x^a + C = \begin{cases} 0 & \text{在A上} \\ >0 & \text{在A的上一侧} \\ <0 & \text{在A的下一侧} \end{cases}$


P.3 J.X Wang 2015-11-17.

设 $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$ 对应的矢量为 $\begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}$ 则

$$\begin{pmatrix} V_{d+1} \\ \vdots \\ V_{d+n} \end{pmatrix} = -a \begin{pmatrix} V_1^{\circ} \\ V_2^{\circ} \\ \vdots \\ V_d^{\circ} \end{pmatrix} \qquad a = \begin{pmatrix} a_{11} & \cdots & a_{d1} \\ \vdots & & \\ a_{1n} & \cdots & a_{dn} \end{pmatrix} \cdots (5)$$

即 a 为 P.2 中 Eq(4) 中除掉第一行第一列后的 a.

从 Eq(5) 出发，可以在 $[V_{d+1}, \cdots V_{d+n}]$ 中找到一组新的基矢 $[V_1'^{\circ}, V_2'^{\circ}, \cdots V_d'^{\circ}]$，并从 a 中找到对应的矩阵 b.

$$\begin{pmatrix} V_1'^{\circ} \\ V_2'^{\circ} \\ \vdots \\ V_d'^{\circ} \end{pmatrix} = -b \begin{pmatrix} V_1^{\circ} \\ V_2^{\circ} \\ \vdots \\ V_d^{\circ} \end{pmatrix} \qquad b = \begin{pmatrix} & \end{pmatrix}_{d\times d} \cdots (6)$$

则剩下的矢量为 $[V_{d+1}, \cdots V_{d+n'}]$.

$$\begin{pmatrix} V_{d+1} \\ \vdots \\ V_{d+n'} \end{pmatrix} = -c \begin{pmatrix} V_1^{\circ} \\ V_2^{\circ} \\ \vdots \\ V_d^{\circ} \end{pmatrix} = -c(-b)^{-1} \begin{pmatrix} V_1'^{\circ} \\ V_2'^{\circ} \\ \vdots \\ V_d'^{\circ} \end{pmatrix} = -a' \begin{pmatrix} V_1'^{\circ} \\ V_2'^{\circ} \\ \vdots \\ V_d'^{\circ} \end{pmatrix}$$

$$V_1, V_2, \ldots V_d, V_{d+1}, \ldots V_{d+n}$$

是组成d维空间d+n棱锥的矢量，我们可以选出d个基矢V_1,...V_d 矩阵"a"为0的元素越多，简并度越高。

Thanks for your attention!