Muon g-2 at Four Loops

Tao Liu

in collaboration with

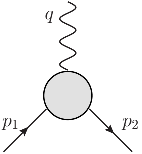
Alexander Kurz, Peter Marquard, Alexander V. Smirnov,
Vladimir A. Smirnov, Matthias Steinhauser

University of Alberta, Karlsruhe Institute of Technology

Overview

Anomalous magnetic moment

$$V_{int} = -\vec{\mu} \cdot \vec{B}, \quad \vec{\mu} = g\left(\frac{e}{2m}\right)\vec{s}.$$



$$= \quad -ie\,\bar{\psi}(p_2)\left(\gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right)\psi(p_1)$$

$$F_1(0) = 1 \quad F_2(0) = \frac{g-2}{2} \equiv a_l$$

$$a_\mu^{exp} = 116592089(63) \times 10^{-11}$$ [PDG]

$$a_\mu^{th} = 116591828(49) \times 10^{-11}$$

Note that $a_\mu(4\ loop\ QED) \sim 381 \times 10^{-11}$ [Aoyama, Hayakawa, Kinoshita, Nio 2012]

Experiment: Fermilab E989 and J-PARC

Aim: 4-loop analytically

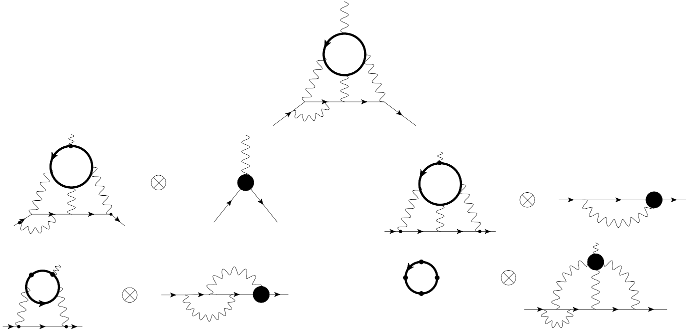
$n_{el}^2 + n_{el}^3$ terms	[Lee, Marquard, Smirnov, Smirnov, Steinhauser 2013]	
Heavy lepton	[Kurz, Liu, Marquard, Steinhauser 2013]	
		[PHIPSI13, Steinhauser]
		[RADCOR2013, Marquard]
		[EPS-HEP2013, Marquard]
Hadronic NNLO	[Kurz, Liu, Marquard, Steinhauser 2014]	
		[LL2014, Liu]
		[LoopFest2014, Kurz]
		[Tau2014, Marquard]
Electron lbl	[Kurz, Liu, Marquard, Smirnov, Smirnov, Steinhauser 2015]	
		[SFB-TR-9 2015, Marquard]
		[FCCP2015, Steinhauser]
Electron VAP	[Kurz, Liu, Marquard, Smirnov, Smirnov, Steinhauser 2016]	
		[LL2016, Steinhauser]
		[LoopFest2016(in August), Liu]

Calculation

- `QGRAF`: generate Feynman diagrams [Nogueira]
- `q2e`: bridge between QGRAF and exp [Harlander, Seidensticker, Steinhauser]
- `exp/asy/in hause`: asymptotic expansion with mass hierarchy [Harlander, Seidensticker, Steinhauser]/ [Pak, Smirnov; Jantzen, Smirnov, Smirnov]
- `FORM`: calculate diagrams [Vermaseren]
- `FIRE/Crusher`: reduction to master integrals [Smirnov]/[Marquard, Seidel]
- `FIESTA`: MIs evaluations [Smirnov]

Graphical example I

Diagram with τ loop:



4-loop tadpole

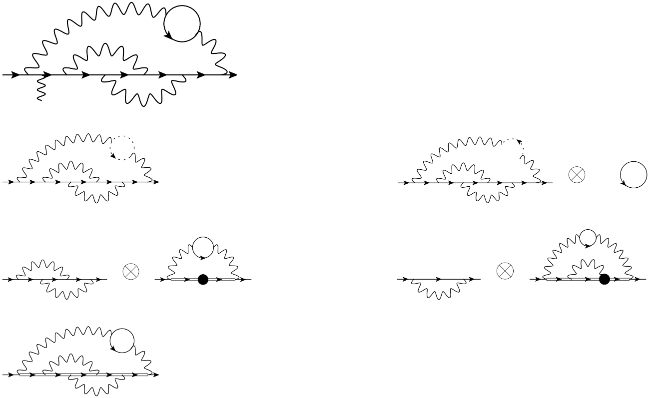
2-loop tadpole $\otimes$ 2-loop on-shell

3-loop tadpole $\otimes$ 1-loop on-shell

1-loop tadpole $\otimes$ 3-loop on-shell

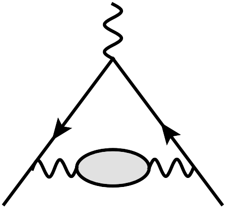
Graphical example II

Diagram with electron loop:



4-loop on-shell appear!
4-loop linear integrals with $\frac{1}{2\ell\cdot q}$ appear!

Hadronic contributions



LO: [Davier et al 2010] [Hagiwara et al 2011] [Jegerlehner et al 2011] [Benayoun et al 2012]

NLO: [Krause 1997] [Hagiwara et al 2011] [Greynat, Rafael 2012]

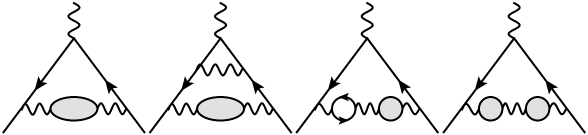
LO lbl: [Nyffeler 2009] [Melnikov et al 2014] ...

NLO lbl: [Colangelo et al 2014]

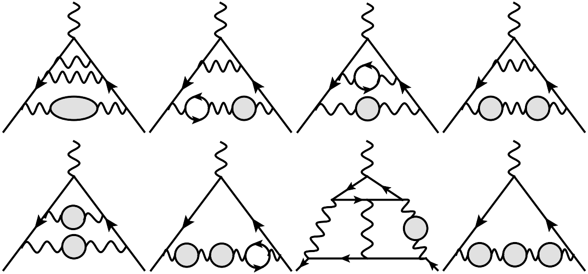
- Analyticity: $\Pi_{\mu\nu} = i(q_\mu q_\nu - g_{\mu\nu})\Pi(q^2)$, $\Pi(q^2) = -\frac{q^2}{\pi}\int_{m_\pi^2}^{\infty}\frac{ds}{s}\frac{\text{Im}\Pi(s)}{q^2-s}$.
- Optical theorem: $R(s) = \frac{\sigma(e^+e^- \to hadrons)}{\sigma_0(e^+e^- \to \mu^+\mu^-)} = 12\pi\text{Im}\Pi(s)$
- LO: $a_\mu = \frac{1}{3}(\frac{\alpha}{\pi})^2\int_{m_\pi^2}^{\infty} ds\frac{R(s)K(s)}{s}$
- $K(s)$: asymptotic expansion with $s \gg m_\mu^2 \gg m_e^2$.
- $R(s)$: numerical value[Nomura, Teubner] plus resonances like J/Ψ

Feynman diagrams

LO + NLO:

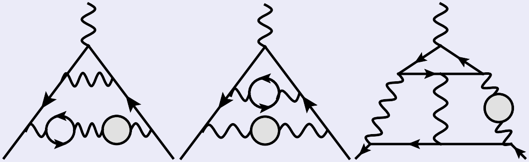


NNLO:



NNLO kernel functions

- Electron loops contribute asymptotic regions with $m_e^2 = \ell^2 \ll p^2 = m_\mu^2$.
- Denominator expansion: $\frac{1}{(\ell+p)^2 - m_\mu^2} = \frac{1}{\ell^2 + 2\ell\cdot p} = \frac{1}{2\ell\cdot p}\sum_{n=0}^{\infty}\left(\frac{-\ell^2}{2\ell\cdot p}\right)^n$
- Contributions calculated up to $\frac{m_e^2}{m_\mu^2}$

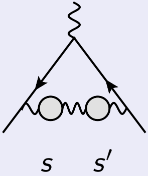


Multi-hadronic insertions

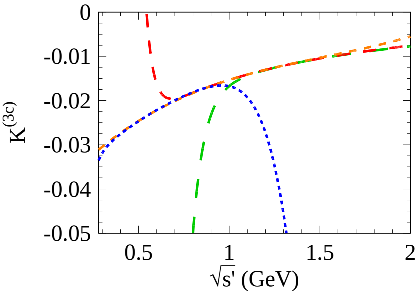
NLO:

$$K(s,s') = \int_0^1 dx \frac{x^4(1-x)}{[x^2+(1-x)\frac{s}{m_\mu^2}][x^2+(1-x)\frac{s'}{m_\mu^2}]}$$

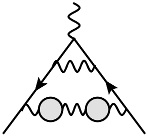
Construct an interpolating function from regions $s \gg s'$, $s \approx s'$ and $s \ll s'$. a_μ differs by less than 1% from exact and approximate kernels.

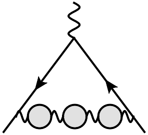


NNLO:

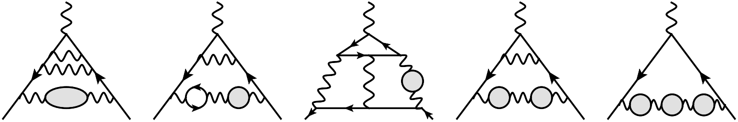


Blue: $s \gg s'$;
Green: $s \ll s'$;
Orange and red: $s \approx s'$;
$\sqrt{s} = 1$ GeV





NNLO results



$10^{11} \times a_\mu =$ 8.0 -4.1 $+9.1$ -0.6 $+0.005$

$a_\mu^{\text{had,NNLO}} = 12.4 \pm 0.1 \times 10^{-11}$

- The above result is of the same order of magnitude as the uncertainty of LO hadronic contribution.
- It reduces the deviation from the experimental value by 0.2σ.

Leptonic contributions

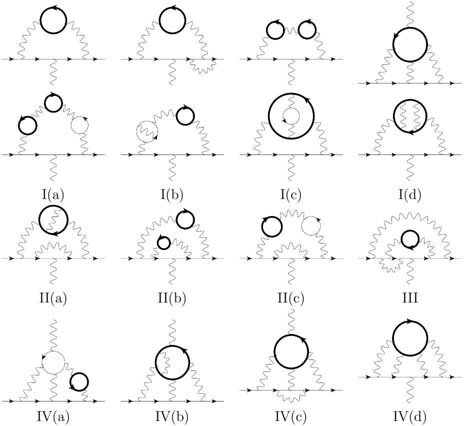
$$a_\mu = A_1 + A_2\left(\frac{m_\mu}{m_e}\right) + A_2\left(\frac{m_\mu}{m_\tau}\right) + A_3\left(\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}\right)$$

2ℓ [Elend 1966]

3ℓ [Laporta, Remiddi 1993; Laporta 1993; ...]

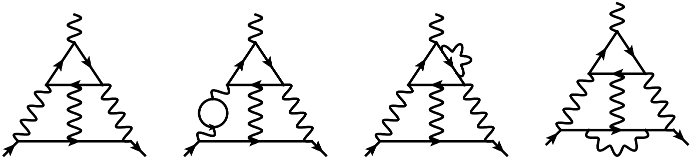
4ℓ [Kinoshita, Nio 2005]

5ℓ [Aoyama, Hayakawa, Kinoshita, Nio 2012]



asymptotic expansion $m_\tau^2 \gg m_\mu^2 \gg m_e^2$

Electronic LBL



IV(a) IV(b) IV(c)

- $m_e = 0$ not possible, IR poles $\longrightarrow$ expansion by regions [Beneke, Smirnov 1998; Smirnov] .
- $\ell \sim m_\mu$ or $\ell \sim m_e$

3-loop example

3-loop OS integrals
2-loop OS $\otimes$ 1-loop tadpole
1-loop OS $\otimes$ 2-loop tadpole
1-loop OS $\otimes$ 2-loop linear

3-loop linear integrals

4-loop LBL

- all the 3-loop integrals are known analytically
  $\longrightarrow$ all analytical CTs
- amplitude: linear combination of MIs
- 4-loop OS-shell integrals:
  $\simeq 70 = 40_{\texttt{ana/high prec.}} + 30_{\texttt{num}}$ [Marquard,Smirnov,Smirnov,Steinhauser 2015]
- 4-loop linear shell integrals:
  $\simeq 70 = 20_{\texttt{ana/high prec.}} + 50_{\texttt{num}}$
- two different sets of auxiliary propagators used for each topology
  several different basis of MIs to get stable results [Kurz, Liu]

One example for IV(b)

$A_2^{(8),\mathrm{IV(b)}}$ $\qquad x = m_e/m_\mu \simeq 1/206.7682843$

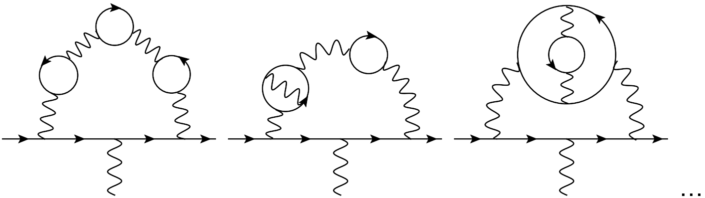
$$
\begin{aligned}
=& 27.395 \pm 0.014 + (4.93482 \pm 0.00003)\ell_x \\
&+ x\left[-0.81 \pm 1.22 + 59.0235\ell_x\right] \\
&+ x^2\left[142.5 \pm 7.6 + 40.6546\ell_x + 20.5582\ell_x^2 - 9.6167\ell_x^3 + 0.8333\ell_x^4\right] \\
&+ x^3\left[62.11 \pm 2.89 + 132.7421\ell_x - 40.9406\ell_x^2\right]
\end{aligned}
$$

$$
\begin{aligned}
=& 27.395 \pm 0.014 + (-26.3105 \pm 0.0002) \\
&+ [-0.0039 \pm 0.0059 - 1.5219] \\
&+ [0.003334 \pm 0.0001769 - 0.005070 + 0.01367 + 0.03409 + 0.01575] \\
&+ [0.000007 \pm 0. - 0.000080 - 0.000132]
\end{aligned}
$$

$$[1.084 \pm 0.014] + [-1.5259 \pm 0.0059] + [0.06177 \pm 0.00018] + [-0.0002047]$$

$$= -0.380 \pm 0.016$$

Non-LBL diagrams



- n_e^2 and n_e^3 gauge invariant subset
- $m_e = 0$, $\bar{\alpha}(\mu) \to \mathrm{Log}(\mu/m_\mu)$
- $\alpha^{OS} \to \mathrm{Log}(\mu/m_e)$
- analytical results at the leading order of m_e

  [Laporta,1993] [Aguila et al 2008] ... [Lee,Marquard,Smirnov,Smirnov,Steinhauser 2013]
- expansion by regions as LBL case for all

  [Kurz,Liu,Marquard,Smirnov,Smirnov,Steinhauser 2016]

τ-loop contributions

- $A_2^{(8)}(\frac{m_\mu}{m_\tau})$ and $A_3^{(8)}(\frac{m_\mu}{m_\tau}, \frac{m_e}{m_\mu})$
- the hardest region to be tadpoles which known analytically

$$
\begin{aligned}
A_2^{(8)} \quad = \quad & \left(\frac{m_\mu}{m_\tau}\right)^2 \left(\frac{37448693521}{2286144000} + \frac{89603}{16200}P_4 + \frac{52}{675}P_5 + \frac{4\pi^2\zeta_3}{15}\right. \\
& + \frac{5771\ln(2)\pi^4}{32400} - \frac{3851\pi^2}{3600} - \frac{25307\zeta_5}{1440} - \frac{37600399\pi^4}{27216000} \\
& + \frac{35590996657\zeta_3}{508032000} \\
& \quad \left. + \ln\frac{m_\mu^2}{m_\tau^2}\left(-\frac{38891}{12150} + \frac{19\pi^2}{135} + \frac{3\zeta_3}{2}\right) + \frac{359}{1080}\ln^2\frac{m_\mu^2}{m_\tau^2}\right) \\
& + \left(\frac{m_\mu}{m_\tau}\right)^3 \frac{\pi^2}{90} + \ldots + \mathcal{O}\left(\left(\frac{m_\mu}{m_\tau}\right)^8\right)
\end{aligned}
$$

Results I

group	$10^2 \cdot A_2^{(8)}(\tau)$	
	our work	ref.(old result)
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	−0.6292808(6)	−0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	−2.316756(5)	−2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)	0.612661(5)	0.6106(31)
IV(c)	−1.83010(1)	−1.823(11)

$$\begin{array}{ccc} 2\ell & 3\ell & 4\ell \end{array}$$

$$10^{11} \times a_\mu(\tau) = 42.13 + 0.45 + 0.12$$

$$10^{11} \times a_\mu\Big|_{\mathrm{univ},5\ell} = 0.06$$

Results II

	$A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau)$	
group	our work	ref.(old result)
I(a)	0.00320905(1)	0.003209(0)
I(b) + I(c)	0.00442289(2)	0.004422(0)
II(b) + II(c)	−0.02865753(1)	−0.028650(2)
IV(a)	0.08374757(9)	0.083739(36)

The discrepancy comes from the mass ratio of m_μ/m_τ.

Results III

$A_2^{(8)}(m_\mu/m_e)$	our work	ref.(old result)
I(a)	7.745136	7.74547 ± 0.00042
I(b)+I(c)	9.2054 ± 0.0060	9.20632 ± 0.00071
I(d)	-0.2303 ± 0.0024	-0.22982 ± 0.00037
II(a)	-2.77885	-2.77888 ± 0.00038
II(b)+II(c)	-13.895796 ± 0.000013	-13.89457 ± 0.00088
III	10.800 ± 0.022	10.7934 ± 0.0027
IV(a)	123.78 ± 0.22	123.78551 ± 0.00044
IV(b)	-0.38 ± 0.08	-0.4170 ± 0.0037
IV(c)	2.94 ± 0.30	2.9072 ± 0.0044
IV(d)	-4.32 ± 0.30	-4.43243 ± 0.00058

- Note that the uncertainties in the second column are multiplied by a factor five.
- $A_2^{(8)} = 126.34(38) + 6.53(30) = 132.86(48)$
- $0.5 \times (\alpha/\pi)^4 \approx 1.5 \times 10^{-11}$

Summary

- $A_\mu^{(8)} = A_\mu^{(8)}|_{\text{univ.}} + 132.86(48) + 0.0424941(53) + 0.062722(10)$
- Agreement with Kinoshita's results.
- Systematic improment possible.
- Hardronic vacuum polization contributions at NNLO.

Thanks for your attention!