# On-Shell Methods: Loop

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# Analytic structure at loop Level

Where can we look for analytic constraint ?

$$A^L = g^{L-1} \int d^4\ell^1 \cdots d^4\ell^L \frac{n(\ell_i, k_j, \epsilon)}{\ell_i^2(\ell_i - k_j)^2 \cdots (\ell_i - \ell_j)^2}$$

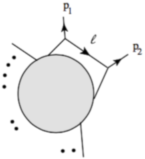
- Integrated level: Discontinuity on multi-Riemann sheet ?

Complication: The integrated results are polluted with singularities IR and UV

$$A^1 = \frac{IR}{\epsilon^2} + \frac{UV + IR}{\epsilon} + Finite$$

# Analytic structure at loop Level

IR-divergences:




$$\rightarrow \int \frac{d\ell^+ \, d\ell^- \, d^2\ell_\perp}{(\ell^2 - 2\ell^+ p_1^-)\, \ell^2 \,(\ell^2 - 2\ell^- p_2^+)}$$

- Soft Divergence: $\ell^+ \rightarrow t\,, \quad \ell^- \rightarrow t\,\ell^-\,, \quad \vec{\ell}_\perp \rightarrow t\,\vec{\ell}_\perp$

$$\int \frac{t^{D-1}\,dt}{t^4} = \int \frac{dt}{t^{5-D}} \quad \rightarrow \quad \int \frac{t^{3-2\epsilon}\,dt}{t^4} = \int \frac{dt}{t^{1+2\epsilon}} = \frac{1}{2\epsilon} + O(\epsilon^0)$$

- Colinear Divergence: $\ell^+ \rightarrow t^2\,, \quad \ell^- \rightarrow \ell^-\,, \quad \vec{\ell}_\perp \rightarrow t\,\vec{\ell}_\perp$

$$\int \frac{t^{D-1}\,dt}{t^4} = \int \frac{dt}{t^{5-D}} \quad \rightarrow \quad \int \frac{t^{3-2\epsilon}\,dt}{t^4} = \int \frac{dt}{t^{1+2\epsilon}} = \frac{1}{2\epsilon} + O(\epsilon^0)$$

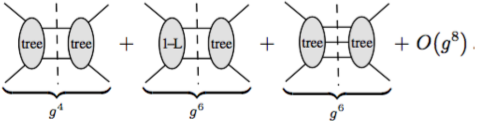
# Analytic structure at loop Level

IR-divergences: Soft and Colinear divergences can occur simultaneously:

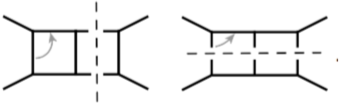
$L$-loop $\rightarrow \frac{1}{\epsilon^{2L}}$

The loop IR divergences are necessary for cancelations!

$$\frac{d\sigma}{d\Omega} \propto \sum_i \langle g_1 g_2 | S | i \rangle \langle i | S | g_3 g_4 \rangle =$$




The soft singularities of 1-loop $A_4$ cancels against colinear singularity of $A_5$



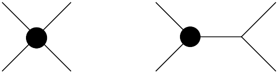
Implies that IR-divergences must be proportional to $A^{\text{tree}}$! $A(+,+,\cdots,+,+)$ and $A(-,+,\cdots,+,+)$ are finite at one-loop

## Analytic structure at loop Level

UV-divergences:

$$\int d^D\ell \frac{\ell^N}{(\ell^2)^P} \sim \int d|\ell| \frac{|\ell|^{N+D-1}}{|\ell|^{2P}}, \qquad \frac{1}{(\ell+k)^2} \to \frac{1}{\ell^2}$$

Divergences correspond to renormalization of local operators:



- $$\mathcal{L} = \frac{1}{\kappa^2}\int \mathrm{dx^D R}, \quad \mathcal{L} = \frac{1}{g^2}\int \mathrm{dx^D F^2}$$
  Implies $[\kappa^2] = M^{2-D}$, $[g^2] = M^{4-D}$
- One can construct infinite number of invariant operators
  $$R^3, R^4, R^5, D^2R^2, D^2R^3, D^2R^4, \cdots$$
  $$F^3, F^4, F^5, D^2F^2, D^2F^3, D^2F^4, \cdots$$
- Loops amplitude are dressed $(\kappa^2)^{L-1}$ (gravity) $(g^2)^{L-1}$ (YM)

# Analytic structure at loop Level

$$A^1 = \frac{IR}{\epsilon^2} A^{Tree} + \frac{UV(A^{Tree} \text{ or } \mathcal{O}) + IR A^{Tree}}{\epsilon} + Finite$$

Finite terms can contain branch cuts

$$\log(x) = \int_0^x \frac{dt}{t}, \quad \log(x)\log(y) = \left(\int_0^x \frac{dt}{t}\right)\left(\int_0^y \frac{ds}{s}\right), \quad \mathrm{Li}_2(x) = \int_0^x \frac{dt}{t} \int_0^t \frac{du}{1-u}$$

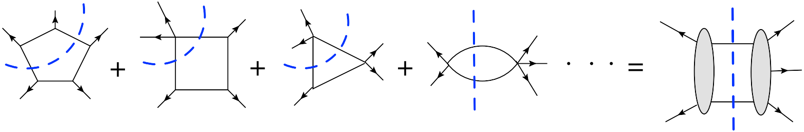
As well as pure pole terms: requires a detailed understanding of possible functions that may appear

# Analytic structure at loop Level

Unitarity Loop amplitudes are not rational, but integrands are

$$\int d^D\ell \frac{n}{\cdots(\ell^2)\cdots(\ell+k_i)^2\cdots}$$

Putting multiple propators on-shell factorizes the Feynman diagrams into tree-amplitudes $\leftarrow$ use it!



# Analytic structure at loop Level

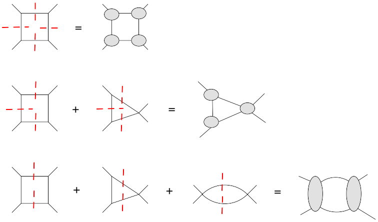
Unitarity Loop amplitudes are not rational, but integrands are

- Find a given integral basis to expand the amplitude

$$A^{\text{1-loop}} = \sum_i C_D^{(i)} I_D^{(i)} + \sum_j C_{D-1}^{(j)} I_{D-1}^{(j)} + \cdots + \sum_k C_2^{(k)} I_2^{(k)} + \text{rational terms}\,.$$

  with kinematic dependent *C*s

- Fix the *C*s by matching it with possible generalized unitarity cuts:



# Analytic structure at loop Level

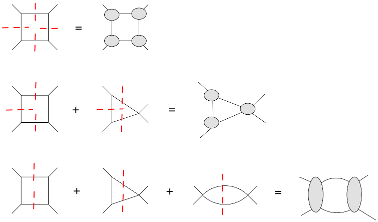
## Unitarity

- IR and UV provides consistency condition on $C$

$$I_{4;4}(p_1,p_2,p_3,p_4) = \frac{\gamma_\Gamma}{su}\left(\frac{2}{\epsilon^2}\Big[(-\mu^{-2}s)^{-\epsilon}+(-\mu^{-2}u)^{-\epsilon}\Big]-\ln^2\left(\frac{s}{u}\right)-\pi^2\right)+O(\epsilon),$$

$$I_{3;4}(p_1,p_2,p_3+p_4) = \frac{\gamma_\Gamma}{\epsilon^2}(-\mu^{-2}s)^{-1-\epsilon}+O(\epsilon),$$

$$I_{2;4}(p_1+p_2,p_3+p_4) = \gamma_\Gamma\left(\frac{1}{\epsilon}-\ln(-\mu^{-2}s)+2\right)+O(\epsilon),$$



# Analytic structure at loop Level

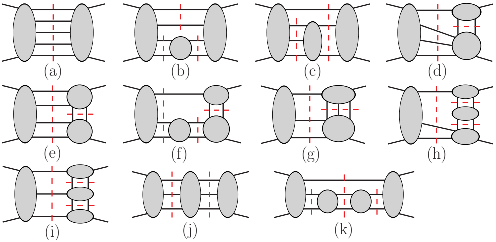
## Unitarity

- Rational terms: do not contribute to four-dimensional unitarity cut

$$\int d^4\ell d^{-2\epsilon}\mu \frac{\mu^2}{(\ell^2+\mu^2)((\ell+k)^2+\mu^2)\cdots} \sim \frac{\epsilon}{\epsilon}$$

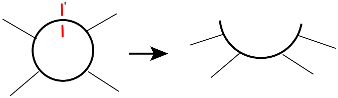
  Compute it in $D$-dimensions (well defined for gauge theories)

- Can be applied to higher loops:

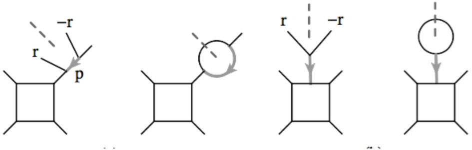


# Analytic structure at loop Level

What about single cuts?



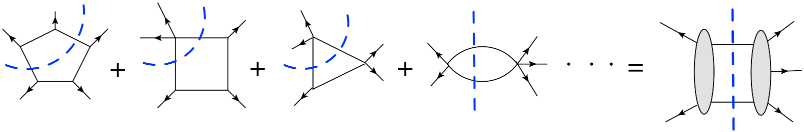
However, forward limits are singular:




- Singularities cancel for supersymmetric theories, all loop integrand for $\mathcal{N} = 4$ SYM
- Massive fourward limits $\rightarrow$ Qcuts

# Analytic structure at loop Level

Unitarity



Loop amplitudes are not rational, but integrands are

$$\int d^D\ell \frac{n}{\cdots(\ell^2)\cdots(\ell+k_i)^2\cdots}$$

- One-loop amplitudes are constructed via tree-amplitude. NLO corrections for QCD processes (BlackHat) Bern, Dixon, Kosower, Diana, Febres Cordero, Forde, Gleisberg, Hoeche, Ita, Maitre, Ozeren
- Multi-loop corrections are iteratively constructed, presented in a simple form: scalar integrands.
- Provides a tool for all loop proofs in QFT.

# Anomalies

$$\mathcal{A}_n = \; C_n \,(\text{branch cuts}) + R_n$$

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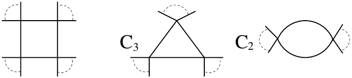
$$\mathcal{A}_n = \underbrace{C_n\,(\text{branch cuts})}_{\text{UV, IR - divergences}} + \underbrace{R_n}_{?}$$

What is the point of rational terms ?

# Anomalies

$$\mathcal{A}_n = \underbrace{C_n\,(\text{branch cuts})}_{\text{UV, IR - divergences}} + \underbrace{R_n}_{?}$$

$$= C_4 \quad C_3 \quad C_2 \quad + \underbrace{R_n}_{?}$$



How can quantum corrections break susy?

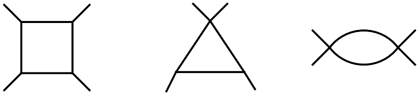
# Gauge anomalies

Consider 1-loop 4-pt

$$\mathcal{A}_4 = Tr(1234)A(1234) + Tr(1342)A(1342) + Tr(1423)A(1423) + \text{flip}$$

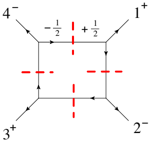
The color-ordered amplitude can be conveniently written as:

$$A(1234) = C_4 I_4 + C_{3s} I_{3s} + C_{3t} I_{3t} + C_{2s} I_{2s} + C_{2t} I_{2t} + R$$



## Four-dimensional

$$-\frac{t^4 s^2}{u^4}$$

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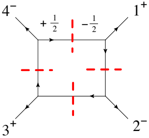


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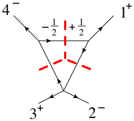


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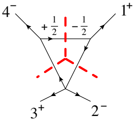
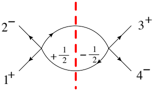
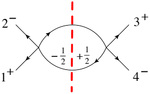



$$\frac{t(su - 6st - 2ut)}{6u^3}$$




$$\frac{t(4s^2 + 2t^2 - 7su)}{6u^3}$$



# Four-dimensional: (Parity-even)

Parity-even:

$$A^{\text{even}}(1^+2^-3^+4^-) = A^{\text{tree}}\Bigg\{ -\frac{st(s^2+t^2)}{2u^4}\left(\log^2 x + \pi^2\right) + \left[\left(\frac{s-t}{3u} - \frac{st(s-t)}{u^3}\right)\right]\log x - \frac{(-s)^{-\epsilon}+(-t)^{-\epsilon}}{3\epsilon} + R^{\text{even}}\Bigg\}$$

$$x = \frac{s}{t}, \quad u = (p_1+p_3)^2$$

Locality requires these spurious poles to be absent.

$$A^{\text{even}}\Big|_{u\to 0} = A^{\text{tree}}\left\{-\frac{s^2}{u^2} - \frac{s}{u} + \mathcal{O}(u^0)\right\}.$$

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Parity-odd:

$$A^{\rm odd}(1^+2^-3^+4^-) = A^{\rm tree}\left\{\frac{st(s^2-t^2)}{2u^4}\left(\log^2 x+\pi^2\right)-\left(\frac{2st}{u^2}\right)\log x + R^{\rm odd}\right\}.$$

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$$A^{\rm tree}R^{\rm odd} = A^{\rm tree}\frac{s-t}{2u}$$

The cancelation of spurious singularities introduces a new factorization channel !

$$Res_s[A^{\rm tree}\frac{s-t}{2u}] = \frac{1}{2}A_3A_3, \quad Res_t[A^{\rm tree}\frac{s-t}{2u}] = -\frac{1}{2}A_3A_3$$

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# Four-dimensional (Parity-odd)

Parity-odd:

$$\mathcal{R}^{\text{odd}} = \frac{\langle 24\rangle^2[13]^2}{2stu}\left[(s-t)\,Tr(1234)+(u-s)\,Tr(1342)+(t-u)\,Tr(1423)\right.$$
$$\left.+(s-u)\,Tr(1243)+(u-t)\,Tr(1324)+(t-s)\,Tr(1432)\right].$$

Let us consider the residue for the $s \to 0$

$$\frac{\langle 24\rangle^2[13]^2}{2su}\left[-Tr(1234)-Tr(1342)+Tr(1243)+Tr(1432)\right]$$

$$Tr(1432)-Tr(1234)+(1\leftrightarrow 2) = d^{1a4}f^{23}{}_a + d^{13a}f^{24}{}_a + d^{1a2}f^{34}{}_a + (1\leftrightarrow 2) = 0\,.$$

$$d^{abc}f^{de}{}_a = 0\,.$$

The non-abelian box-anomaly

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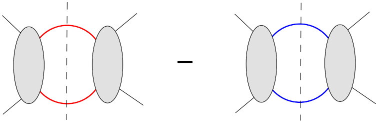
## Six-dimensions (Parity-odd)

$d, \dot{d} \in$ SU(2)$\times$SU(2):

$$\begin{aligned} \text{Chiral}: \quad A_4^{\text{tree}}(g_{a_1\dot{a}_1} g_{a_2\dot{a}_2} q_{a_3} q_{a_4}) &= \frac{\langle 1_{a_1} 2_{a_2} 3_{a_3} 4_{a_4}\rangle [1_{\dot{a}_1}|3|2_{\dot{a}_2}]}{ts}, \\ \overline{\text{Chiral}}: \quad A_4^{\text{tree}}(g_{a_1\dot{a}_1} g_{a_2\dot{a}_2} q_{\dot{a}_3} q_{\dot{a}_4}) &= \frac{[1_{\dot{a}_1} 2_{\dot{a}_2} 3_{\dot{a}_3} 4_{\dot{a}_4}]\langle 1_{a_1}|3|2_{a_2}\rangle}{ts}. \end{aligned}$$

$$A^{\text{odd}}(1,2,3,4) = C_4 I_4 + C_{3s} I_{3s} + C_{3t} I_{3t} + C_{2s} I_{2s} + C_{2t} I_{2t} + R^{\text{odd}}$$



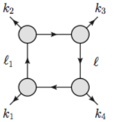
# Six-dimensions (Parity-odd)

The computation:

$$(\lambda_\ell)^A{}_a = \begin{pmatrix} -\kappa\lambda_j & \frac{(1-y)}{c}\lambda_j + \lambda_i \\ c\tilde\lambda_j + y\tilde\lambda_i\,, & \frac{\tilde\kappa}{c}\tilde\lambda_i \end{pmatrix}, \quad \kappa = \frac{m}{\langle ij\rangle}\,, \quad \tilde\kappa = \frac{\tilde m}{[ij]}\,,$$

$$(\tilde\lambda_\ell)_{A\dot a} = \begin{pmatrix} \kappa'\lambda_j & \frac{(1-y)}{c}\lambda_j + \lambda_i \\ -c\tilde\lambda_j - y\tilde\lambda_i & \frac{\tilde\kappa'}{c}\tilde\lambda_i \end{pmatrix}, \quad \kappa' = \frac{\tilde m}{\langle ij\rangle}\,, \quad \tilde\kappa' = \frac{m}{[ij]}\,,$$

$(k^{(6)})^2 = (k^{(4)})^2 + m\tilde m$ with $m = k_4 + ik_5\,, \quad \tilde m = m^*$




$$(\ell + k_3)^2\,(\ell_1 + k_1)^2 A((-\ell_1)_a\, 2_{2\dot 1}\, 3_{2\dot 2}\, \ell_b)\; A((-\ell)^b\, 4_{1\dot 1}\, 1_{1\dot 2}\, \ell_1^a)$$

$$\to C_4 = \left(\frac{2m\tilde m}{t} - 1\right)\langle 23\rangle^2[24]^2\left(\frac{m\tilde m}{u} + \frac{st}{2u^2}\right)$$

$$m\tilde m = -16\,\frac{Gram(\ell, k_1, k_2, k_3, k_4')}{(stu)^2}$$

with $k_4' = \epsilon^{\mu\nu\rho\sigma} k_{1\,\mu} k_{2\,\nu} k_{3\,\rho}$

# Six-dimensions (Parity-odd)

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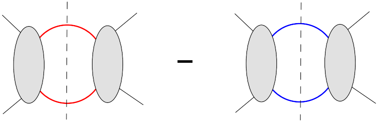
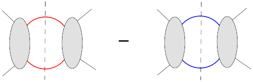


$$C_4 = \frac{(s-t)}{6u^2} F^4, \quad C_{3s} = -\frac{(s-t)}{6tu^2} F^4, \quad C_{2s} = \frac{F^4}{stu},$$

$$F^4 \equiv \langle 4_d | p_2 p_3 | 4_{\dot{d}}] (\langle 1_a | 2_{\dot{b}}] \langle 2_b | 3_{\dot{c}}] \langle 3_c | 1_{\dot{a}}] + \langle 2_b | 1_{\dot{a}}] \langle 1_a | 3_{\dot{c}}] \langle 3_c | 2_{\dot{b}}]) + (\sigma_i)\text{cyclic},$$

$$F^4 = \Big( (\epsilon_4 \cdot p_1) s - (\epsilon_4 \cdot p_3) t \Big) F_1 \wedge F_2 \wedge F_3 + (\sigma_i)\text{cyclic}$$

# Six-dimensions (Parity-odd)



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$$\begin{aligned} I_3[K^2] &= \frac{1}{2\epsilon} + \frac{1}{2}\left(3 - \gamma_E - \log[K^2]\right), \\ I_2[K^2] &= -\frac{K^2}{6\epsilon} + \frac{K^2}{18}\left(-8 + 3\gamma_E + 3\log[K^2]\right), \\ I_4(1,2,3,4) &= -\frac{\log^2 x + \pi^2}{2u}. \end{aligned}$$

$$A^{\text{odd}}(1,2,3,4) = F^4\left(\frac{(t-s)(\pi^2 + \log^2 x)}{12u^3} + \frac{\log x}{3u^2} + \frac{s-t}{18stu} + R^{\text{odd}}\right)$$

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There are no factorization channels if $str(T^4) = 0$, no $R^{\text{odd}}$

$$str(T^4) \neq 0 \rightarrow \begin{cases} str(T^4) = tr(tt)tr(tt) \\ Res_u[-\frac{F^4}{18ut}] = A_3 A_3 \end{cases}$$

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We are not done yet

$$\mathcal{A}_4 \xrightarrow{u=0} -\frac{F^4}{18ut}\left(tr(t_1 t_2)(t_3 t_4) + tr(t_1 t_3)(t_2 t_4) + tr(t_1 t_4)(t_3 t_2)\right) + \mathcal{O}(u^0)\,.$$

Clearly only $tr(t_1 t_3)(t_2 t_4)$ makes sense for the $u$ channel pole

$$\mathcal{R} = F^4 tr\left(tr(t_1 t_2)(t_3 t_4)\frac{u-t}{18stu} + tr(t_1 t_3)(t_2 t_4)\frac{t-s}{18stu} + tr(t_1 t_4)(t_3 t_2)\frac{s-u}{18stu}\right)\,.$$

With the above rational term, we now find:

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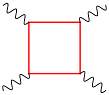
Where did this factor come from?



$$R^{anom} = -\frac{1}{18}\left[\left(\frac{(\epsilon_1 \cdot k_2)}{s} + \frac{(\epsilon_1 \cdot k_3)}{u} + \frac{(\epsilon_1 \cdot k_4)}{t}\right) F_2 \wedge F_3 \wedge F_4 + (\text{cyclic})\right]$$

The GS mechanism

$$\mathcal{L} = \epsilon^{abcdef} B_{ab} tr(F_{cd}F_{ef}) + H_{abc}H^{abc}, \quad H_{abc} = d_{[a}B_{bc]} + tr(A_{[a}d_b A_{c]})$$

Adding the GS term for $tr(t_1 t_3)tr(t_2 t_4)$

$$\begin{aligned} R^{anom} + GS &= \frac{-1}{18}\left[\left(\frac{(\epsilon_1 \cdot k_2)}{s} - 2\frac{(\epsilon_1 \cdot k_3)}{u} + \frac{(\epsilon_1 \cdot k_4)}{t}\right) F_2 \wedge F_3 \wedge F_4 + (\text{cyclic})\right] \\ &= F^4 \frac{t-s}{18stu} \end{aligned}$$

# Six-dimensions

$$\mathcal{R}=F^4 tr\left(tr(t_1t_2)(t_3t_4)\frac{u-t}{18stu}+tr(t_1t_3)(t_2t_4)\frac{t-s}{18stu}+tr(t_1t_4)(t_3t_2)\frac{s-u}{18stu}\right).$$

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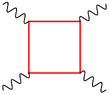

$$R^{anom} = -\frac{1}{18}\left[\left(\frac{(\epsilon_1\cdot k_2)}{s}+\frac{(\epsilon_1\cdot k_3)}{u}+\frac{(\epsilon_1\cdot k_4)}{t}\right)F_2\wedge F_3\wedge F_4+(\text{cyclic})\right]$$

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# Summary:

From the on-shell point of view, anomaly is simply the tension between unitarity and locality

- To reconcile the tension, the spectrum requires a new particle $\rightarrow$ the longitudinal degree of freedom.
- For higher dimensions, this new particle is a two form, Green-Schwarz mechanism
- The EFT is fine, just that there's a hidden UV cut off.