Lessons from gravitationally coupled fundamental scalars

Yu-tin Huang
w
Nima Arkani-Hamed (IAS), Tzu-Chen Huang (NTU) (In progress)

National Taiwan University

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Preulde

Non-renormalizable theories:

$$\mathcal{L} = \mathcal{L}_{marginal} + \alpha^n \mathcal{L}_n$$

- Determined by the details of the UV completion.
- IR imposes no constraint other than global symmetries

Prelude

Not true:

The existence of a UV completion → c_i of higher dimension operators must be Positive

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi

- Euler-Hiesenberg

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{c_2}{\Lambda^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \ldots$$

- DBI

$$\mathcal{L} = -f^4\sqrt{1-(\partial y)^2} = f^4\Big[-1 + \frac{(\partial y)^2}{2} + \frac{(\partial y)^4}{8} + \ldots\Big]$$

- String theory

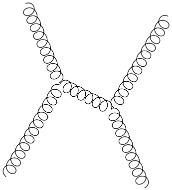
$$\mathcal{M}^{\text{Regge}}(s, t\to 0) = -\psi_2(1)\, s^4 + \frac{-\psi_4(1)}{192}\, s^6 + \frac{-\psi_6(1)}{92160}\, s^8 + \ldots$$

Prelude

Causality imposes further constraint:

$$\mathcal{L} = \int d^D x \sqrt{g}(R + \alpha_1 R^2 + \alpha_2 R^3 + \cdots)$$

Causality: if $\alpha_1, \alpha_2 \neq 0$, a particle traveling pass a shock wave will experience time advancement Camanho, Edelstein, Maldacena, Zhiboedov



$$\sim \frac{t^2 + \alpha_1^2 t^4 + \alpha_2^2 t^6}{s}$$

Can only be cured by an infinite $J > 2$ massive particles. (Assumes $\Lambda << M_p$)

Prelude

Can Unitarity impose further constraints? We've been here before

Prelude

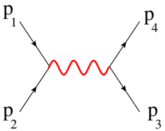
Fermi's four fermion theory

$$A(\phi_1, \phi_2, \bar{\phi}_3, \bar{\phi}_4) \sim -G_F s \sim E^2$$

Unitarity violation

$$s > \sqrt{\frac{16\pi}{G_F}} \sim 2.0 TeV$$

There is a new particle!



$$A(\phi_1, \phi_2, \bar{\phi}_3, \bar{\phi}_4) \sim -G_F s \to G_F M^2 \frac{s}{s - M^2}$$

Unitarity then requires

$$\frac{G_F M^2}{8\pi} \leq \frac{1}{2} \to M < 1 TeV$$

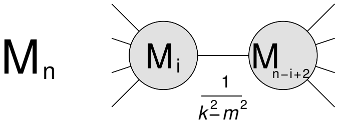
$M_W \sim 80 GeV$

Prelude

Multi-faceted consequence of unitarity:

- Froissart bound $M|_{t\to 0} < s \log^2 s$
- Optical theorem $\sigma = Im\left[M|_{t\to 0}\right]/E$

There presence of new states allow for constraints on factorization:



Does tree unitarity impose strong constraint on the spectrum of completion?

Prelude

Naively no

Example: Generalized Veneziano amplitudes Coons

$$\frac{\Gamma[1-t]\Gamma[1-s]}{\Gamma[2+u]} \to \prod_{n=0}^{\infty} \frac{u+\lambda_i+\lambda_0}{(s-\lambda_i)(t-\lambda_i)}$$

where $\lambda_n = \frac{\sigma^n - 1}{\sigma - 1} + \lambda_0$, free of ghosts for $\sigma < 1$

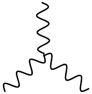
Similar result from deformed propagators Ho, Tian

$$\sum_{n=1}^{\infty} \frac{c_n}{k^2 - m_n^2} \to \sum_{n=1}^{\infty} \frac{c_n}{k^2} \sum_{\delta} \left(\frac{m_n^2}{k^2}\right)^{\delta}$$

$$\sum_n c_n m_n^2 = 0$$

Prelude

Would the existence of massless particles with three-point interactions be more constraining ?



$$\mathcal{L} = \int d^D x \sqrt{g}(R + \cdots), \quad D > 2$$

$$\mathcal{L} = \int d^D x F^2 + \cdots , \quad D > 4$$

$$\mathcal{L} = \int d^D x (\partial\phi)^2 + \phi^3 + \cdots , \quad D > 6$$

Prelude

For gravity

$$\mathcal{M}_4|_{s\to\infty} \sim \frac{s^2}{t} \sim E^2$$

- Assuming that the high energy behaviour is tamed while weakly coupled (true for many known examples in nature)
- Implies new degrees of freedom come in at tree-level
- Implies new degrees of freedom $M << M_{Plank}$

How constraining is the space of possible solutions?

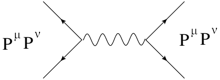
Prelude

With SUSY VERY

Unitarizing Gravity

Consider scalar exchange via gravity coupling:

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2 + t^2 + u^2)^2}{stu}$$



As $s \to \infty$ s/t fixed, same violation of unitarity bound at Plank scale

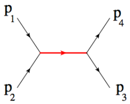
$$\sqrt{s} > 10^{19} GeV$$

Unitarizing Gravity

Let's consider modifying by adding one massive propagator

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

Unitarity imposes constraint on $f(s,t,u)$:



$$\to A_3(\phi_1, \phi_2, h^\ell) \times A_3(h^\ell, \phi_3, \phi_4)$$

Since $A_3(\phi_1, \phi_2, h^\ell) \sim i c_\ell (p_1 - p_2)^{\mu_1} (p_1 - p_2)^{\mu_2} \cdots (p_1 - p_2)^{\mu_\ell} \epsilon_{\mu_1 \mu_2 \cdots \mu_\ell}$ the residue must take the simple form:

$$[(p_1 - p_2) \cdot (p_3 - p_4)]^{2n} = (t-u)^{2n}$$

with $u = -t - s$ the residue must be a definite positive function in t:

$$n(t) = \sum_i a_i t^i, \quad a_i > 0$$

Unitarizing Gravity

There are also massless poles

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

The residue for massless poles are generated from local operators in the EFT description

$$\mathcal{L} \sim R\phi^2 + (\nabla\phi)^2 + R^3 + R^2$$

Such interactions are limited, and unique on-shell

$$\partial^\nu \phi \partial^\mu \phi \epsilon_{\mu\nu}$$

Unitarizing Gravity

The rules of the game:

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(s,t,u)_{2,sym}}{(s-m^2)(t-m^2)(u-m^2)}$$

- At low energies, i.e. if $\sqrt{s}$ is smaller than the new mass scale M, then the S-matrix must reduce to the known amplitude.
- At intermediate scales, i.e. $0 \leq s \leq M^2$, the residues of the poles must be definite negative function of t:

$$n(t) = \sum_i a_i t^i, \quad a_i > 0$$

- The factorization pole of $s = 0$, the residue must be the original tree residue $-t^2$

Unitarizing Gravity

$$Ans_1 = -\frac{(s^2+t^2+u^2)^2}{stu}\frac{a_1(t^2+u^2+s^2)+1}{(s-1)(t-1)(u-1)}$$

Enforcing massless residues,

$$Res[Ans_1]_{s=0} : -\frac{4t^2(1+2a_1t^2)}{(1-t)(1+t)} \rightarrow -4t^2$$

This fixes $a_1 = -\frac{1}{2}$

$$Ans_1 = -\frac{(s^2+t^2+u^2)^2}{stu}\frac{1-(s^2+t^2+u^2)/2}{(s-1)(t-1)(u-1)}$$

However, consider the residue at $s = 1$

$$Res[Ans_1]_{s=1} = \frac{1}{(t-1)(t+2)}\,.$$

The residue changes sign when $t \sim 1$

Unitarizing Gravity

$$Res[Ans_1]_{s=0} : -\frac{4t^2(1+2a_1 t^2)}{(1-t)(1+t)} \to -4t^2$$

$$Res[Ans_1]_{s=1} = \frac{1}{(t-1)(t+2)} .$$

- Unitarity requirements on the massless and massive poles are inconsistent with each other.
- Unitarity disaster is now at the scale of the massive particle.

Adding more particles end up with the same result No solution

Unitarizing Gravity

The persistent failure for ever larger spectrum hints at a fundamental tension in our constraints

The root of the problem

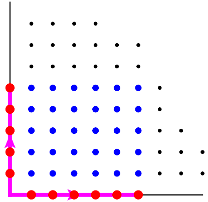
$$M \sim \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(s,t)}{(s-m_1)(t-m_2)\cdots}\Big|_{s=m_1} \rightarrow \frac{(s^2+t^2+u^2)^2}{stu} \frac{f(m_1,t)}{(t-m_2)\cdots}$$

Unitarity requires the function $f(m_1, t)$ to have a zero when $t = m_2$, and all other t-channel poles.

The root of the problem

$f(s,t)$ is a bounded polynomial function that has zero for each pair of $(s,t) = (m_i, m_j)$!

$$f(m_1, m_1) = f(m_1, m_2) = \cdots = f(m_i, m_j) = 0$$



There are more zeros than poles!

Unitarizing Gravity

Permutation invariance only makes things worst!

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
4	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3
5	1	0	1	1	2	2	5	4	8	9	13	15	23	24	34
6	1	0	1	2	4	6	13	19	36	58	97	149	244	364	558
7	1	0	1	2	4	8	20	36	83	169	344	680	1342	2518	4695
8	1	0	1	2	5	10	28	59	152	364	885	2093	4930	11199	25021
9	1	0	1	2	5	10	31	72	205	557	1565	4321	11942	32131	84927
10	1	0	1	2	5	11	33	81	246	722	2222	6875	21497	66299	202179

Unitarizing Gravity

Assume maximal susy

$$A^{\text{tree}} = \frac{\delta^{16}(Q)}{stu}$$

Requires modification of s, t and u channel by a single function:

$$Ansatz = A^{\text{tree}} \frac{n(s,t)}{\prod_i (s-a_i)(t-a_i)(u-a_i)}$$

Since there are requisite zeros in s, t, u, the simplest solution is:

$$Ansatz = A^{\text{tree}} \frac{\prod_{i,j}(u+b_{ij})(t+b_{ij})(s+b_{ij})}{\prod_i (s-a_i)(t-a_i)(u-a_i)}$$

with $b_{ij} = (a_i + a_j)$

Unitarizing Gravity

Now consider the residue of the massless pole:

$$-4t^2 \frac{\prod_{i,j}(-t+b_{ij})(t+b_{ij})(b_{ij})}{\prod_l(-a_l)(t-a_l)(-t-a_l)}$$

Massless constraint requires 1-1 map between b_{ij}s and a_l.

For a given a_i there will be a $b_{ii}=2a_i$, which requires the presence of $a_{i+1}=2a_i$ in the propagators. This then induces a $b_{ii+1}=3a_i$, which again requires the presence of $3a_i$ in the propagators......

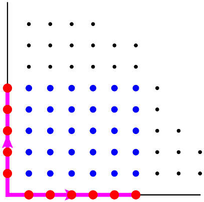
The minimum solution is:

$$A^{\text{tree}} \prod_{i\in N}\left(\frac{(s+i\alpha)(t+i\alpha)(u+i\alpha)}{(s-i\alpha)(t-i\alpha)(u-i\alpha)}\right)$$

Unitarizing Gravity

$$A^{\text{tree}} \prod_{i \in N} \left(\frac{(s+i\alpha)(t+i\alpha)(u+i\alpha)}{(s-i\alpha)(t-i\alpha)(u-i\alpha)} \right)$$

this solution solves another problem



The only way possible is if some zeros are shared by multiple double poles!
For single massive pole m_1^2, two zeros

$$(s=0, t=m_1^2) \to (u=-m_1^2), \quad (s=m_1^2, t=m_1^2) \to (u=-2m_1^2)$$

Let $(u=-2m_1^2)$ be shared

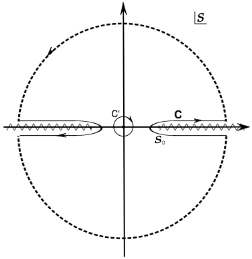
$$(s=0, t=2m_1^2) \to (u=-2m_1^2), \quad (s=2m_1^2, t=0) \to (u=-2m_1^2)$$

Again m^2 has spread across integers

Perturbative completion

Consider the amplitude for some fixed $t = t^*$, which we express in the form of a dispersion relation

$$M(s, t^*) = \int_{v=s} dv \frac{M(v, t^*)}{v - s} = -\frac{Res[M(v, t^*)]|_{v=v^*}}{v^* - s}$$



The residues in the complex s-plane lies on the real axis, where poles in the positive region are s-channel resonance, and negative region are from u-channel resonance. Due to permutation invariance, the residue of a given s-channel resonance, say $s = n$, there will be the opposite of the u-channel resonance in the negative s-branch, $s = -n - t^*$

$$M(s, t^*) = -\sum_{n=0}^{\infty} \frac{Res[M(v, t^*)]|_{v=n}(2n + t^*)}{(n - s)(n + t^* + s)}$$

Perturbative completion

$$M(s,t^*) = -\sum_{n=0}^{\infty} \frac{Res[M(v,t^*)]|_{v=n}(2n+t^*)}{(n-s)(n+t^*+s)}$$

Now, consider the case where $t^* = -2$, then we have:

$$\begin{aligned} M(s,-2) \quad = \quad & -\frac{Res[M(v,-2)]|_{v=0}(-2)}{(-s)(-2+s)} - \frac{Res[M(v,-2)]|_{v=1}(0)}{(1-s)(-1+s)} \\ & -\frac{Res[M(v,-2)]|_{v=2}(2)}{(2-s)(s)} - \sum_{n=3}^{\infty} \frac{Res[M(v,-2)]|_{v=n}(2n-2)}{(n-s)(n-2+s)} \,. \end{aligned} \quad (1)$$

There are no poles at $s = 0, 1$! For $t = -n$ the poles of $s = 0, 1, \cdots, n$ are missing

$$Res[M(s,t)]|_{s=0} = \prod_{i=1}^{\infty}(t+i)$$

But this is impossible for bounded high-energy behavior $\rightarrow$ The S-matrix must have zeros in the unphysical channel, at $s = -n$

Perturbative completion

The S-matrix must have zeros in the unphysical channel, at $s = -n$

$$M_4 \sim \frac{\prod_i (s+i)(t+i)(u+i)}{\prod_i (s-i)(t-i)(u-i)} \sim \frac{\Gamma[-s+1]\Gamma[-t+1]\Gamma[-u+1]}{\Gamma[s+1]\Gamma[t+1]\Gamma[u+1]}$$

Unitarizing Gravity

The solution is given as:

$$A(s,t)\,\frac{\prod_{i=1}^{\infty}(s+i)(t+i)(u+i)}{\prod_{i=0}^{\infty}(s-i)(t-i)(u-i)}$$

$$(s^2+t^2+u^2)\,\frac{\Gamma[-\alpha' u]\Gamma[-\alpha' t]\Gamma[-\alpha' s]}{\Gamma[1+\alpha' t]\Gamma[1+\alpha' u]\Gamma[1+\alpha' s]}$$

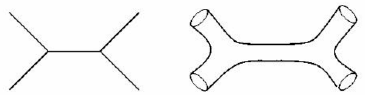


Fig.3: Particle scattering processes (left), string scattering processes (right).

Type II superstring

Generalizations

Another solution: unitarize each massless pole separately:

$$\frac{(s^2+t^2+u^2)^2}{stu} = (s^2+t^2+u^2)\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}\right)$$

The absence of massless poles in other channels leads to the absence of zeros

$$(s^2+t^2+u^2)\frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+t]\Gamma[1+u]}\left(\frac{tu}{1+s}+\frac{su}{1+t}+\frac{st}{1+u}\right)$$

Heterotic superstring

Unitarizing Gravity

How unique is the answer ?

Gegenbauerology

There is much stronger constraint for the massive residues

$$A_3(\phi_1, \phi_2, h^\ell) A_3(\phi_3, \phi_4, h^\ell) = -(t-u)^{2k}$$

$$n(t) = \sum_\ell a_\ell t^\ell, \qquad a_\ell > 0$$

Project the residue into irreducible representation

$$n(t) = n\left(-\frac{s(1-\cos\theta)}{2}\right) = \sum_\ell c_\ell C_\ell^\alpha(\cos\theta), \quad c_\ell > 0$$

where $\alpha = (D-3)/2$

$$\frac{1}{(1-2x\cos\theta + x^2)^\alpha} = \sum_{\ell=0}^{\infty} x^\ell C_\ell^\alpha(\cos\theta),$$

Gegenbauerooligy

$\sum_\ell c_\ell C_\ell^\alpha(\cos\theta), \quad c_\ell > 0 \quad \rightarrow$ positive function

- A positive function in D-dimensions is positive in lower dimensions ($D = 2 \cos n\theta$)
- If f($\cos\theta$) is a positive function then

$$|f(x)| < f(1)$$

- Rescaling makes the function "more positive"

$$C_\ell^\alpha((1+\epsilon)x) - C_\ell^\alpha(x) = \sum_\ell a_\ell C_\ell^\alpha, \quad a_\ell > 0$$

since

$$\frac{d^n}{dx^n}\frac{1}{(1-2tx+t^2)^\alpha} = 2^n t^n \frac{\prod_{i=1}^n (a-1+i)}{(1-2tx+t^2)^{\alpha+n}}$$

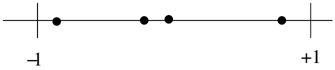
Gegenbaueroligy

What are the possible form of the residue?

$$f(x) = \prod_i (x + a_i)$$

- Permutation invariance $t = -\frac{s(1-x)}{2}$ $u = -\frac{s(1+x)}{2}$: the zeros distribute symmetrically
$$f(x) = f(-x)$$
- The zeros must satisfy $-1 \leq a_i \leq 1$, if not, rescale $x \to xa_i$ should give a more positive function. But

$$(x - a_i) \to a_i(x - 1) \quad \leftarrow \text{negative function}$$



A new look at string theory

Consider open superstring residue for $s = n, n \in odd$ ($x = \cos\theta$):

$$(t+1)(t+2)\cdots(t+n-1) \quad \rightarrow \quad \left(x^2-\frac{1}{n^2}\right)\left(x^2-\frac{9}{n^2}\right)\cdots\left(x^2-\frac{(n-2)^2}{n^2}\right)$$

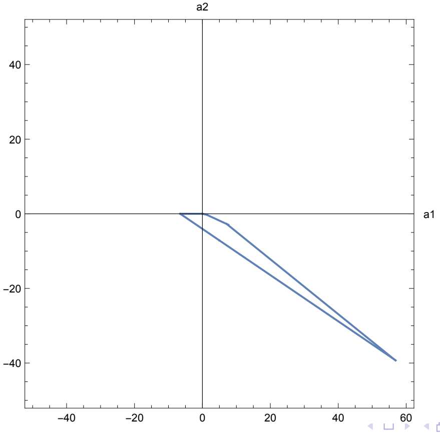
This must be a positive function

- For $n = 3$:
$$\left(x^2 - \frac{1}{9}\right) \leftrightarrow c_2 C_2^D + c_0 C_0^D = c_2(x^2 - \frac{1}{D}) + c_0$$
  The critical dimension is $D = 9$
- As $n \to \infty$ it's a boundary positive function
- The numerator is a prime positive function near critical dimensions

Gegenbaueroligy

$$Ansatz = M^{\text{String}}(1 + \sigma_3(a_1 + a_2\sigma_2))$$
$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

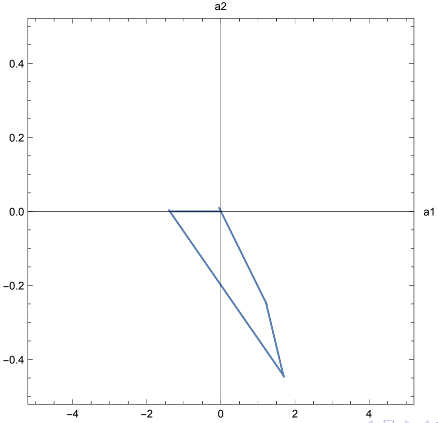
s=1



Gegenbaueroligy

$$Ansatz = M^{\text{String}}(1 + \sigma_3(a_1 + a_2\sigma_2))$$
$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

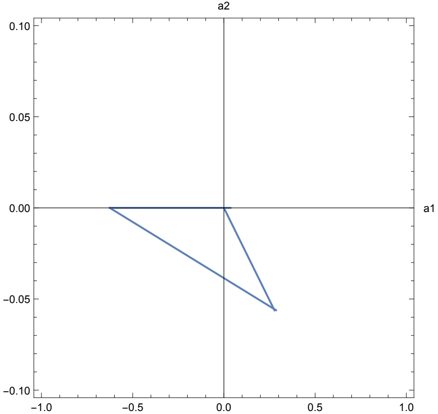
s=2



Gegenbaueroligy

$$Ansatz = M^{\text{String}}(1 + \sigma_3(a_1 + a_2\sigma_2))$$
$$\sigma_3 = stu, \quad \sigma_2 = (s^2 + t^2 + u^2)$$

s=3



Gegenbaueroligy

$$|f(\cos\theta)| \leq f(1)$$

But

$$stu = \frac{s^3}{4}(1 - \cos\theta^2)$$

So anything with an *stu* factor cannot be positive!!!!

Gegenbaueroligy

A general solution:

$$A_4 = A_4^{\text{Tree}} \frac{f(\sigma_3, \sigma_2)}{\prod_i (s - a_i)(t - a_i)(u - a_i)}$$

$$\sigma_3 = stu, \;\; \sigma_2 = (s^2 + t^2 + u^2)$$

Massless residue fixes the purely σ_2 part:

$$f(\sigma_3, \sigma_2) = \{String\} + \sigma_3 \Delta$$

As $n \to \infty$ $\quad \{String\} \to$ near negative!

Conclusions

- Assuming tree-unitarization while gravity is weakly coupled, and infinite tower of massive spin is necessary
- Their mass^2 must have integer spacing.
- Positivity of Gegenbauer expansion renders the solution almost unique

A new look at string theory

$$(t+1)(t+2)\cdots(t+n-1) \quad \rightarrow \quad \left(x^2-\frac{1}{n^2}\right)\left(x^2-\frac{9}{n^2}\right)\cdots\left(x^2-\frac{(n-2)^2}{n^2}\right)$$

Does there exists a proof of positivity without the world-sheet?

- Since

$$xC_n^4 = \frac{1}{2}\left(C_n^4 + C_{n-2}^4\right)$$

$$M_1 M_2 \cdots M_{n-1}, \qquad M = \begin{pmatrix} a_i & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & a_i & \frac{1}{2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{2} & a_i \end{pmatrix}, \qquad a_i = \frac{n-2i}{n}$$

their product yields a positive matrix.

Gegenbaueroligy

- Since

$$xC_n^4 = \frac{1}{2}\left(C_n^4 + C_{n-2}^4\right)$$

$$M_1 M_2 \cdots M_{n-1}, \qquad M = \begin{pmatrix} a_i & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & a_i & \frac{1}{2} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{2} & a_i \end{pmatrix}, \qquad a_i = \frac{n-2i}{n}$$

Equivalently