On shell Diagrams and Amplitude

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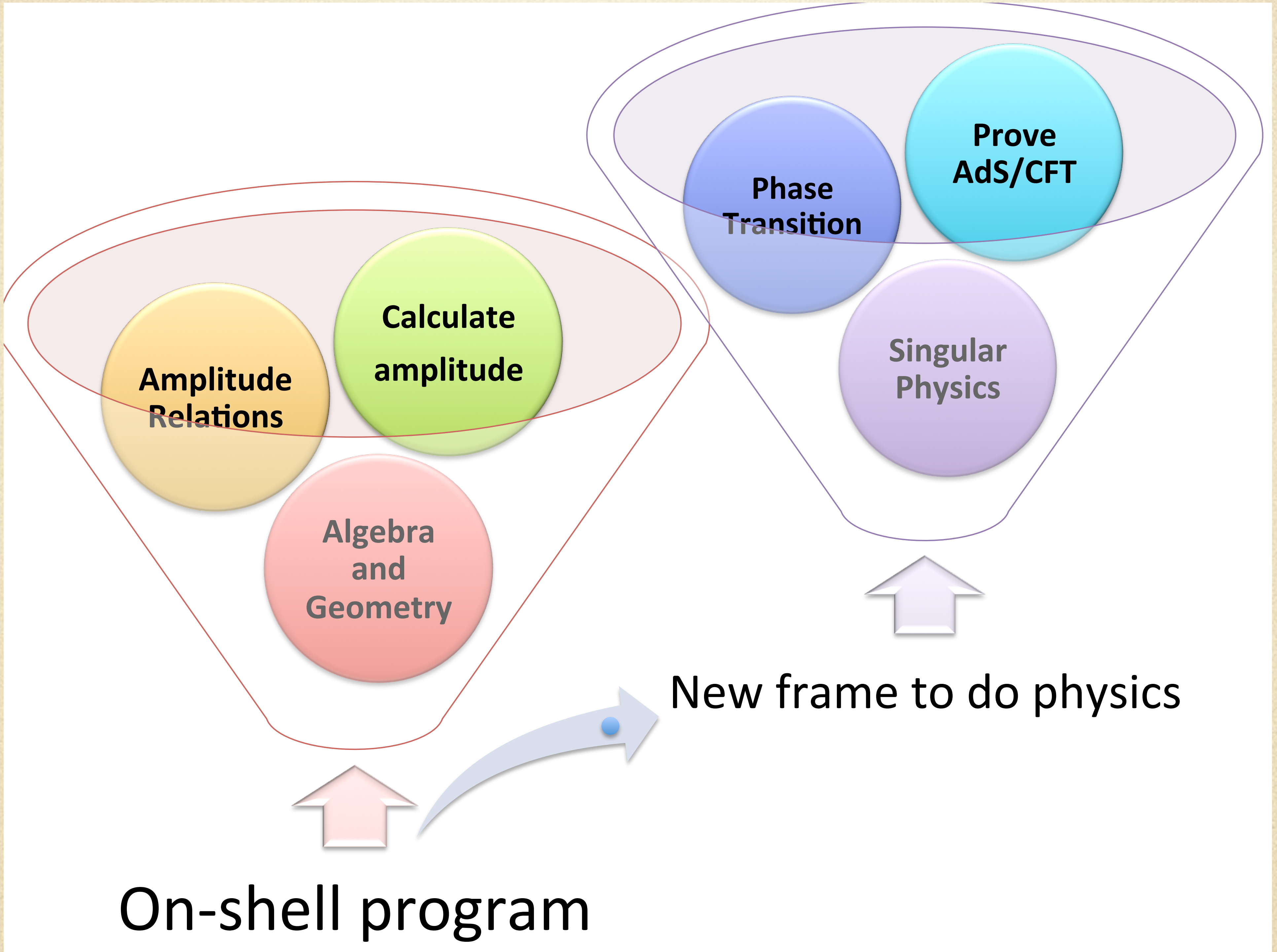
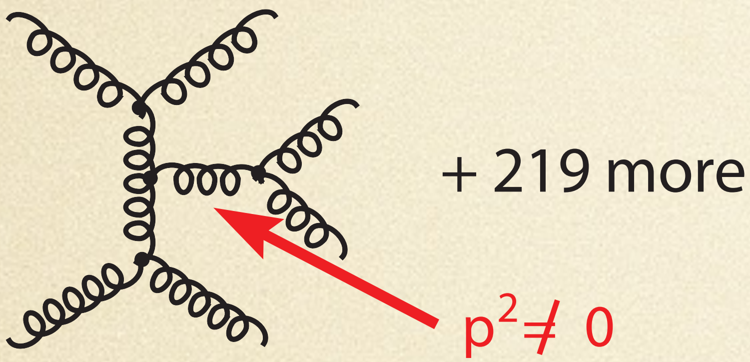
Arkani-Hamed 1205.5605, 1412.8475

Chen et al: :1401.6610,1411.3889, 1507.03214

Franco et all: 1502.02034

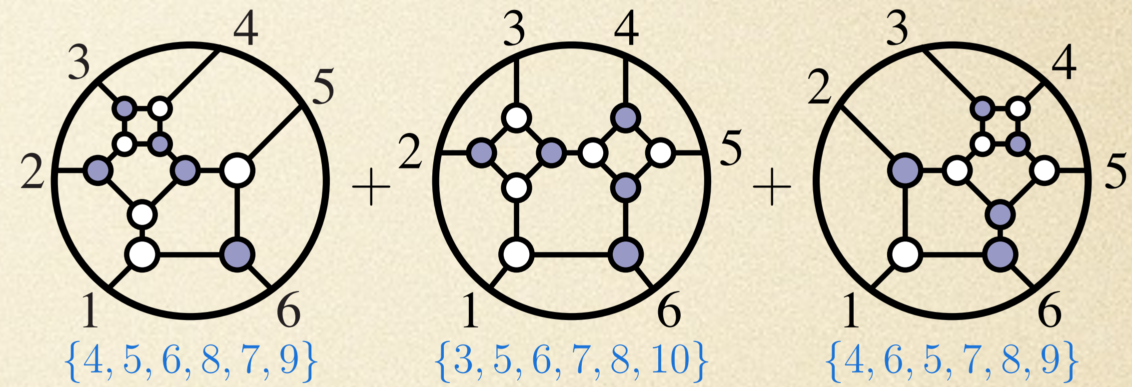

Prove
AdS/CFT
Phase
Transition
Singular
Physics
Calculate
amplitude
Amplitude
Relations
Algebra
and
Geometry
New frame to do physics
On-shell program

Traditional way



- **off-shell states**
- **Gauge-dependent**
- **locality**
- **Explosive growth of number of terms**

Modern on-shell method



- **on-shell states**
- **Gauge invariant**
- **non-locality**
- **Slow growth of number of diagrams/terms**

How to justify a good/worse representation?

- Simple
- Make the symmetry obvious

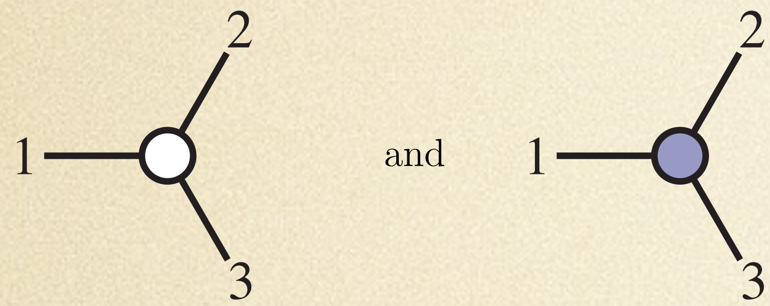
Example: Why people like Dimensional regularization?

It preserve the gauge symmetry and Lorenz group symmetry

Why we need on-shell diagram and top-form?

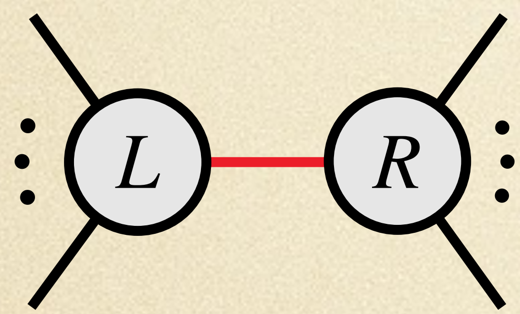
- It make the gauge symmetry and Yangian symmetry obvious

On-shell diagrams in N=4 SYM (1212.5605)–Nima Arkani-Hamed et al

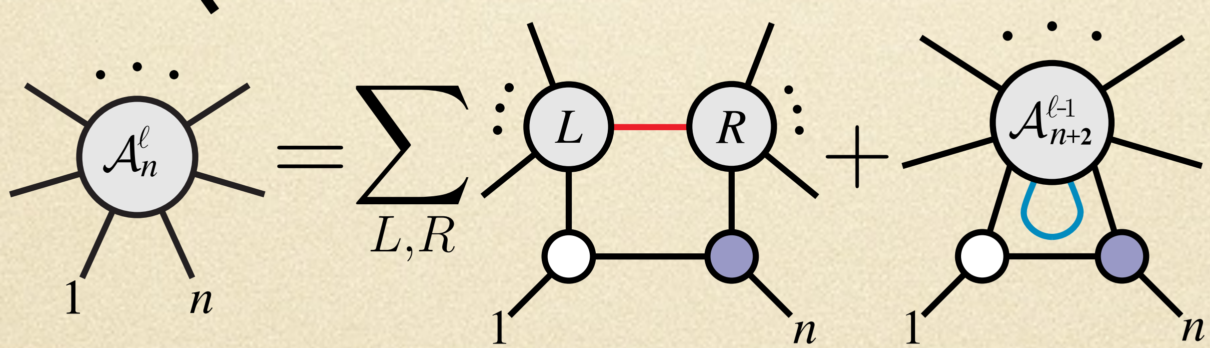


$$\mathcal{A}_3^1 = \int \frac{d^{1\times3}C}{vol(GL(1))}\frac{\delta}{(1)(2)(3)}$$

$$\mathcal{A}_3^2 = \int \frac{d^{2\times3}C}{vol(GL(2))}\frac{\delta}{(12)(23)(34)}$$



$$\int \frac{d^2\lambda d^2\tilde{\lambda}}{vol(GL(1))} d^4\tilde{\eta}$$



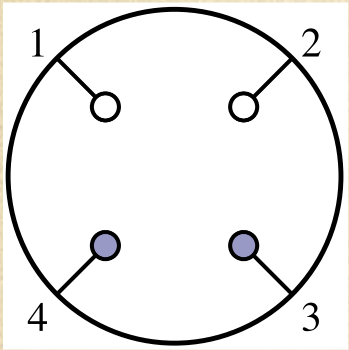
$$\mathcal{A}(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a; \lambda_b, \tilde{\lambda}_b, \tilde{\eta}_b) = \frac{d\alpha}{\alpha}\mathcal{A}_0(\lambda_{\hat{a}}, \tilde{\lambda}_{\hat{a}}, \tilde{\eta}_{\hat{a}}; \lambda_{\hat{b}}, \tilde{\lambda}_{\hat{b}}, \tilde{\eta}_{\hat{b}})$$

A warm up example: Four point amplitude

Adding Bridge (a,b)

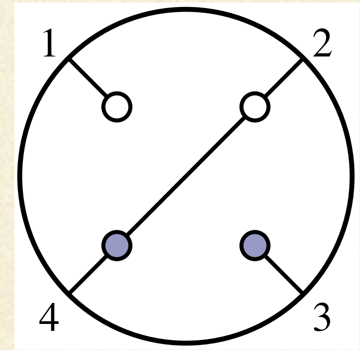
$c_b \to c_b + \alpha c_a$

Grassmannian Matrix



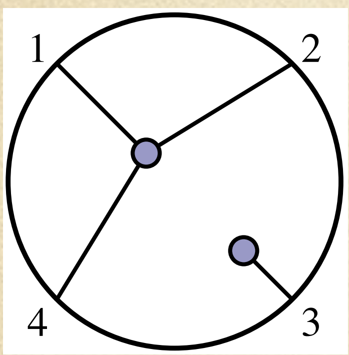
$$\begin{matrix} & c_1 & c_2 & c_3 & c_4 \\ \begin{matrix}1\\2\end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}} \end{matrix}$$

$\xrightarrow{(2,4)}$



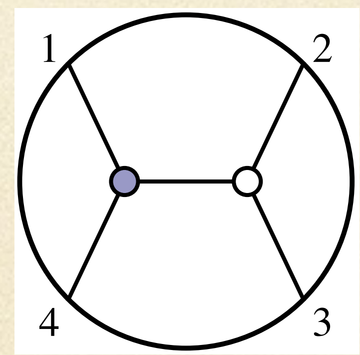
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_1 \end{pmatrix}$$

$(1,2)$



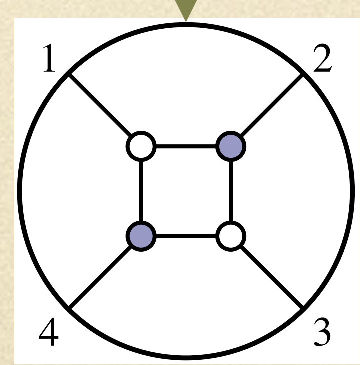
$$\begin{pmatrix} 1 & \alpha_2 & 0 & 0 \\ 0 & 1 & 0 & \alpha_1 \end{pmatrix}$$

$\xrightarrow{(2,3)}$



$$\begin{pmatrix} 1 & \alpha_2 & \alpha_2\alpha_3 & 0 \\ 0 & 1 & \alpha_3 & \alpha_1 \end{pmatrix}$$

$(1,2)$



$$\begin{pmatrix} 1 & \alpha_2+\alpha_4 & \alpha_2\alpha_3 & 0 \\ 0 & 1 & \alpha_3 & \alpha_1 \end{pmatrix}$$

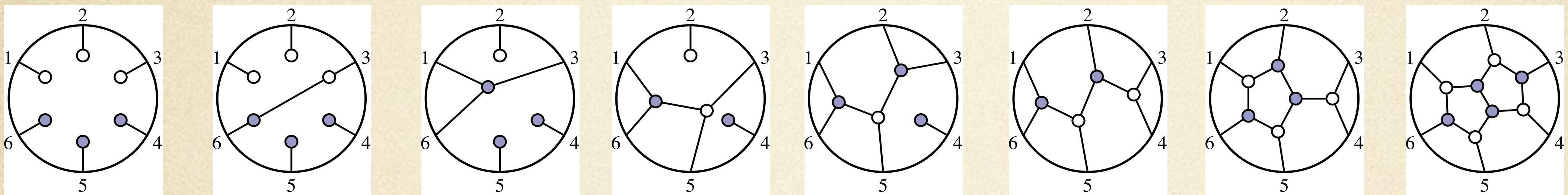
Finally we can get the four point tree-level amplitude

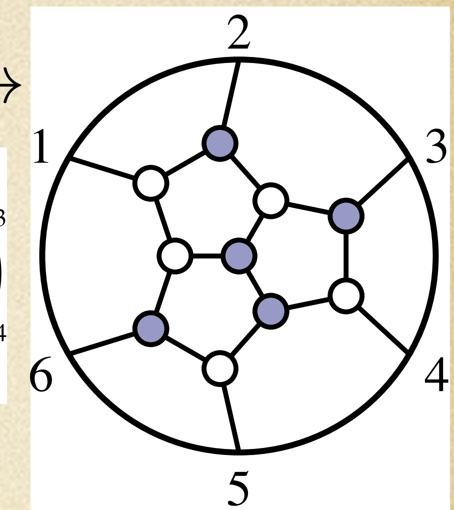
$$\mathcal{A} = \int \frac{dC^{2\times 4}}{\mathrm{Vol}(GL(2))} \frac{\delta^{2\times 4}(C\cdot\widetilde{\eta})}{(12)(23)(34)(41)} \delta^{2\times 2}(C\cdot\widetilde{\lambda})\delta^{2\times(2)}(\lambda\cdot C^{\perp})$$

Six Point amplitude

Adding the BCFW bridges reductively

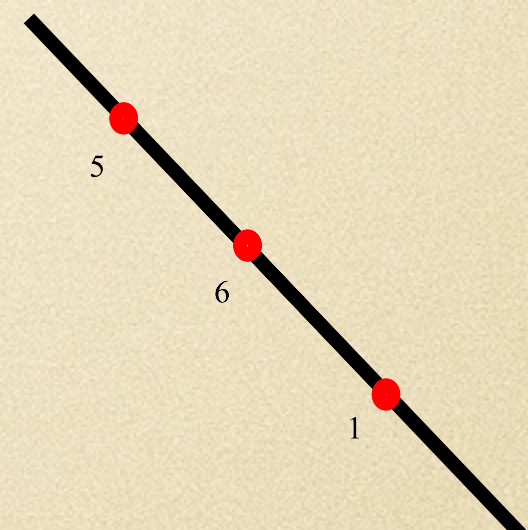
$$(3,6)\to(1,3)\to(3,5)\to(2,3)\to(3,4)\to(1,2)\to(2,3)\to(1,2)\to$$





$$\begin{pmatrix} 1 & \alpha_6+\alpha_8 & \alpha_2+\alpha_6\alpha_7 & \alpha_2\alpha_5 & \alpha_2\alpha_3 & 0 \\ 0 & 1 & \alpha_4+\alpha_7 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 0 & 1 & \alpha_5 & \alpha_3 & \alpha_1 \end{pmatrix}$$

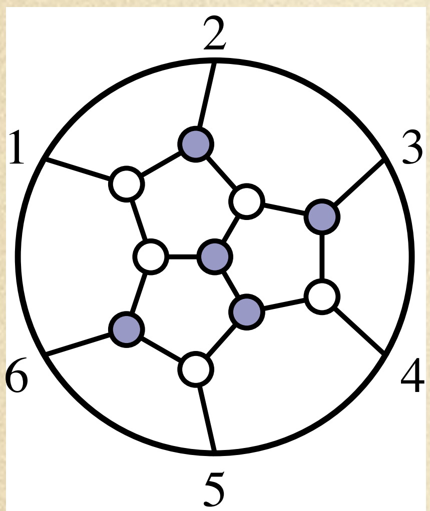
- This only lie in the sub-manifold of the Grassmannian. The constraint is Γ: $(561)=0$

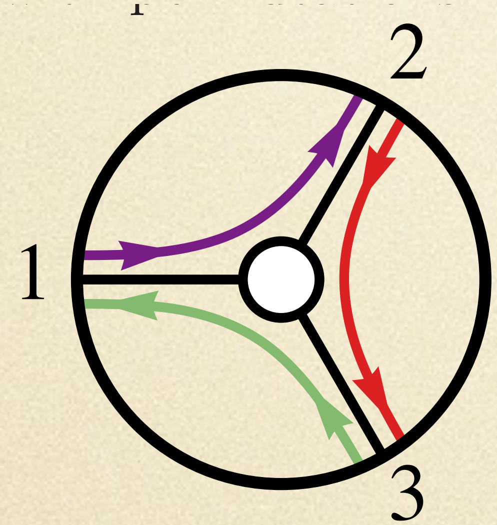


The top-form is just

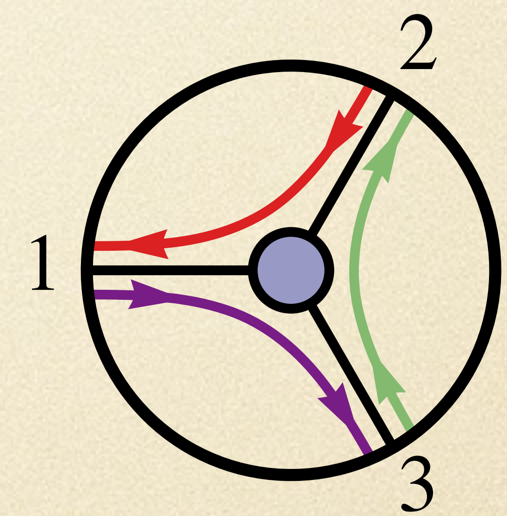
$$\mathcal{A}=\oint_\Gamma \frac{dC^{3\times 6}}{\mathrm{Vol}(GL(k))}\frac{\delta^{3\times 4}(C\cdot\widetilde{\eta})}{(123)(234)(345)(456)(561)(612)}\delta^{3\times 2}(C\cdot\widetilde{\lambda})\delta^{2\times 3}(\lambda\cdot C^{\perp})$$

Geometry and Permutation





$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix}$ and



$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix}$

$$\begin{pmatrix} 1,2,3,4,5,6 \\ 4,5,6,8,7,9 \end{pmatrix} \rightarrow \begin{cases} \Gamma : (561) = 0 \\ f(C) = \dfrac{1}{(123)(234)(345)(456)(561)(612)} \end{cases}$$

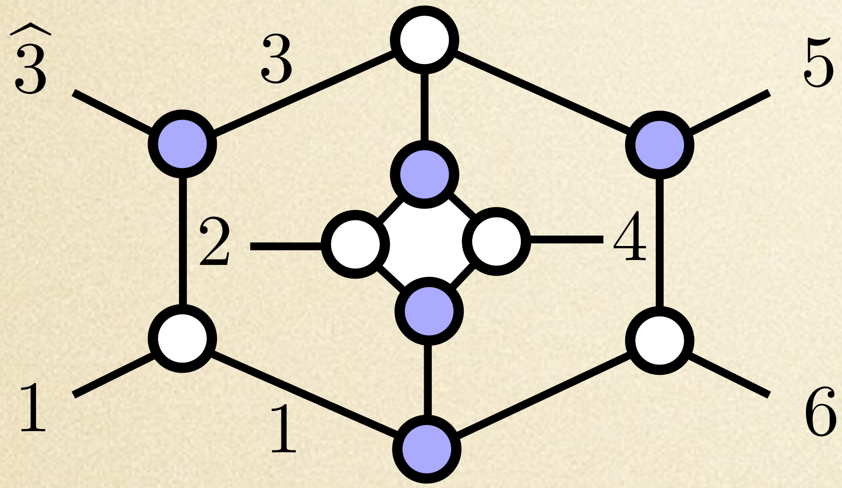
Grassmannian Integral Form

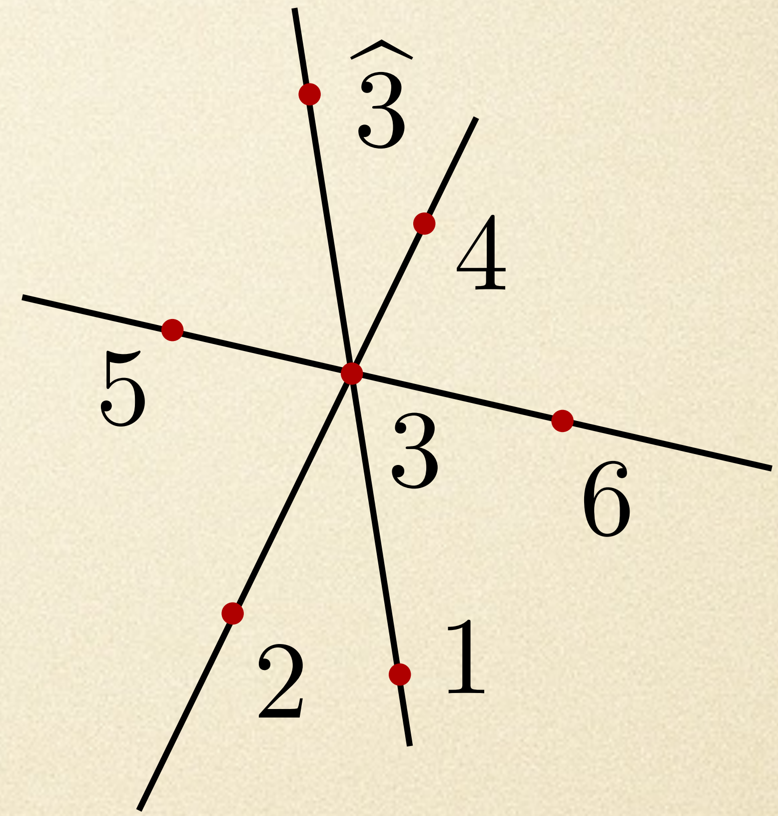
$$\mathcal{A}_n^k = \oint_{C\subset\Gamma} \frac{d^{k\times n}C}{vol(GL(k))} \frac{\delta^{k\times 4}(C\cdot\tilde{\eta})}{f(C)} \delta^{k\times 2}(C\cdot\tilde{\lambda})\delta^{2\times(n-k)}(\lambda\cdot C_\perp^\top),$$

- G(k,n) is the space of k-dimensional planes in
- Γ is constrained by some linear relations among the columns of C
- quadratic constraints $\delta(\sum_{i=1}^n \lambda_i\tilde{\lambda}_i) \xrightarrow{linearize} \delta(C\cdot\tilde{\lambda})\delta(\lambda\cdot C_\perp^\top)]$
- Classify by concepts and tools in algebra geometry (ideal, intersections, Grobner basis, Free modular decomposition)
- f(C) is any rational function of minors of C. For planar diagram, the f(C) are simply

$$(1\cdots k)(2\cdots k+1)\cdots(k12\cdots k-1)$$

Non-planar on-shell diagrams I: tangled geometric constraints





Constraints

$$(234) = 0, (356) = 0$$

Integrand

$$\frac{(361)}{(123)(234)(345)(356)(146)(561)(612)}.$$

(1,3)

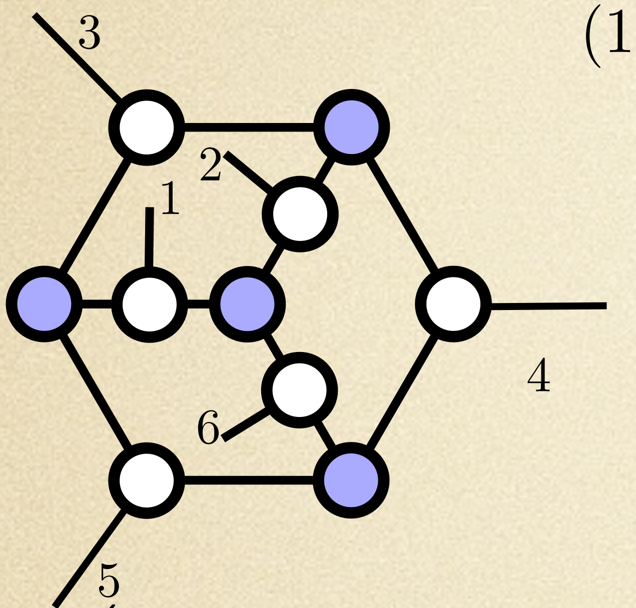
$$\frac{(234)}{(214)} - \frac{(356)}{(156)} = 0$$

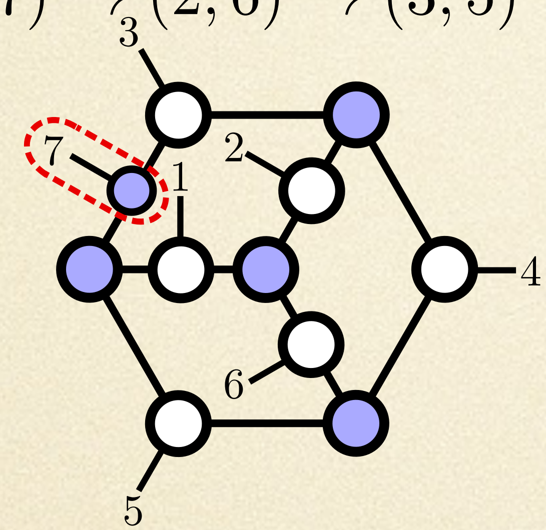
$$\big((1\hat{3}) \cap (24)56\big)$$

$$\frac{(214)^2(361)}{(123)(234)((13)\cap(24)56)((13)\cap(24)45)(146)(561)(612)}$$

Non-planar on-shell diagrams II: No-bridge diagram

$$(1,7) \to (2,6) \to (3,5) \to (2,3 \to (3,4) \to (12) \to (2,3) \to (5,7) \to (3,7)$$





$$\frac{(135)^2}{(123)(234)(345)(456)(156)(357)(157)(126)(137)}.$$

$$(234)^2,\ (456)^2, (126)^2$$

$$\left(\begin{array}{cccccc|c} & & & C & & & 0 \\ & & & & & & 0 \\ \hline 0 & 0 & c_3 & c_4 & c_5 & c_6 & 1 \end{array}\right).$$

$$\begin{array}{ll} (126) = (12)c_6 & (234) = -c_3(24) + c_4(23) \\ (123) = (12)c_3 & (456) = c_4(56) - c_5(46) \\ (156) = -(16)c_5 & (135) = -c_3(15) + c_5(13) \\ (137) = (13) & (345) = c_3(45) - c_4(35) + c_5(34) \\ (357) = (35) & (157) = (15)\ , \end{array}$$

Cycle integrating $(234)^2,\ (456)^2, (126)^2$ for c_3, c_4, c_5, c_6

$c_6 \to 0, c_3 \to \frac{(23)}{(24)} c_4, c_5 \to \frac{(56)}{(46)} c_4$ Now everything is the rank two minor

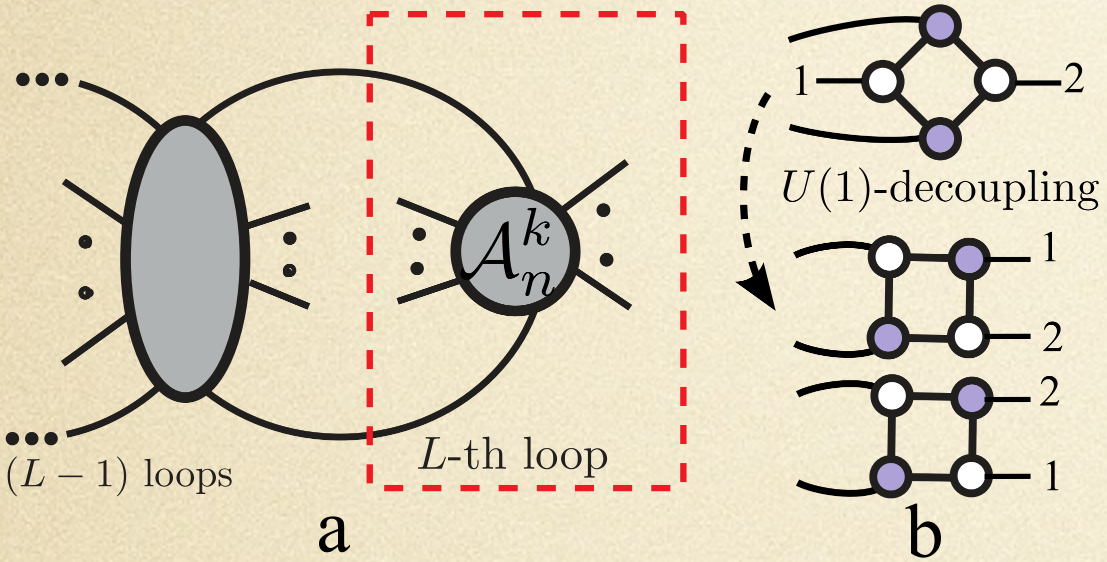
$$\frac{[(56)(13)(24) - (23)(15)(46)]^2}{(23)(12)(24)(26)(45)(34)(16)(56)(35)(15)(13)(46)},$$

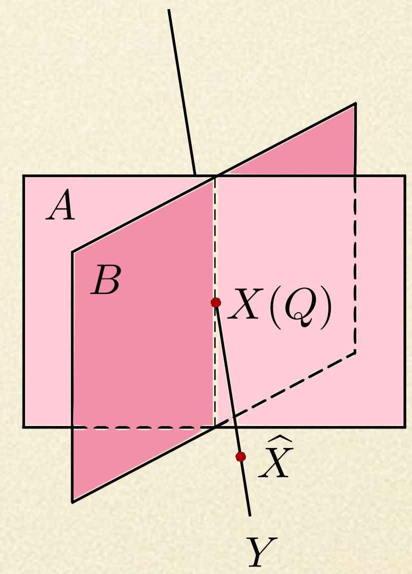
No Constraints left, MHV

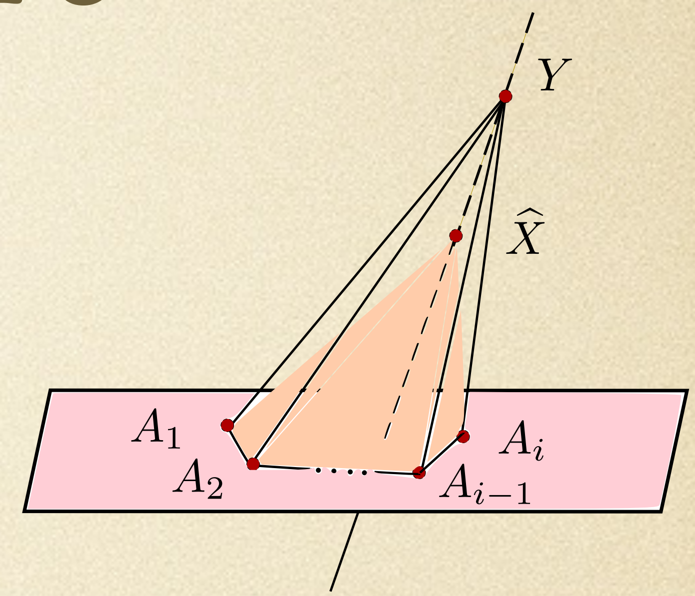
MHV: Any non-planar amp can be taken as summation of planar amps

$$-f_p(125364) + f_p(125463) - f_p(134265) + f_p(126543) - f_p(132465) - f_p(123564),$$

Non-planar on-shell diagrams III: General Procedure







General method to obtain the integrand

$$f(\widehat{C}) = M_0(\widehat{X}) \prod_i M_i(\widehat{X} - \alpha Y) \times \left(\begin{array}{c} \text{minors} \\ \text{without } \alpha \end{array} \right)$$

where

$$\alpha = M_0(\widehat{X})/R(Y)$$

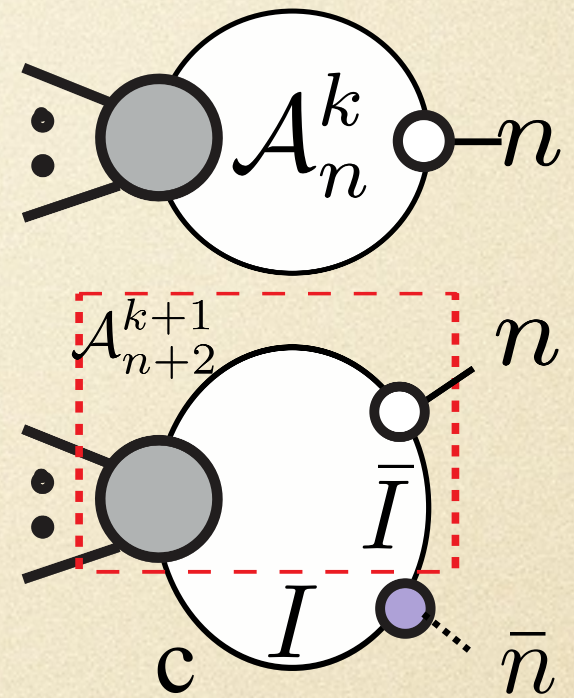
$$F_i(C) \to F_i(\widehat{C}) = F_i(C) + F_i'(C)\alpha = F_i'(\widehat{C})\alpha$$

Constraints

$$F_i(\cdots \widehat{X} - \alpha Y \cdots) = 0$$

Non-planar on-shell diagrams IV: General Procedure

$$C' = \left(\begin{array}{ccccc|c} & & C & & & \begin{array}{c}0\\ \vdots \\ 0\end{array} \\ \hline 0 & \cdots & 0 & c_{k+1} & \cdots \quad c_n & 1 \end{array}\right)$$



$$\int \frac{\mathcal{A}_{n+2}^{k+1} d\Omega_I \langle I\bar{I}\rangle^3}{\langle \bar{n} I\rangle^2 \langle \bar{I}\bar{n}\rangle} \delta([\bar{n}I])\delta^4(\widetilde{\eta}_{\bar{n}} + \frac{\langle \bar{I}\bar{n}\rangle}{\langle \bar{I}I\rangle}\widetilde{\eta}_I), \tag{1}$$

$$\oint_{\bar{\Gamma}} \frac{dC^{k\times n} d\vec{c}}{\mathrm{Vol}(GL(k))} \frac{(1\cdots k)\bar{\delta}}{f(C,c_i)} \delta^2(\sum_{i=k+1}^{n,\bar{n}} c_i\tilde{\lambda}_i)\delta^4(\sum_{i=k+1}^{n,\bar{n}} c_i\widetilde{\eta}_i), \tag{2}$$

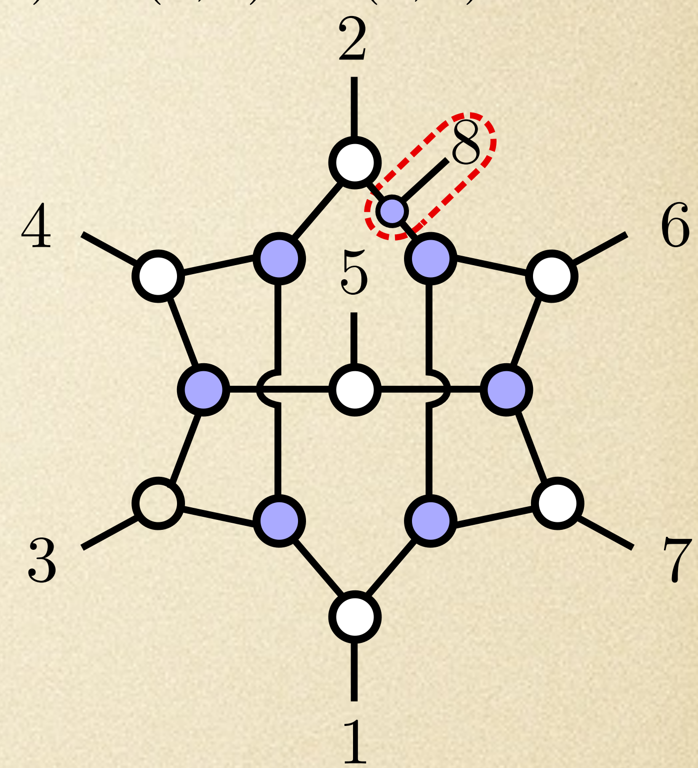
$$\mathcal{A}_n^k = \oint_{\bar{\Gamma}} \frac{dC^{k\times n} dc_{k+1}\cdots dc_n}{\mathrm{Vol}(GL(k))} \frac{(1\cdots k)\delta^{k\times 4}(C\cdot\widetilde{\eta})}{f(C,c_i)} \times$$

$$\times \delta^{k\times 2}(C\cdot\widetilde{\lambda})\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})\delta(\sum_{i=k+1}^{n} c_i \frac{[i1]}{[\bar{n}1]} + 1)$$

An NMHV Example

$(2,8) \to (6,8) \to (4,2) \to (1,2 \to (2,3) \to (2,4) \to (4,5) \to (4,6) \to (6,7) \to (1,6) \to (6,8)$

$$\frac{1}{f_p}\,\frac{(134)(357)[(457)(126)-(456)(127)]^3}{(124)(126)(135)(145)(267)(367)(457)^2}$$



where $f_p = (123)(234)(345)(456)(567)(671)(712).$

For NMHV, there are geometry constraints

$(345)^2, (567)^2.$

After a long calculation we can simplify it

$$\frac{-1}{(124)(126)(257)(134)(345)(567)(367)} \qquad \frac{1}{(124)(127)(167)(346)(235)(345)(567)}$$

$$\frac{-1}{(124)(126)(167)(234)(357)(345)(567)} \qquad \frac{1}{(124)(127)(134)(267)(356)(345)(567)}$$

$$\frac{1}{(123)(126)(143)(467)(275)(345)(567)} \qquad \frac{1}{(123)(127)(143)(267)(456)(345)(567)}$$

$$\frac{1}{(123)(126)(243)(467)(175)(345)(567)} \qquad \frac{1}{(123)(127)(234)(167)(456)(345)(567)}.$$

But non of them correspond to planar amplitude

Thanks