An Effective Field Theory for Jet Processes

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第二届高精度物理学术研讨会

In collaboration with T. Becher, M. Neubert & L. Rothen
(PRL116(2016)192001, arXiv:1605.02737, work in progress)

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DESY

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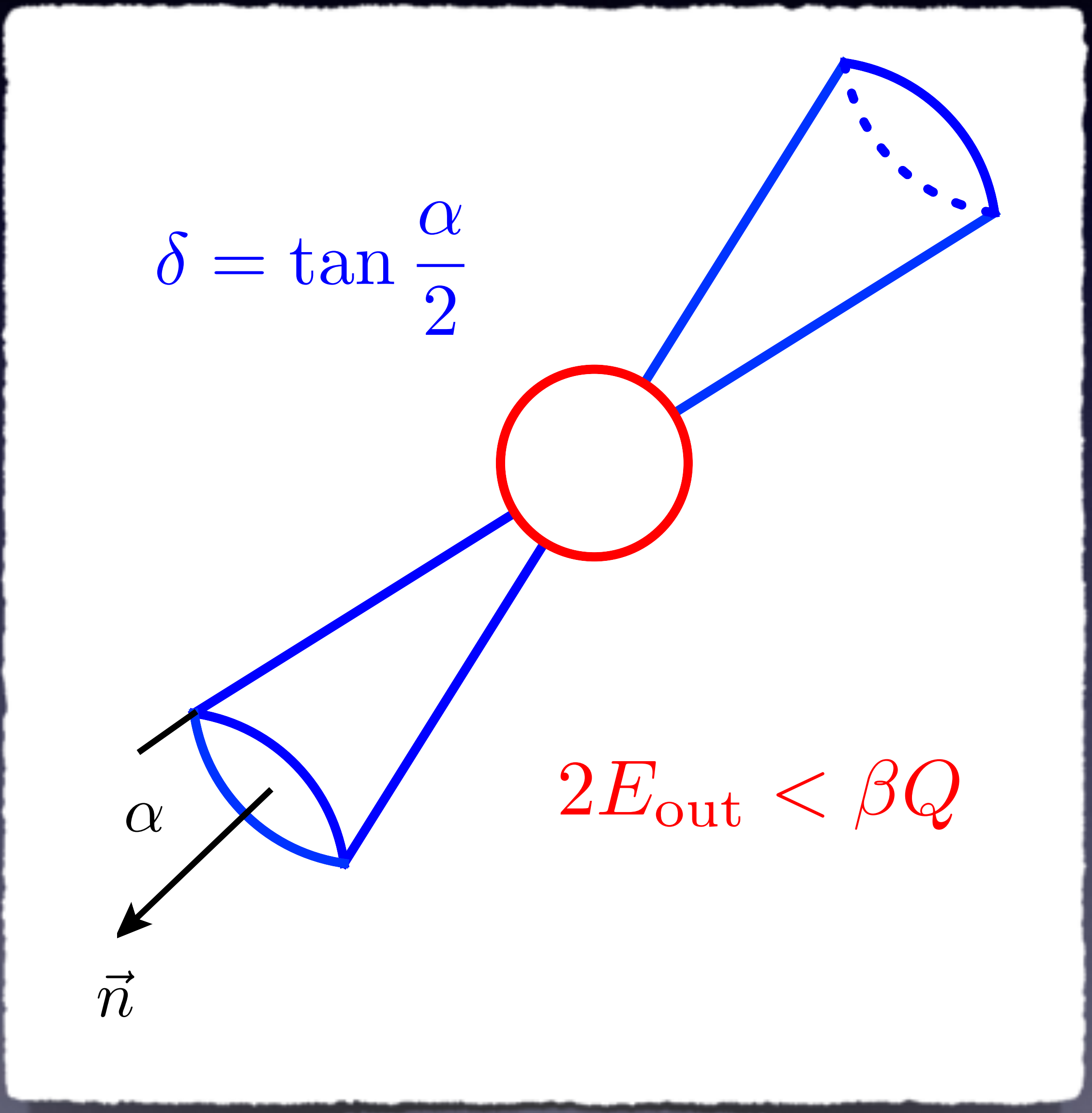
AEC ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

Sterman-Weinberg Jets

(Sterman & Weinberg 1977)



$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{3\pi}\left[-16\ln\delta\ln\beta - 12\ln\delta + 10 - \frac{4\pi^2}{3}\right]$$

IR finite, but problems for small β, δ

- Large log can spoil perturbative expansion
- Scale choice?

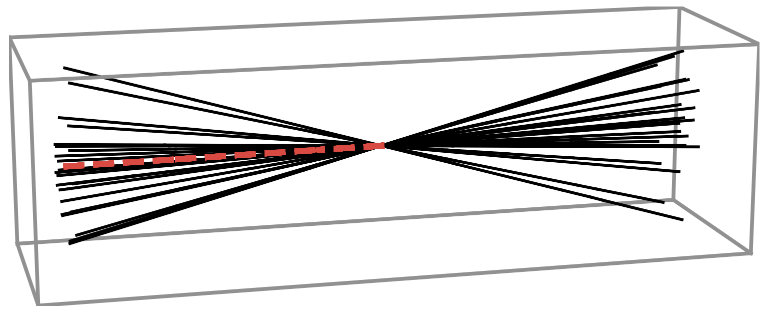
$$\mu = Q,\ Q\beta,\ Q\delta,\ Q\beta\delta\ ?$$

Global Observables v.s. Non-Global Observables

- Global Observables: all radiation in the full phase space contributes to the observables
  - E.g. thrust, broadening, C-parameter, . . .
  - “well” understood: all orders factorization theorems, resummation to high orders

Example: Resummation for Thrust

$$T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_i |\mathbf{p}_i|}$$



$$1 - T \approx \frac{M_1^2 + M_2^2}{Q^2}$$

- The perturbative result for the thrust distribution contains logarithms $\alpha_s^n \ln^{2n} \tau$, where $\tau = 1 - T$
  - Near the end-point $\tau \to 0$, the logarithmic terms dominate.
- Using SCET one can derive the factorization formula

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q^2,\mu) \int dM_1^2 \int dM_2^2 \, J(M_1^2,\mu) J(M_2^2,\mu) S_T(\tau Q - \frac{M_1^2 + M_2^2}{Q}, \mu)$$

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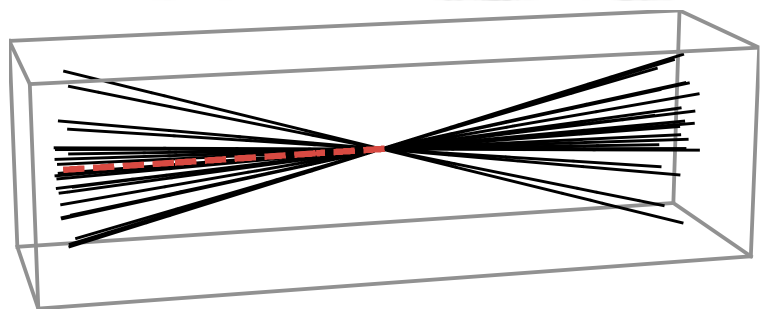

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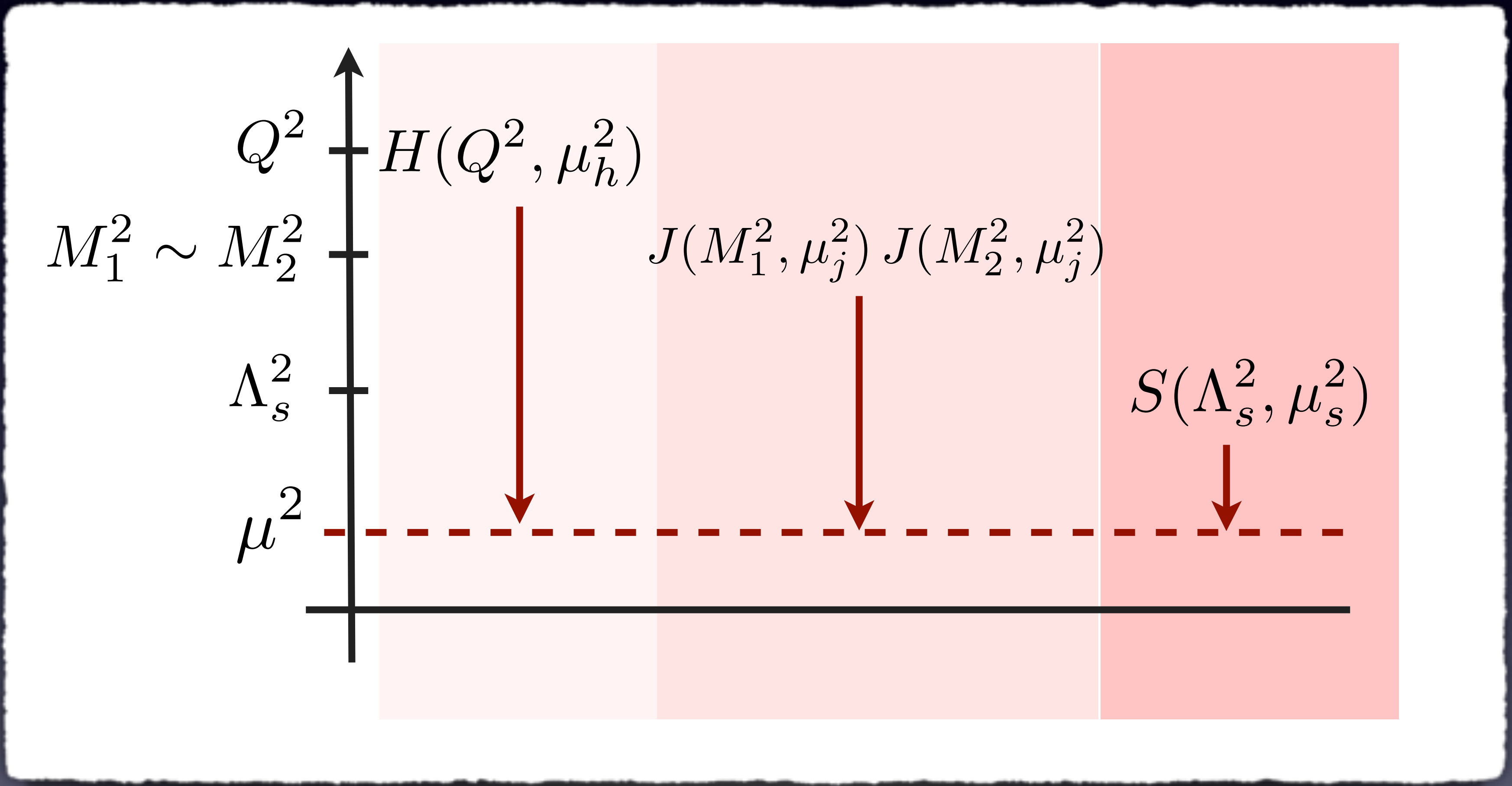
$$Q^2 \quad \gg \quad M_1^2 \sim M_2^2 \sim \tau Q^2 \quad \gg \quad \tau^2 Q^2$$

Hard Collinear Soft

Resummation by RG evolution

Evaluate each part at it characteristic scale, evolve to common reference scale

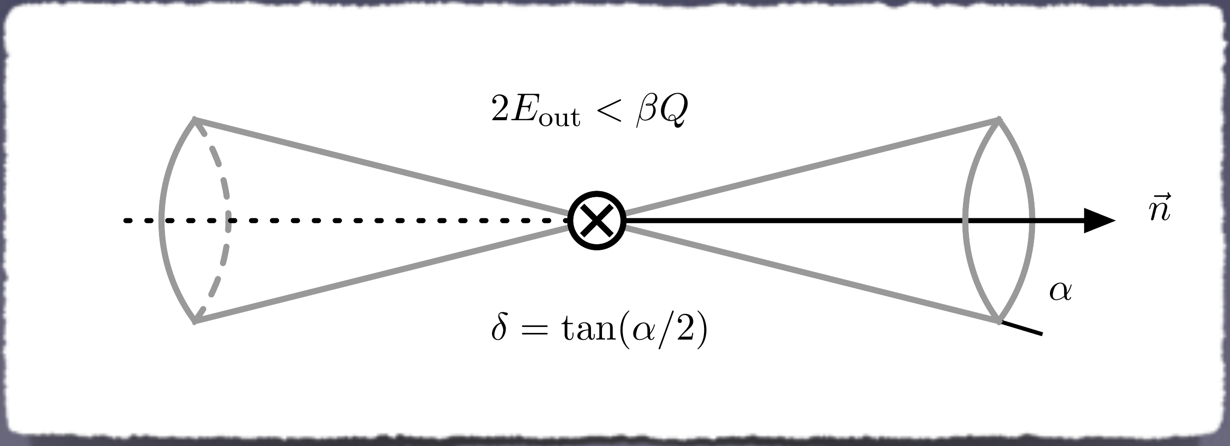


Each contribution is evaluated at its natural scale. No large perturbative logarithms.

- N^3LL resummation (Becher & Schwartz 2008)

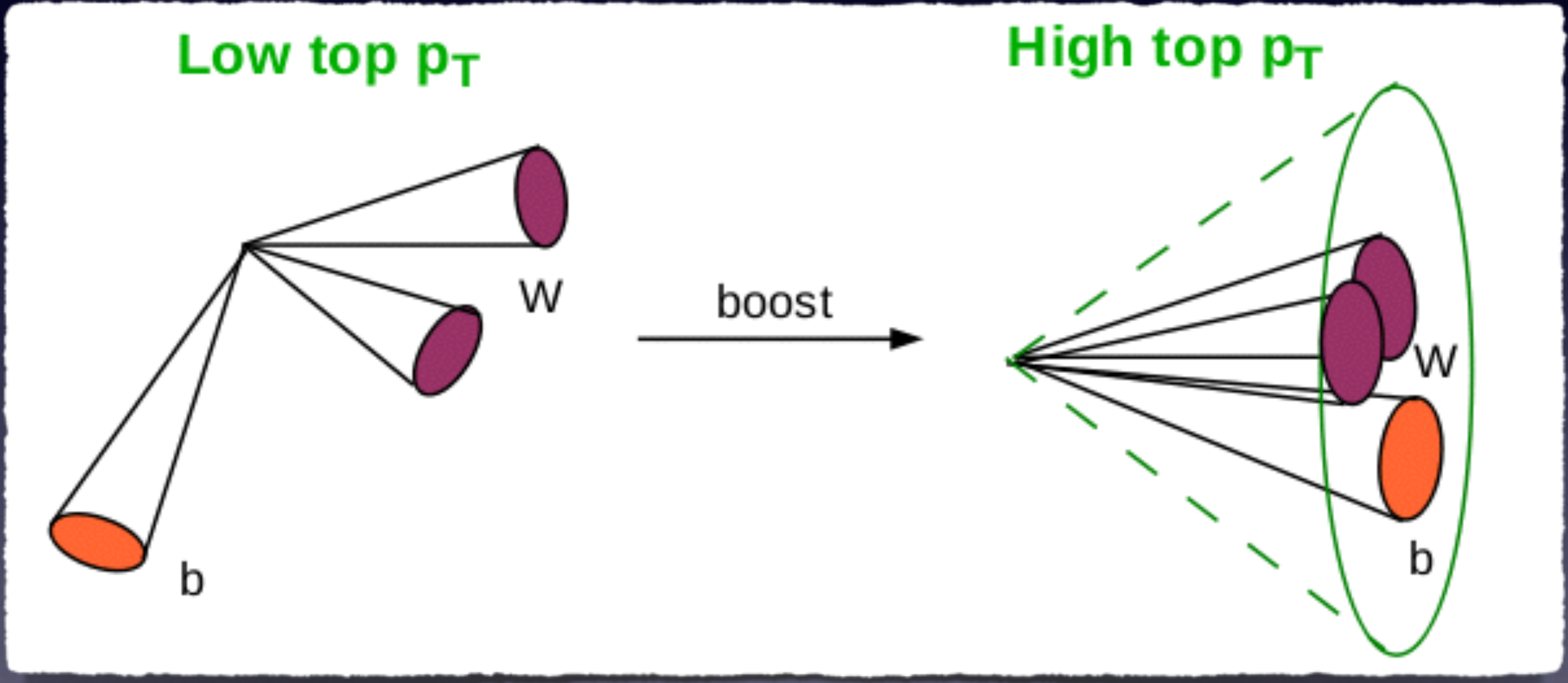
Global Observables v.s. Non-Global Observables

- Global Observables: all radiation in the full phase space contributes to the observables
  - E.g. thrust, broadening, C-parameter, . . .
  - "well" understood: all orders factorization theorems, resummation to high orders
- Non-global observables: radiation in a limited region of the full phase space



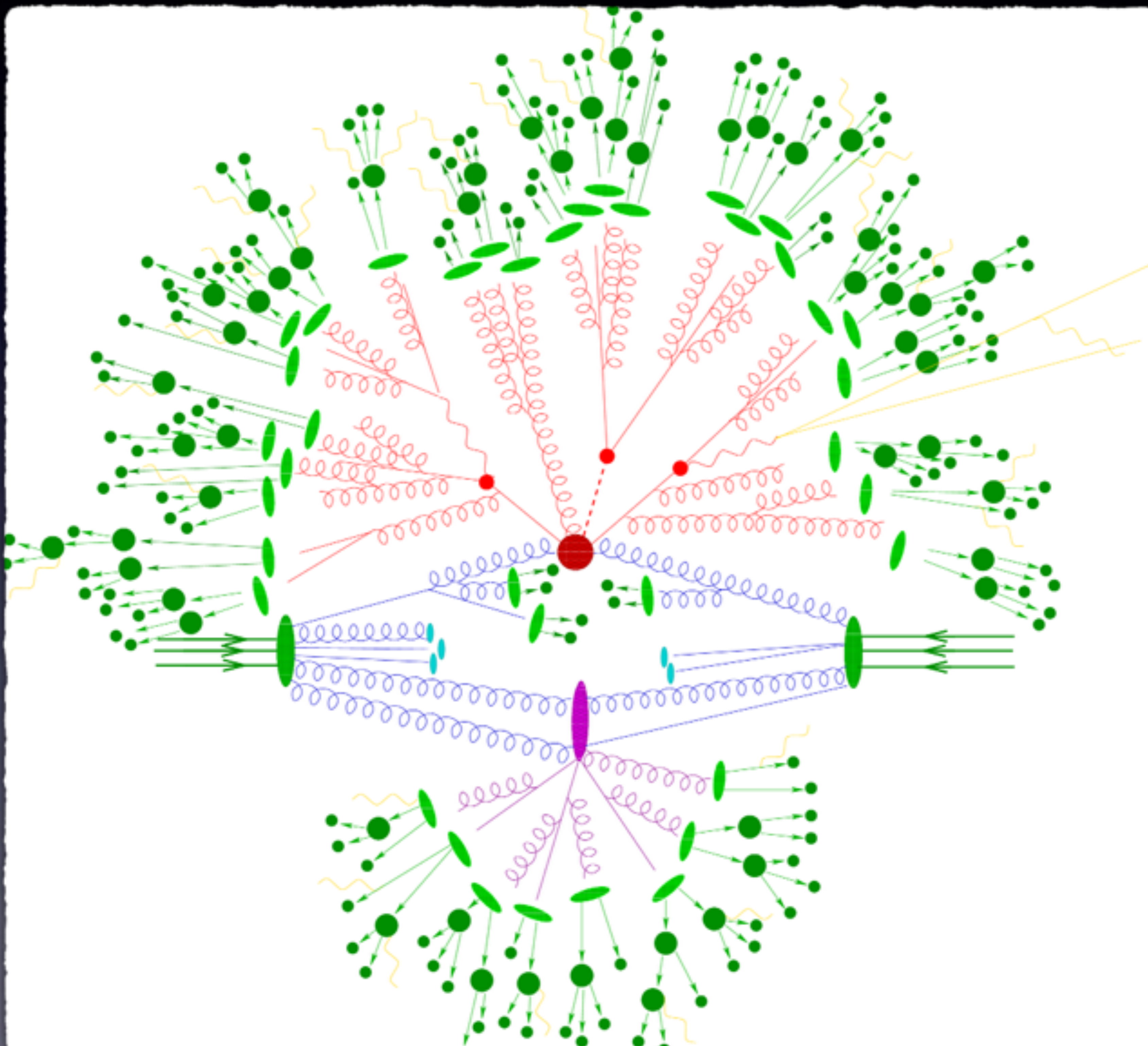
Boosted jets

Analysis of jet substructure can provide important information.



- resummation see Dasgupta et al. '15, '16; Isaacson, Li, Li, Yuan '15

Non-Global Observables @ Hadron Collider



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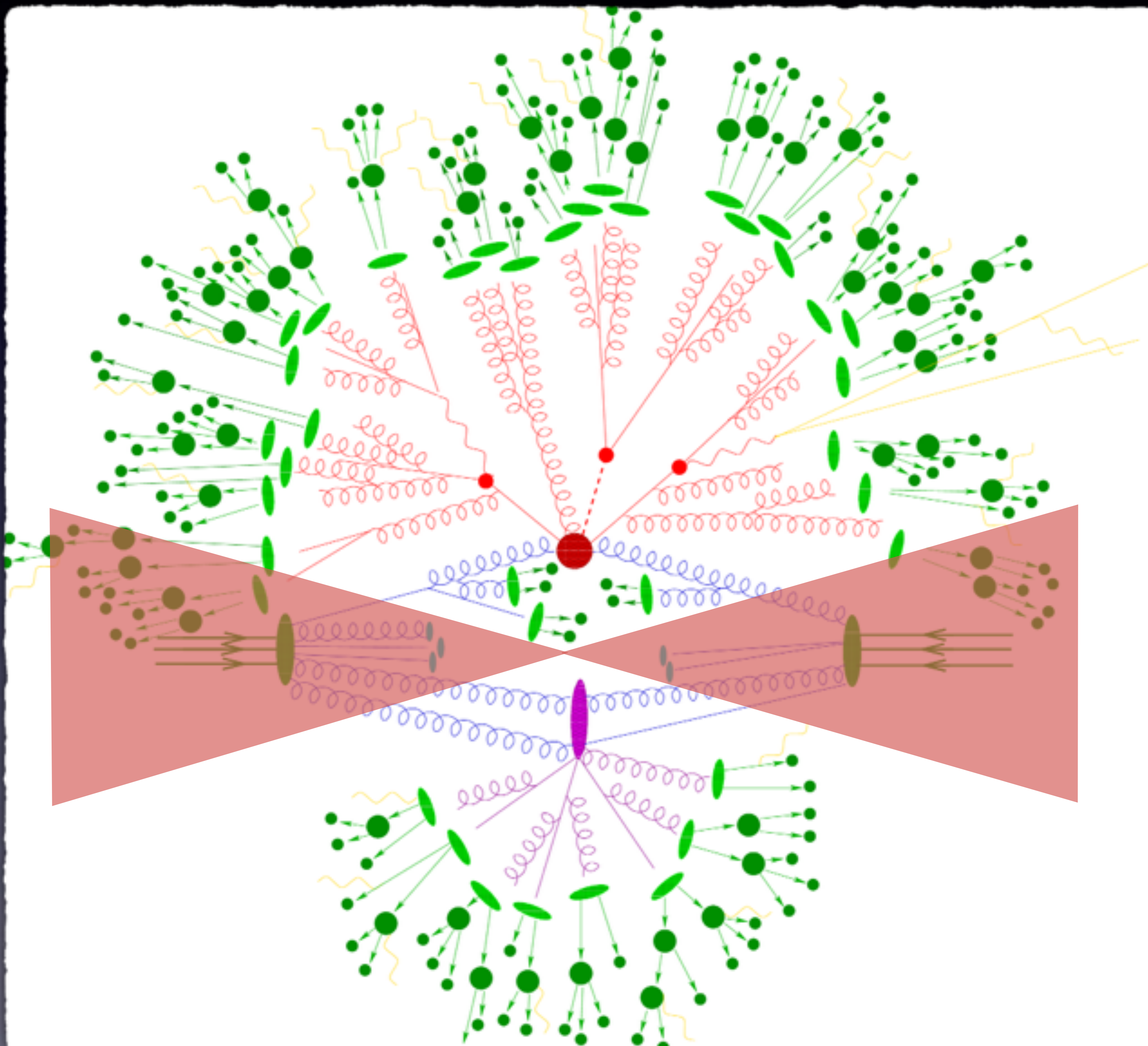


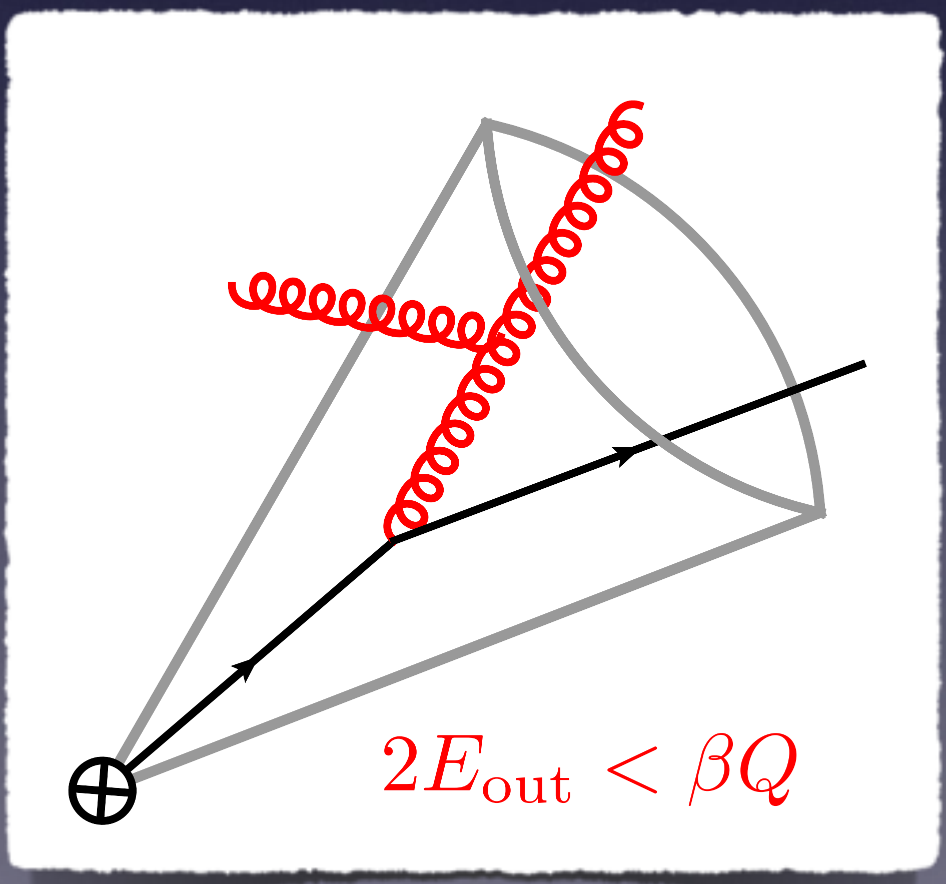
$$|y| < y_{\max}$$

Non-global logarithms (NGLs)

(Dasgupta & Salam 2001)

Observables which are insensitive to emissions into certain regions of phase space involve additional NGLs **not captured** by the usual resummation formula

$$\sigma \sim \mathcal{H} \cdot \mathcal{J}_1 \otimes \mathcal{J}_2 \otimes \mathcal{S}$$



Jet observables involve NGLs because they are insensitive to emissions inside the cone

$$\alpha_s^2 C_F C_A \pi^2 \ln^2 \beta$$

These types of logarithm do not exponentiate in the usual way

Leading-Log resummation

Banfi, Marchesini & Smye 2002

- The leading logarithms arise from configuration in which the emitted gluons are strongly ordered

$$E_1 \gg E_2 \gg \cdots \gg E_m$$

- In the large-Nc limit, multi-gluon emission amplitudes become simple:

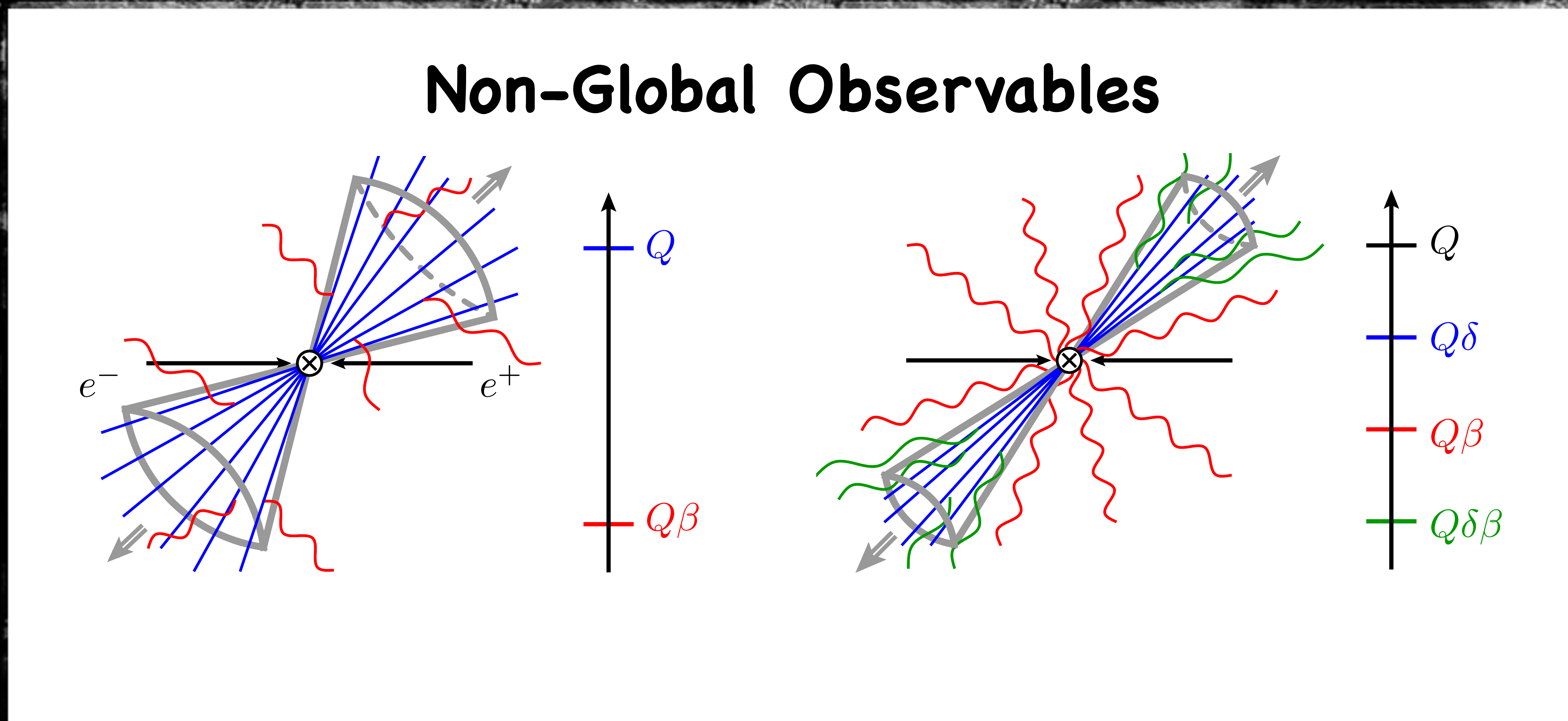
$$N_c^m g^{2m} \sum_{(1\cdots m)} \frac{p_a \cdot p_b}{(p_a \cdot p_1)(p_1 \cdot p_2)\cdots(p_m \cdot p_b)}$$

- Based on this structure, Banfi, Marchesini & Smye derive an integral-differential equation for resuming NG logarithms at LL level in the large-Nc limit:

BMS equation: $$\partial_L G_{ab}(L) = \int \frac{d\,\Omega_j}{4\pi} W_{ab}^j \left[\Theta_{\rm in}^{n\bar{n}}(j) G_{aj}(L) G_{jb}(L) - G_{ab}(L)\right]$$

Some recent progress

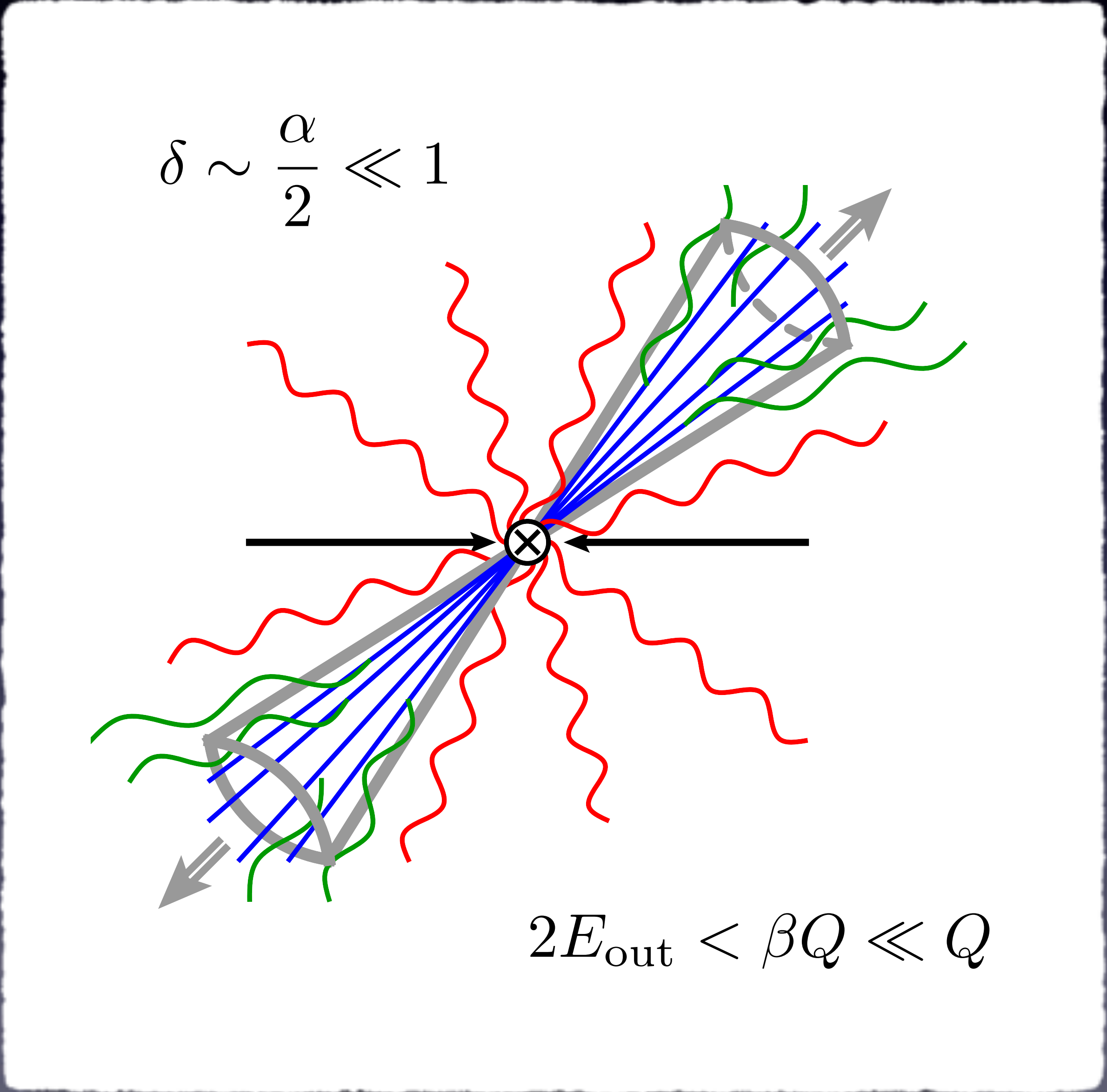
- Resummation of LL NGLs beyond large Nc Weigert ’03; Hatta Ueda ’13 + Hagiwara ’15; Caron-Huot ‘15
- Fixed-order results
  - two-loop hemisphere soft function Kelley, Schwartz, Schabinger & Zhu ’11; Horning, Lee, Stewart, Walsh & Zuberi ’11
  - with jet-cone Kelley, Schwartz, Schabinger & Zhu ’11; von Manteuffel, Schabinger & Zhu ’13
  - LL NGLs 5-loops (BMS eq & finite Nc) Schwartz, Zhu ’14; Delenda, Khelifa-Kerfa ’15
- Resummation for soft subjets Larkoski, Moult & Neill ’15; Neill ’15; Laroski, Moult ’15
- Groomed jet substructure Frye, Larkoski, Matthew & Yan ‘16



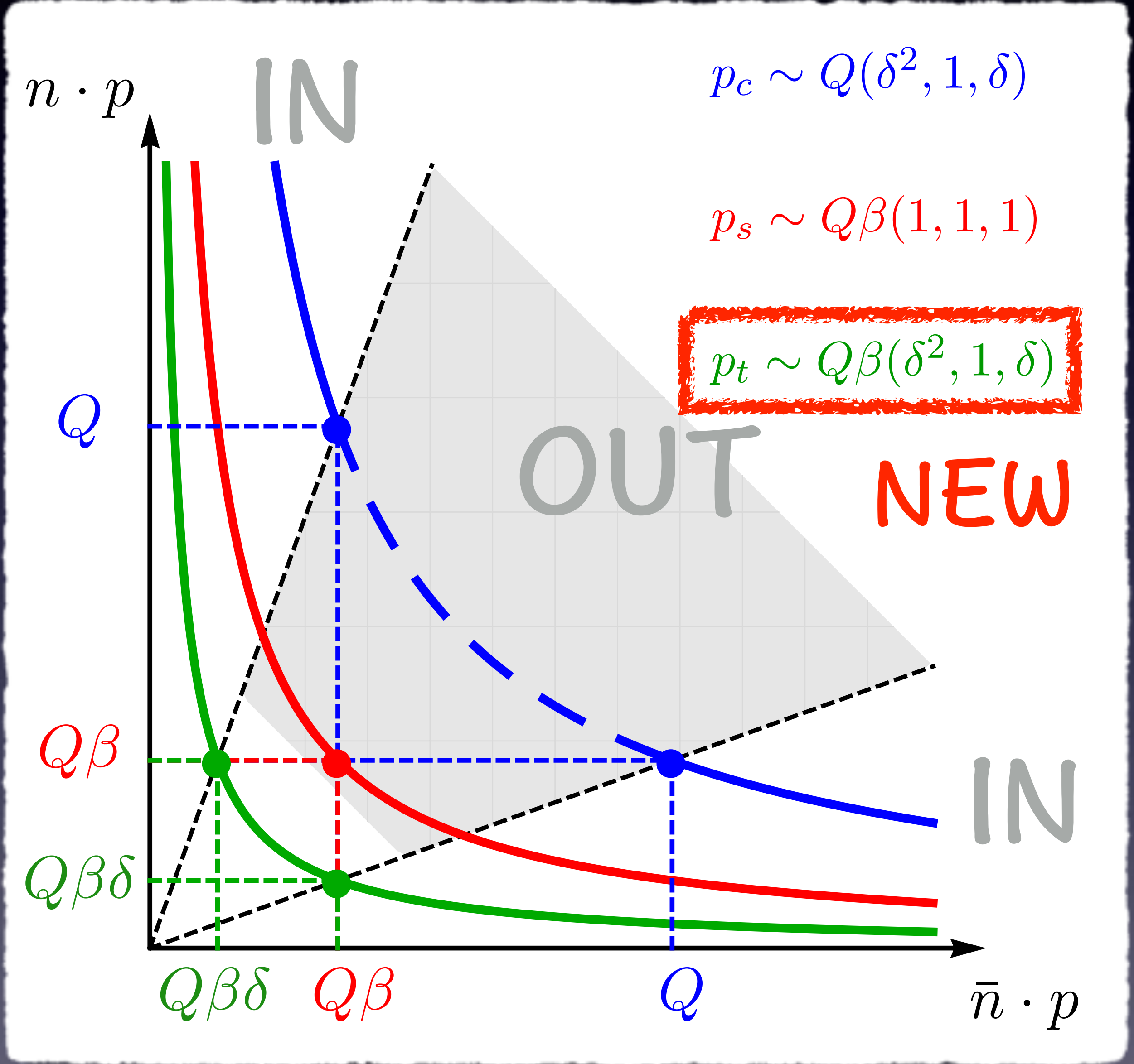
From SCET to Jet Effective Theory

Becher, Neubert, Rothen & DYS, PRL116(2016)192001

EFT for narrow-cone jets



$\beta \sim \delta^2$



Becher, Neubert, Rothen & DYS 1508.06645;
Chien, Hornig & Lee 1509.04287

Check at One-loop

Hard	$\Delta\sigma_h = \frac{\alpha_s C_F}{4\pi}\,\sigma_0 \left(\frac{\mu}{Q}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2} - \frac{6}{\epsilon} + \frac{7\pi^2}{3} - 16\right)$
Collinear	$\Delta\sigma_{c+\bar{c}} = \frac{\alpha_s C_F}{4\pi}\,\sigma_0 \left(\frac{\mu}{Q\delta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2} + \frac{6}{\epsilon} + c_0\right)$
Soft	$\Delta\sigma_s = \frac{\alpha_s C_F}{4\pi}\,\sigma_0 \left(\frac{\mu}{Q\beta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2} - \pi^2\right)$

Check at One-loop

Hard	$\Delta\sigma_h = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}-\frac{6}{\epsilon}+\frac{7\pi^2}{3}-16\right)$
Collinear	$\Delta\sigma_{c+\bar{c}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}+\frac{6}{\epsilon}+c_0\right)$
Soft	$\Delta\sigma_s = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\beta}\right)^{2\epsilon}\left(\frac{4}{\epsilon^2}-\pi^2\right)$
Coft	$\Delta\sigma_{t+\bar{t}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(\frac{\mu}{Q\delta\beta}\right)^{2\epsilon}\left(-\frac{4}{\epsilon^2}+\frac{\pi^2}{3}\right)$

$$\Delta\sigma^{\text{tot}} = \frac{\alpha_s C_F}{4\pi}\sigma_0\left(-16\ln\delta\ln\beta+12\ln\delta+c_0+\frac{5\pi^2}{3}-16\right)$$

Constant c_0 depends on the definition of jet axis:

$c_0 = -3\pi^2+26$ (Sterman-Weinberg)

$c_0 = -5\pi^2/3+14+12\ln 2$ (thrust axis)

Factorization for two-jet cross section

$$\sigma(\beta,\delta) \stackrel{?}{=} \sigma_0\, H(Q,\mu)[J(Q\delta,\mu)]^2\, S(Q\beta,\mu) \otimes U(Q\beta\delta,\mu) \otimes U(Q\beta\delta,\mu)$$

Chien, Hornig & Lee 1509.04287

Factorization for two-jet cross section

Factorization for two-jet cross section

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q)\, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \left\langle \boldsymbol{\mathcal{J}}_m(Q\delta) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_m(Q\delta\tau) \right\rangle \right]^2$$

Factorization for two-jet cross section

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q)\, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m(Q\delta) \otimes \widetilde{\mathcal{U}}_m(Q\delta\tau) \right\rangle \right]^2$$

- color trace
- integration over angles
- hard function
- soft function
- jet function with m patrons at fixed direction
- coft function with m Wilson lines

Factorization for two-jet cross section

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q)\, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m(Q\delta) \otimes \widetilde{\mathcal{U}}_m(Q\delta\tau) \right\rangle \right]^2$$

color trace

integration over angles

hard function

soft function

jet function with m patrons at fixed direction

coft function with m Wilson lines

First all-order factorization theorem for non-global observable.

Achieves full scale separation!

NNLO check

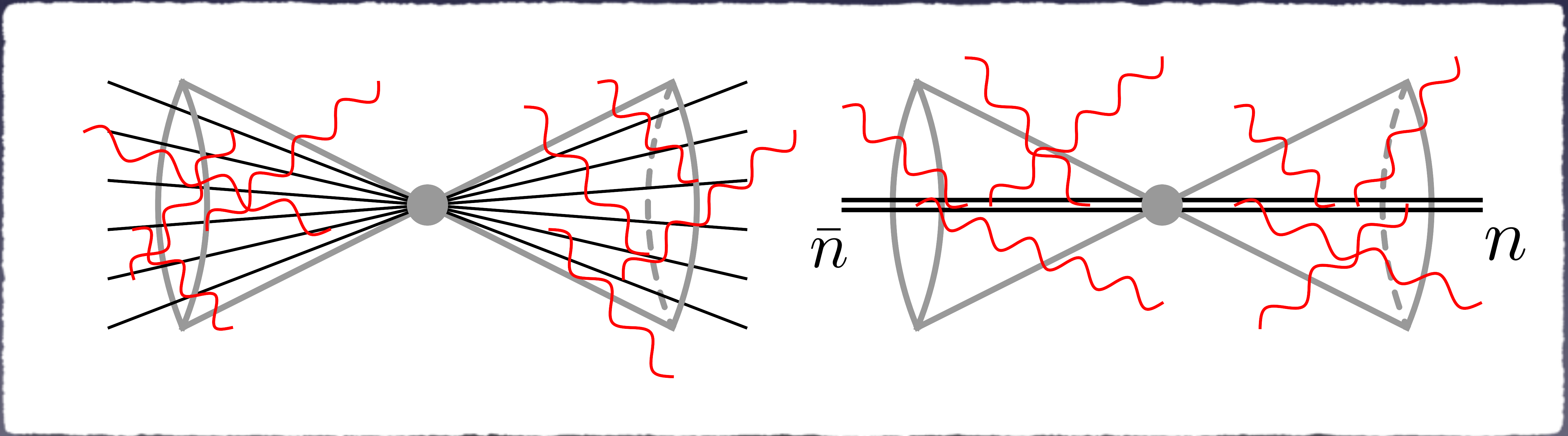
$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon) \Big\langle \boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \Big\rangle^2$$

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon)\, \big\langle \boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon)$$
$$+ \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \big\rangle^2$$

Soft function:

$$S(Q\beta)\,\mathbf{1} = \sum\!\!\!\!\!\!\int_{X_s} \langle 0|\, S^\dagger(\bar{n})\, S(n)\, |X_s\rangle\langle X_s|\, S^\dagger(n)\, S(\bar{n})\, |0\rangle \theta(Q\beta - 2E_{X_s})$$



Coft Radiation

Large-angle soft radiation off a jet of collinear particles does not resolve individual energetic patrons

$$\sum_i Q_i \frac{p_i \cdot \epsilon}{p_i \cdot k} \approx Q_{\rm tot} \frac{n \cdot \epsilon}{n \cdot k}$$

This approximation breaks down for soft radiation collinear to the jet!!!

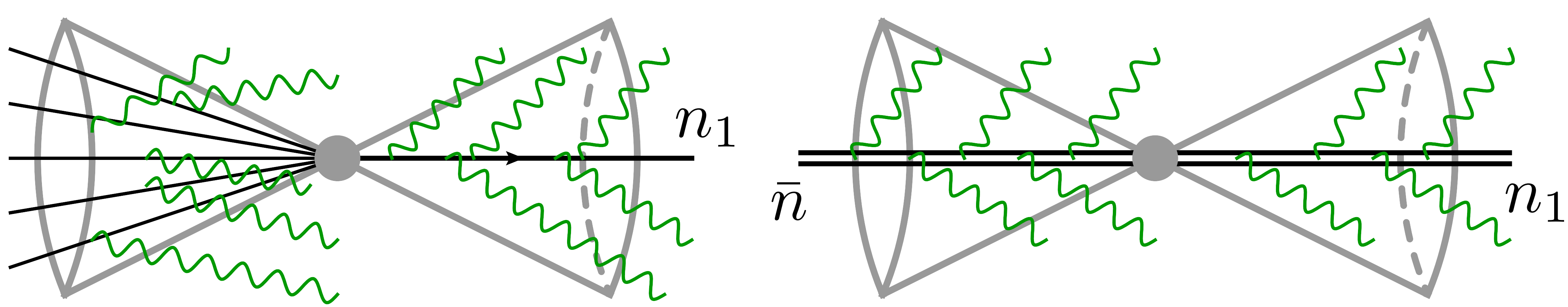
$$k^\mu = \omega n^\mu$$

Typically this small region of phase space does not give an $\mathcal{O}(1)$ contribution.

However it does in the non-global observables

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon)\, \big\langle \mathcal{J}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\mathcal{U}}_1(\{n_1\},Q\delta\tau,\epsilon) + \mathcal{J}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\mathcal{U}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \mathcal{J}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \big\rangle^2$$



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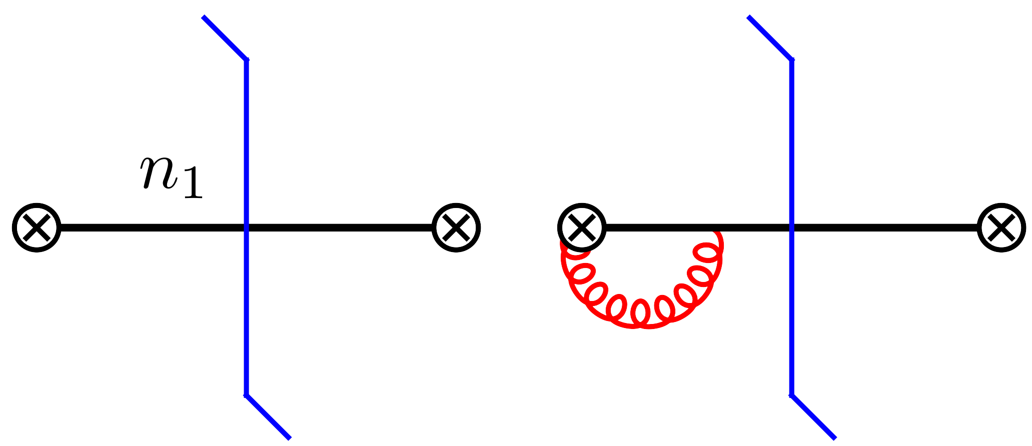
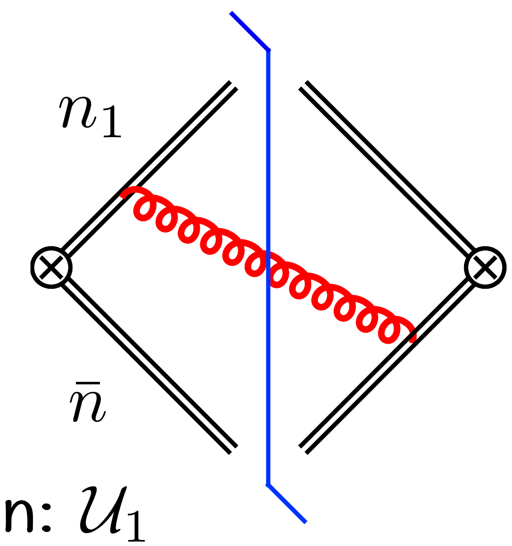
$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon) \Big\langle \boldsymbol{\mathcal{J}}_1(\{n_1\}, Q\delta, \epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\}, Q\delta\tau, \epsilon) + \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\}, Q\delta, \epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\}, Q\delta\tau, \epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\}, Q\delta, \epsilon) \otimes \mathbf{1} + \dots \Big\rangle^2$$



Jet function: $\mathcal{J}_1$



Coft function: $\mathcal{U}_1$

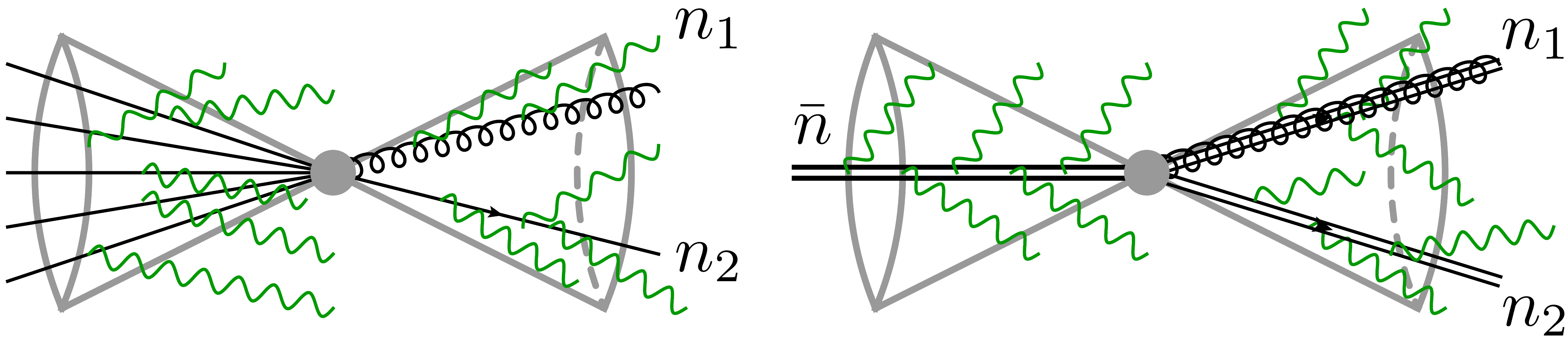
$$\boldsymbol{\mathcal{J}}_1(\hat{\theta}_1, Q\delta, \epsilon) = \delta(\hat{\theta}_1)\, \mathbf{1}$$

$$\widetilde{\boldsymbol{\mathcal{U}}}_1(\hat{\theta}_1, Q\tau\delta, \epsilon) = \mathbf{1} + \frac{C_F \alpha_0}{4\pi}\, e^{-2\,\epsilon L_t}\, u_F(\hat{\theta}_1)\, \mathbf{1}$$

$$\big\langle \boldsymbol{\mathcal{J}}_1 \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1 \big\rangle = \big\langle \widetilde{\boldsymbol{\mathcal{U}}}_1(0, Q\delta\tau, \epsilon) \big\rangle$$

NNLO check

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0\, H(Q,\epsilon)\, \widetilde{S}(Q\tau,\epsilon) \Big\langle \boldsymbol{\mathcal{J}}_1(\{n_1\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_1(\{n_1\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_2(\{n_1,n_2\},Q\delta,\epsilon) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_2(\{n_1,n_2\},Q\delta\tau,\epsilon) + \boldsymbol{\mathcal{J}}_3(\{n_1,n_2,n_3\},Q\delta,\epsilon) \otimes \mathbf{1} + \dots \Big\rangle^2$$



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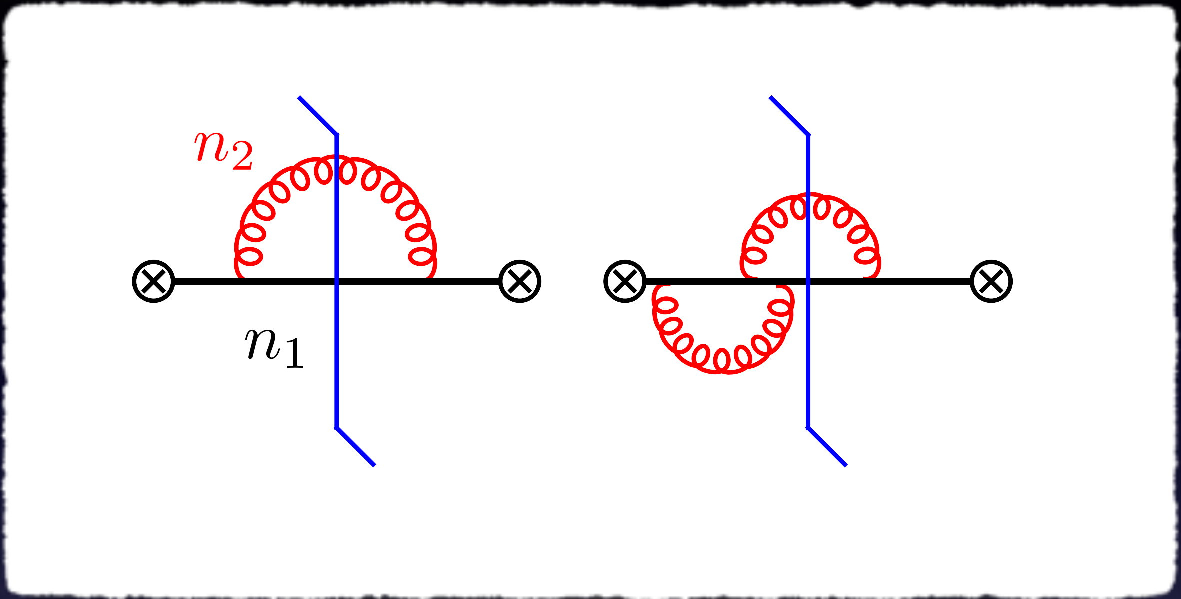
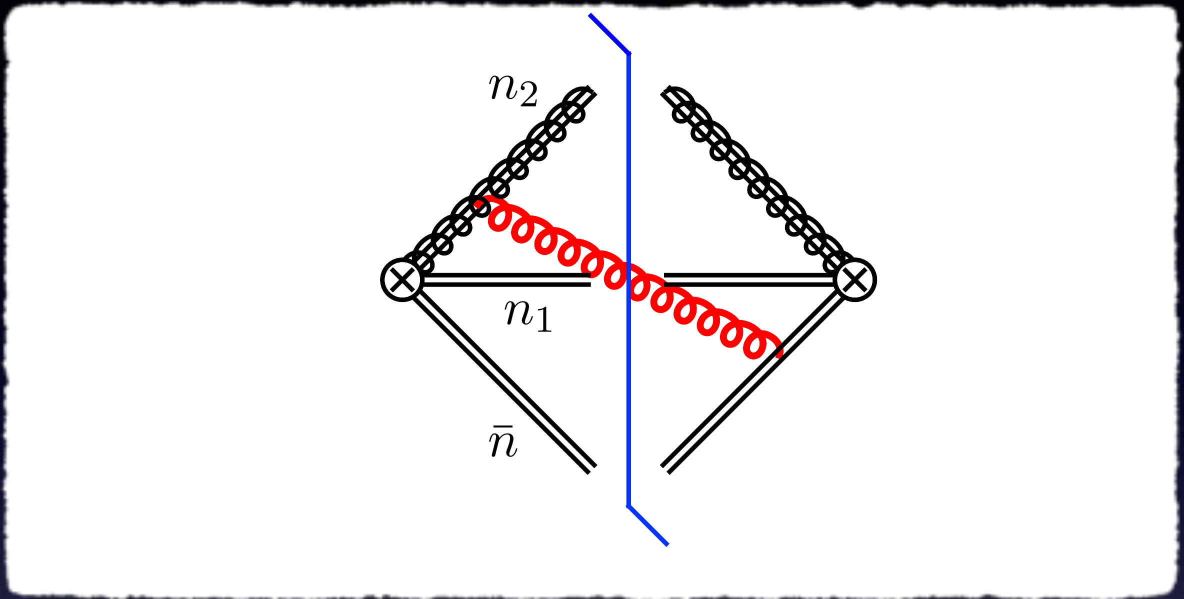
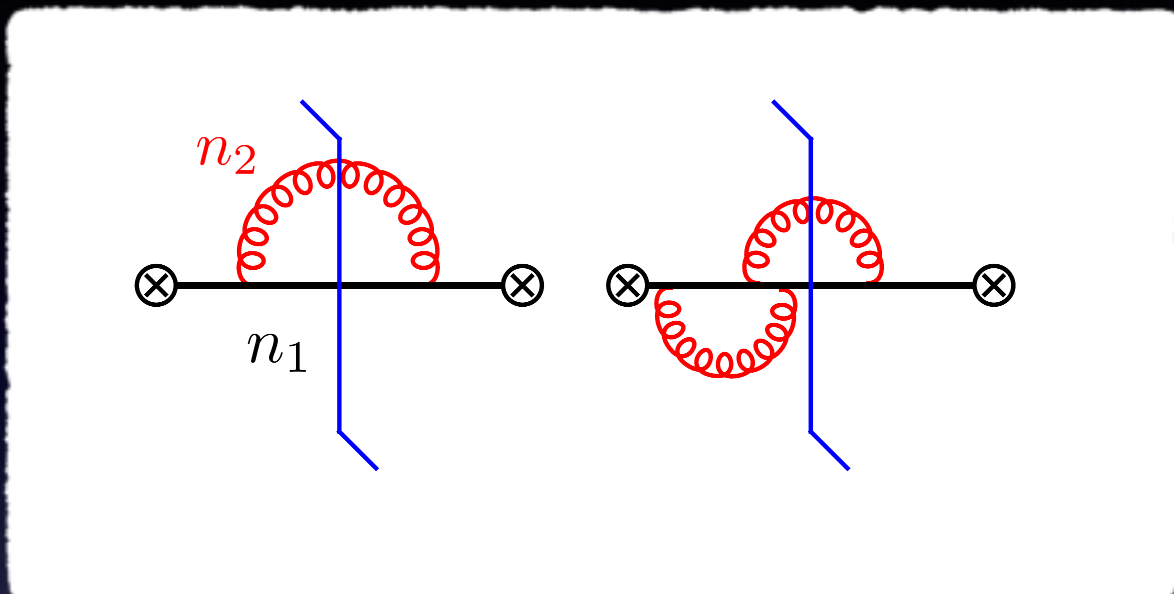


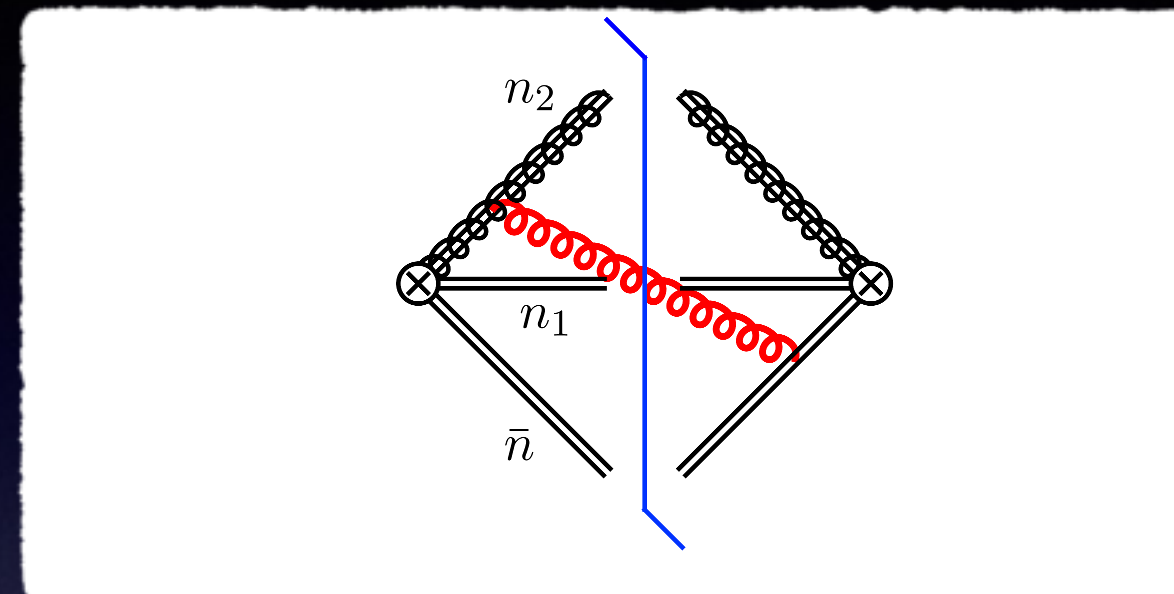


$$\mathcal{J}_2^{(1)}(\hat{\theta}_1, \hat{\theta}_2, \phi_2, Q\delta, \epsilon) = C_F\, \delta(\phi_2 - \pi)\, e^{-2\epsilon L_c}$$
$$\times \Bigg\{ \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 7 - \frac{5\pi^2}{6} + 6\ln 2 \right) \delta(\hat{\theta}_1)\, \delta(\hat{\theta}_2) - \frac{4}{\epsilon}\, \delta(\hat{\theta}_1) \left[\frac{1}{\hat{\theta}_2} \right]_+ + 8\, \delta(\hat{\theta}_1) \left[\frac{\ln \hat{\theta}_2}{\hat{\theta}_2} \right]_+$$
$$+ 4\, \frac{dy}{d\hat{\theta}_2} \left[\frac{1}{\hat{\theta}_1} \right]_+ \frac{1 + 2y + 2y^2}{(1+y)^3}\, \theta(\hat{\theta}_1 - \hat{\theta}_2)$$
$$+ 4\, \frac{dy}{d\hat{\theta}_1} \left[\frac{1}{\hat{\theta}_2} \right]_+ \left(2 \left[\frac{1}{y} \right]_+ - \frac{4 + 5y + 2y^2}{(1+y)^3} \right) \theta(\hat{\theta}_2 - \hat{\theta}_1) + \mathcal{O}(\epsilon) \Bigg\} \mathbf{1}$$

$$\widetilde{\mathcal{U}}_2(\hat{\theta}_1, \hat{\theta}_2, \phi_2, Q\tau\delta, \epsilon) = \mathbf{1} + \frac{\alpha_0}{4\pi}\, e^{-2\,\epsilon L_t} \left[C_F\, u_F(\hat{\theta}_1) + C_A\, u_A(\hat{\theta}_1, \hat{\theta}_2, \phi_2) \right] \mathbf{1}$$

NNLO check





$$\left\langle \mathcal{J}_2^{(1)} \otimes \widetilde{\mathcal{U}}_2^{(1)} \right\rangle = e^{-2\epsilon(L_c+L_t)} \left(C_F^2 M_F + C_F C_A M_A \right)$$

$$\begin{aligned} M_F = &-\frac{4}{\epsilon^4} - \frac{6}{\epsilon^3} + \frac{1}{\epsilon^2}\left(-14 + \frac{2\pi^2}{3} - 12\ln 2 \right) + \frac{1}{\epsilon}\left(-26 - \pi^2 + 10\,\zeta_3 - 32\ln 2 \right) \\ &- 52 - \frac{10\pi^2}{3} - 27\zeta_3 + \frac{11\pi^4}{30} - \frac{4}{3}\ln^4 2 - 8\ln^3 2 - 4\ln^2 2 + \frac{4\,\pi^2}{3}\ln^2 2 \\ &- 52\ln 2 + 4\pi^2 \ln 2 - 28\zeta_3 \ln 2 - 32\,\mathrm{Li}_4\left(\frac{1}{2}\right), \end{aligned}$$

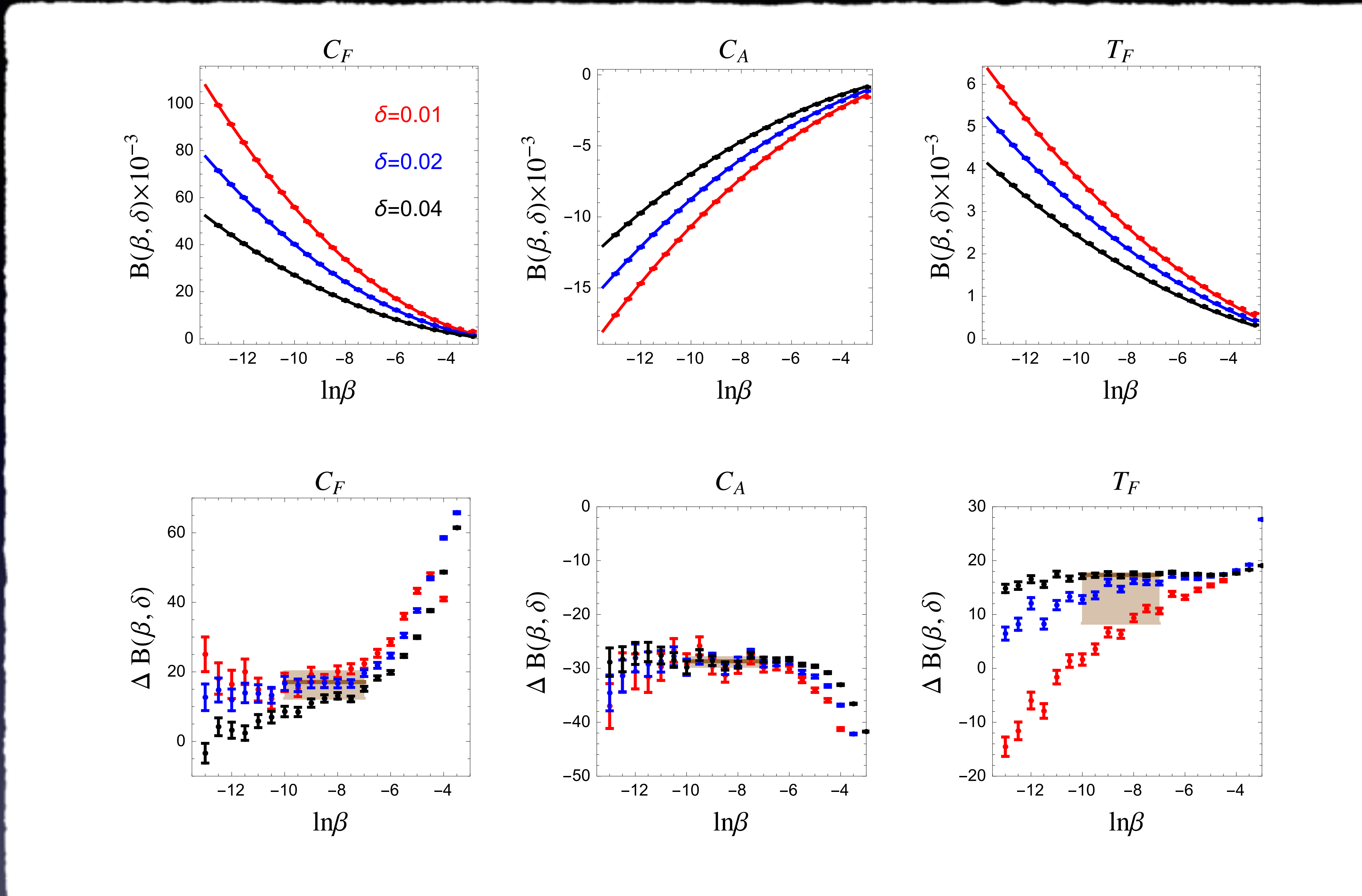
$$\begin{aligned} M_A = &\frac{2\pi^2}{3\epsilon^2} + \frac{1}{\epsilon}\left(-2 + \frac{\pi^2}{2} + 12\,\zeta_3 + 6\ln^2 2 + 4\ln 2 \right) - 4 + \frac{7\pi^2}{6} - 24\zeta_3 - \frac{\pi^4}{6} + \frac{8}{3}\ln^4 2 \\ &- 4\ln^3 2 + 6\ln^2 2 - \frac{8\pi^2}{3}\ln^2 2 - 4\ln 2 + 9\pi^2 \ln 2 + 56\zeta_3 \ln 2 + 64\,\mathrm{Li}_4\left(\frac{1}{2}\right) \end{aligned}$$

NNLO check

$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} A(\beta,\delta) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\beta,\delta) + \ldots$$

$$\begin{aligned}
B(\beta,\delta) = C_F^2 \Bigg[&\left(32\ln^2\beta + 48\ln\beta + 18 - \frac{16\pi^2}{3}\right)\ln^2\delta + \left(-2 + 10\zeta_3 - 12\ln^2 2 + 4\ln 2\right)\ln\beta \\
&+ \left((8 - 48\ln 2)\ln\beta + \frac{9}{2} + 2\pi^2 - 24\zeta_3 - 36\ln 2\right)\ln\delta + c_2^F\Bigg] \\
+ C_F C_A \Bigg[&\left(\frac{44\ln\beta}{3} + 11\right)\ln^2\delta - \frac{2\pi^2}{3}\ln^2\beta + \left(\frac{8}{3} - \frac{31\pi^2}{18} - 4\zeta_3 - 6\ln^2 2 - 4\ln 2\right)\ln\beta \\
&+ \left(\frac{44\ln^2\beta}{3} + \left(-\frac{268}{9} + \frac{4\pi^2}{3}\right)\ln\beta - \frac{57}{2} + 12\zeta_3 - 22\ln 2\right)\ln\delta + c_2^A\Bigg] \\
+ C_F T_F n_f \Bigg[&\left(-\frac{16\ln\beta}{3} - 4\right)\ln^2\delta + \left(-\frac{16}{3}\ln^2\beta + \frac{80\ln\beta}{9} + 10 + 8\ln 2\right)\ln\delta \\
&+ \left(-\frac{4}{3} + \frac{4\pi^2}{9}\right)\ln\beta + c_2^f\Bigg].
\end{aligned}$$

- $\frac{1}{\epsilon^4}, \frac{1}{\epsilon^3}, \frac{1}{\epsilon^2}, \frac{1}{\epsilon}$ divergences have cancelled!
- Two loop constants unknown.

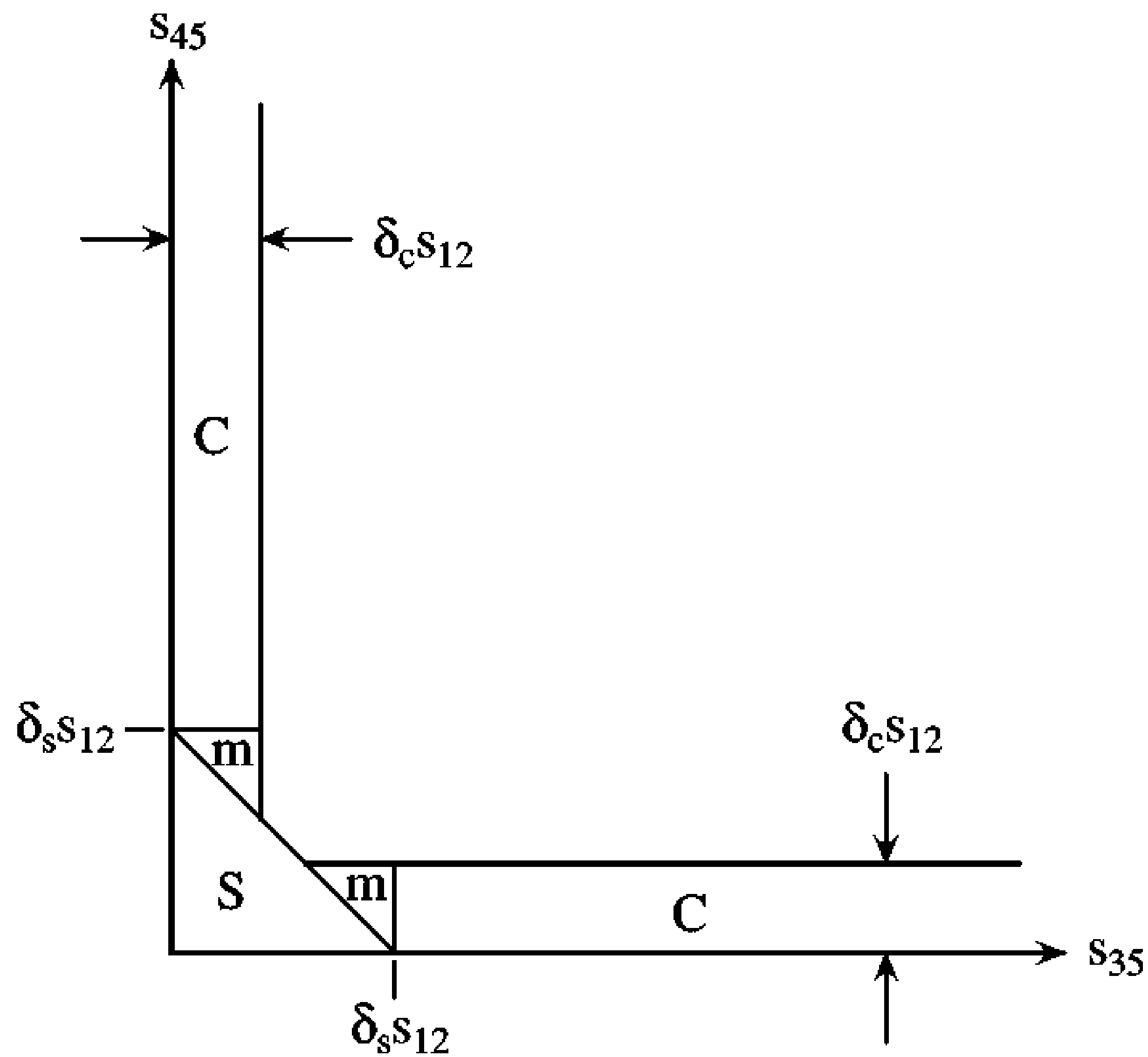


Data point from EVENT2, solid lines are our prediction. Difference yields unknown constants

$$c_2^F = 17.1^{+3.0}_{-4.7}\,, \qquad c_2^A = -28.7^{+0.7}_{-1.0}\,, \qquad c_2^f = 17.3^{+0.3}_{-9.0}\,.$$

Note: EVENT2 suffers from numerical instability in n_f channel.

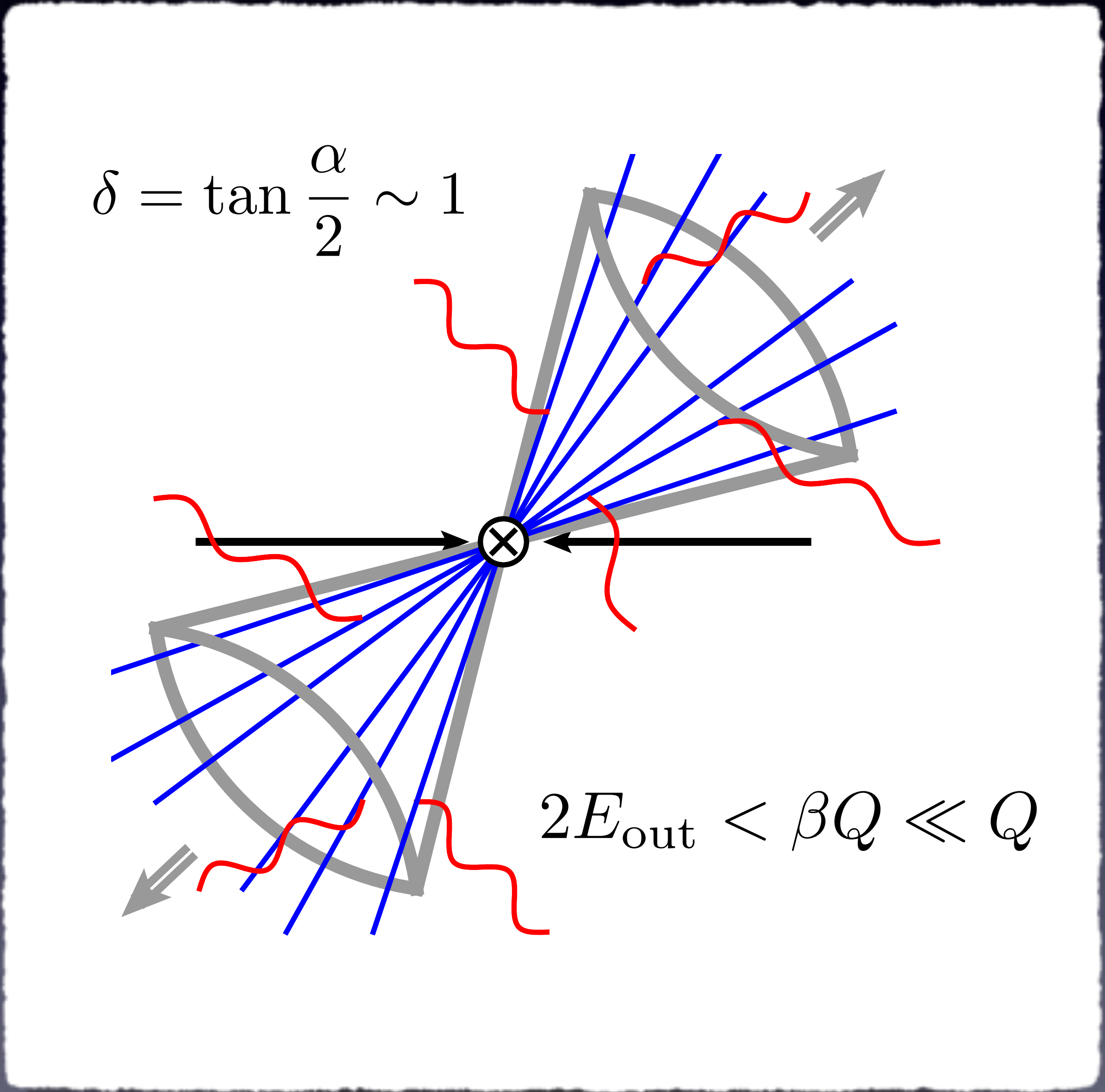
Two cut-off method @ NNLO, N^3LO, N^4LO, . . .

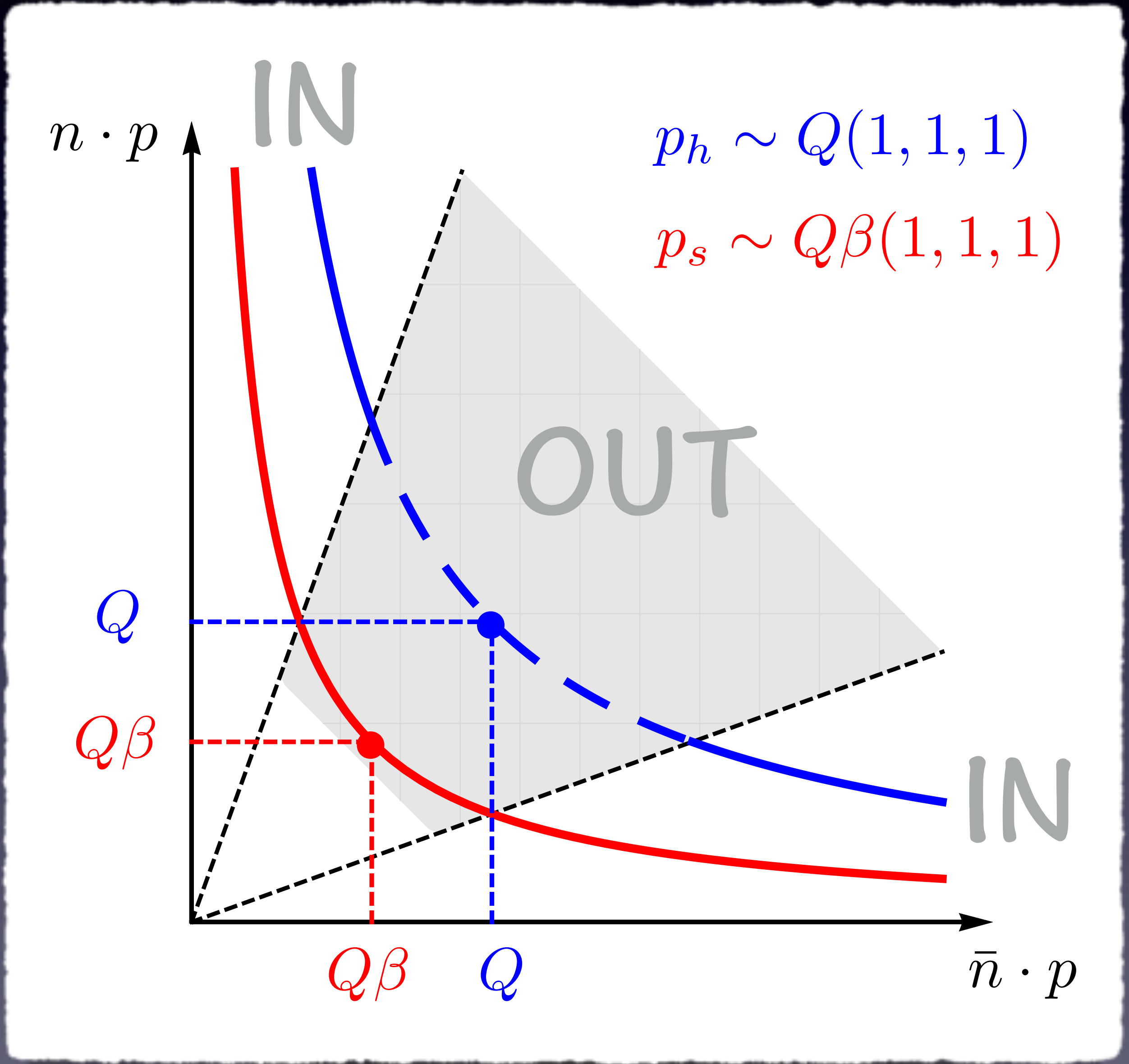


EFT for wide-cone jets

(Becher, Neubert, Rothen, DYS 1605.02737)

$$\mathcal{L} = \mathcal{L}_h + \mathcal{L}_s$$





Factorization

- Then the cross section can be written in factorized form as,

$$\sigma(\beta,\delta) = \sum_{m=2}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\},Q,\delta) \otimes \mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) \rangle$$

- We define the squared matrix element of this operator as

$$\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) = \sum\!\!\!\!\!\!\int_X \langle 0|S_1^\dagger(n_1)\,\ldots\,S_m^\dagger(n_m)|X_s\rangle\langle X_s|S_1(n_1)\,\ldots\,S_m(n_m)|0\rangle\,\theta\,(Q\beta - 2E_{\rm out})$$

- The hard functions are obtained by integrating over the energies of the hard particles, while keeping their direction fixed

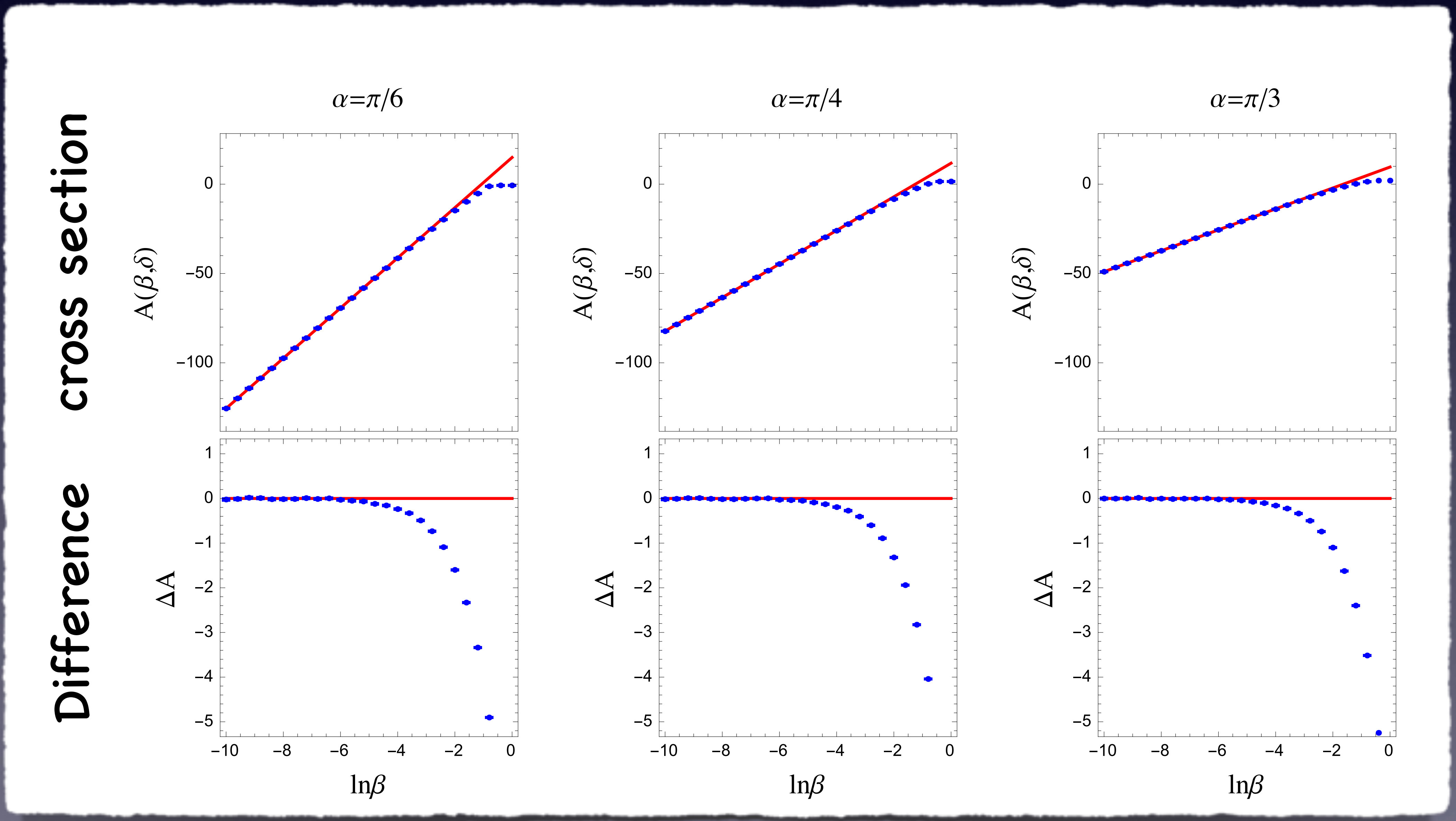
$$\mathcal{H}_m(\{\underline{n}\},Q,\delta) = \frac{1}{2Q^2}\sum_{\rm spins}\prod_{i=1}^{m}\int\frac{d\omega_i\,\omega_i^{d-3}}{(2\pi)^{d-2}}|\mathcal{M}_m\rangle\langle\mathcal{M}_m|\delta\Big(Q-\sum_{i=1}^{m}\omega_i\Big)\delta^{d-1}(\vec{p}_{\rm tot})\,\Theta_{\rm in}^{n\bar{n}}(\{\underline{p}\})$$

- $\otimes$ indicates integration over the direction of the energetic partons

$$\mathcal{H}_m(\{\underline{n}\},Q,\delta)\otimes\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta) = \prod_{i=1}^{m}\int\frac{d\Omega(n_i)}{4\pi}\,\mathcal{H}_m(\{\underline{n}\},Q,\delta)\,\mathcal{S}_m(\{\underline{n}\},Q\beta,\delta)$$

One-loop coefficient v.s. EVENT2

$$A(\beta,\delta) = C_F\left[-8\ln\delta\ln\beta - 1 + 6\ln 2 - 6\ln\delta - 6\,\delta^2 + \left(\frac{9}{2} - 6\ln 2\right)\delta^4 - 4\,\mathrm{Li}_2(-\delta^2) + 4\,\mathrm{Li}_2(\delta^2)\right]$$

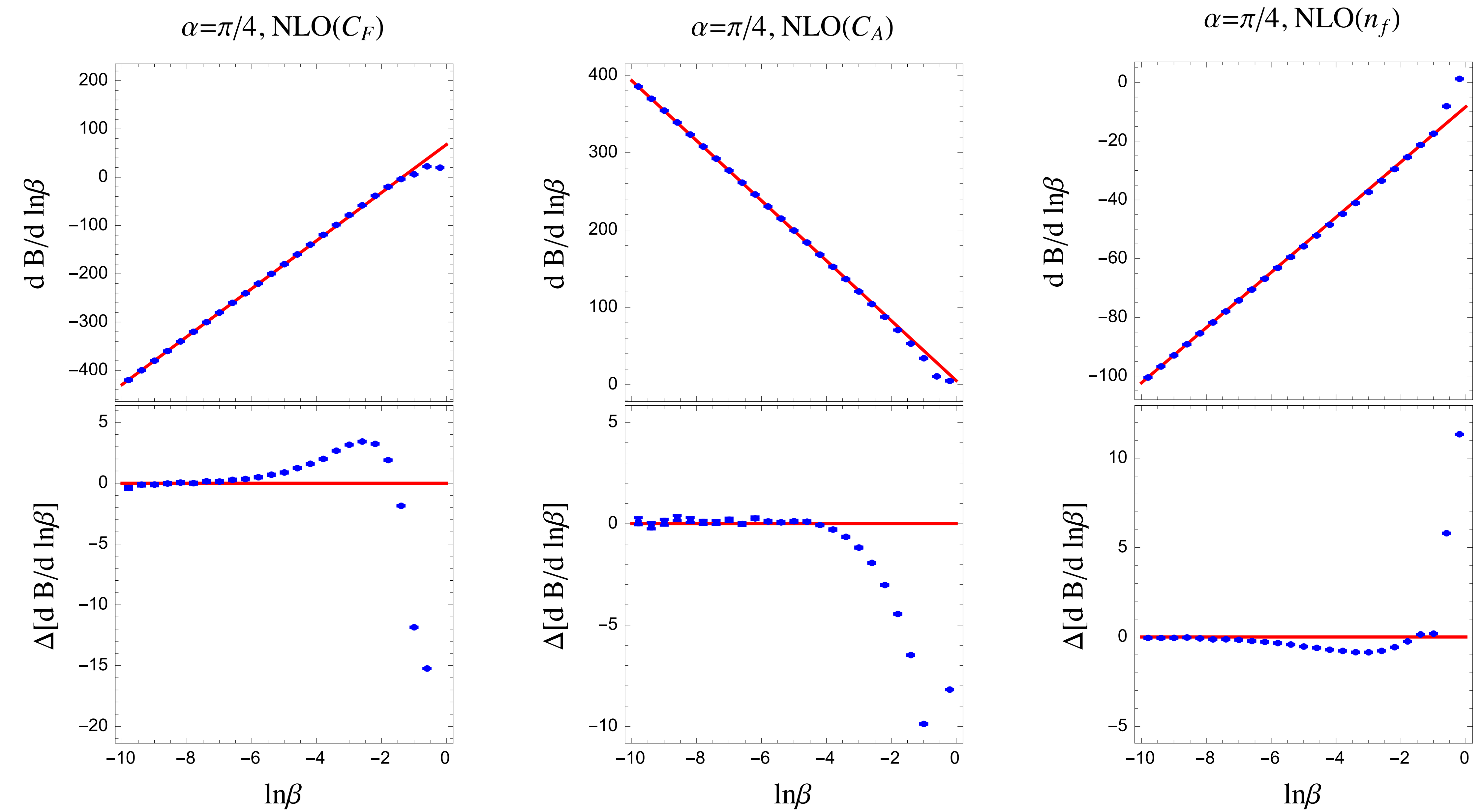


Two-loop coefficient v.s. EVENT2

$$B(\beta,\delta) = C_F^2 B_F + C_F C_A B_A + C_F T_F n_f B_f$$

$$\begin{aligned}
B_A = {} & \left[\frac{44}{3}\ln\delta - \frac{2\pi^2}{3} + 4\,\mathrm{Li}_2(\delta^4)\right]\ln^2\beta + \left[\frac{4}{3\,(1-\delta^4)} - \frac{16\ln\delta}{3\,(1-\delta^4)} + \frac{16\ln\delta}{3\,(1-\delta^4)^2}\right. \\
& -\frac{4}{3}\ln^3\left(1-\delta^2\right) - \frac{20}{3}\ln^3\left(1+\delta^2\right) + 32\ln\delta\ln^2\left(1-\delta^2\right) - 4\ln\left(1+\delta^2\right)\ln^2\left(1-\delta^2\right) \\
& -4\ln^2\left(1+\delta^2\right)\ln\left(1-\delta^2\right) + 64\ln\delta\ln^2\left(1+\delta^2\right) - 64\ln^2\delta\ln\left(1+\delta^2\right) \\
& +\frac{88}{3}\ln\delta\ln\left(1-\delta^2\right) - \frac{16}{3}\pi^2\ln\left(1-\delta^2\right) + 44\ln\delta\ln\left(1+\delta^2\right) + \frac{16}{3}\pi^2\ln\left(1+\delta^2\right) \\
& +\frac{44\ln^2\delta}{3} - \frac{16}{3}\pi^2\ln\delta - \frac{268\ln\delta}{9} + \frac{88\,\mathrm{Li}_2\left(\delta^4\right)}{3} - 4\,\mathrm{Li}_3\left(\delta^4\right) + 8\,\mathrm{Li}_3\left(-\frac{\delta^4}{1-\delta^4}\right) \\
& +8\ln 2\,\mathrm{Li}_2\left(\delta^4\right) - \frac{88\,\mathrm{Li}_2\left(\delta^2\right)}{3} - \frac{22}{3}\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) + \frac{22}{3}\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) + 32\,\mathrm{Li}_3\left(1-\delta^2\right) \\
& +32\,\mathrm{Li}_3\left(\frac{\delta^2}{1+\delta^2}\right) + 32\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^2\right) + 32\ln\delta\,\mathrm{Li}_2\left(\delta^2\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^2\right) \\
& +32\ln\delta\,\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{1}{1+\delta^2}\right) - 32\ln\delta\,\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) \\
& +32\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\frac{\delta^2}{1+\delta^2}\right) - 8\ln\left(1-\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) + 8\ln\left(1+\delta^2\right)\mathrm{Li}_2\left(\delta^4\right) - 24\,\zeta_3 \\
& \left.-\frac{2}{3} - \frac{4}{3}\pi^2\ln 2 - M_A^{[1]}(\delta)\right]\ln\beta + c_2^A(\delta),
\end{aligned}$$

Two-loop coefficient v.s. EVENT2



- **We reproduce ALL logs at two loops**

LL Resummation

Cross section @ LL:

$$\frac{\sigma^{\mathrm{LL}}}{\sigma_0} = 1 - 8\frac{\alpha_s}{2\pi} C_F \ln\delta \ln\beta + \left(\frac{\alpha_s}{2\pi}\right)^2 \left[32\, C_F^2 \ln^2\delta + \frac{4}{3}\, C_F C_A \left(11 \ln\delta - \frac{\pi^2}{2} + 3\,\mathrm{Li}_2(\delta^4)\right) - \frac{16}{3} C_F T_F n_f \ln\delta\right] \ln^2\beta$$

Exponentiate the one-loop logarithms:

$$\exp\left[16\, C_F \ln\delta \int_{\alpha(Q\beta)}^{\alpha(Q)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha}{4\pi}\right] = 1 - 8\frac{\alpha_s}{2\pi} C_F \ln\delta \ln\beta + \left(\frac{\alpha_s}{2\pi}\right)^2 \left(32\, C_F^2 \ln^2\delta + \frac{44}{3}\, C_F C_A \ln\delta - \frac{16}{3} C_F T_F n_f \ln\delta\right) \ln^2\beta$$

LL Resummation

Cross section @ LL:

Non-Global Log!!!

$$\frac{\sigma^{\rm LL}}{\sigma_0} = 1 - 8\frac{\alpha_s}{2\pi}C_F \ln\delta \ln\beta + \left(\frac{\alpha_s}{2\pi}\right)^2 \left[32\, C_F^2 \ln^2\delta + \frac{4}{3}\, C_F C_A \left(11\ln\delta - \frac{\pi^2}{2} + 3\,{\rm Li}_2(\delta^4)\right) - \frac{16}{3} C_F T_F n_f \ln\delta\right] \ln^2\beta$$

(Dasgupta & Salam 2002)

Exponentiate the one-loop lo

$$\mathcal{S}_2 = -4C_F C_A \left[\frac{\pi^2}{12} + (\Delta\eta)^2 - \Delta\eta \ln\left(e^{2\Delta\eta} - 1\right) - \frac{1}{2}{\rm Li}_2\left(e^{-2\Delta\eta}\right) - \frac{1}{2}{\rm Li}_2\left(1 - e^{2\Delta\eta}\right)\right]$$

$$\exp\left[16\, C_F \ln\delta \int_{\alpha(Q\beta)}^{\alpha(Q)} \frac{d\alpha}{\beta(\alpha)}\frac{\alpha}{4\pi}\right] = 1 - 8\frac{\alpha_s}{2\pi}C_F\ln\delta\ln\beta + \left(\frac{\alpha_s}{2\pi}\right)^2\left(32\, C_F^2\ln^2\delta + \frac{44}{3}\, C_F C_A \ln\delta - \frac{16}{3}C_F T_F n_f \ln\delta\right)\ln^2\beta$$

Renormalization

- We renormalise the bare hard function

$$\boldsymbol{\mathcal{H}}_m(\{\underline{n}\}, Q, \delta, \epsilon) = \sum_{l=2}^{m} \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu)\, \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)$$

- The Z-factor has the form $Z^H(\{\underline{n}\}, \epsilon, \mu) \sim \begin{pmatrix} 1 & \alpha_s & \alpha_s^2 & \alpha_s^3 & \dots \\ 0 & 1 & \alpha_s & \alpha_s^2 & \dots \\ 0 & 0 & 1 & \alpha_s & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

- By consistency, matrix Z^H must render the soft function finite

$$\boldsymbol{\mathcal{S}}_l(\{\underline{n}\}, Q\beta, \delta, \mu) = \sum_{m=l}^{\infty} \boldsymbol{Z}^H_{lm}(\{\underline{n}\}, Q, \delta, \epsilon, \mu)\, \hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \epsilon)$$

- We verify that Z^H renormalises the two-loop soft function

$$\mathcal{S}_2(\mu) = Z^H_{22}\, \mathcal{S}_2(\epsilon) + Z^H_{23}\, \hat{\otimes}\, \mathcal{S}_3(\epsilon) + Z^H_{24}\, \hat{\otimes}\, 1 + \mathcal{O}(\alpha_s^3)$$

- and the general one-loop soft function

$$\alpha_s z_{m,m}(\{\underline{n}\}, \epsilon, \mu) + \int \frac{d\Omega(n_{m+1})}{4\pi}\, \alpha_s z_{m,m+1}(\{\underline{n}, n_{l+1}\}, \epsilon, \mu) + \mathcal{S}_m(\{\underline{n}\}, \epsilon) = \text{finite}$$

LL resummation

Expand RG equation order by order

$$\mathcal{S}_2^{(1)} = L\left[\boldsymbol{R}_2 + \boldsymbol{V}_2\right]$$
$$= -\left(\frac{\alpha_s}{\pi} N_C L\right) \int_\Omega \mathbf{3}_{\rm Out} W_{12}^3,$$
$$\mathcal{S}_2^{(2)} = L^2\left[\boldsymbol{R}_2\left(\boldsymbol{R}_3 + \boldsymbol{V}_3\right) + \boldsymbol{V}_2\left(\boldsymbol{R}_2 + \boldsymbol{V}_2\right)\right]$$
$$= \frac{1}{2!}\left(\frac{\alpha_s}{\pi} N_c L\right)^2 \int_\Omega \left[-\mathbf{3}_{\rm In}\mathbf{4}_{\rm Out}\left(P_{12}^{34} - W_{12}^3 W_{12}^4\right) + \mathbf{3}_{\rm Out}\mathbf{4}_{\rm Out} W_{12}^3 W_{12}^4\right],$$
$$\mathcal{S}_2^{(3)} = L^3\left[\boldsymbol{R}_2\left[\boldsymbol{R}_3(\boldsymbol{R}_4 + \boldsymbol{V}_4) + \boldsymbol{V}_3(\boldsymbol{R}_3 + \boldsymbol{V}_3)\right] + \boldsymbol{V}_2\left[\boldsymbol{R}_2(\boldsymbol{R}_3 + \boldsymbol{V}_3) + \boldsymbol{V}_2(\boldsymbol{R}_2 + \boldsymbol{V}_2)\right]\right]$$
$$= \frac{1}{3!}\left(\frac{\alpha_s}{\pi} N_c L\right)^3 \int_\Omega \Big[\mathbf{3}_{\rm In}\mathbf{4}_{\rm Out}\mathbf{5}_{\rm Out}\left[P_{12}^{34}\left(W_{13}^5 + W_{32}^5 + W_{12}^5\right) - 2W_{12}^3 W_{12}^4 W_{12}^5\right]$$
$$- \mathbf{3}_{\rm In}\mathbf{4}_{\rm In}\mathbf{5}_{\rm Out} W_{ab}^1\left[\left(P_{13}^{45} - W_{13}^4 W_{13}^5\right) + \left(P_{32}^{45} - W_{32}^4 W_{32}^5\right) - \left(P_{12}^{34} - W_{12}^4 W_{12}^5\right)\right]$$
$$- \mathbf{3}_{\rm Out}\mathbf{4}_{\rm Out}\mathbf{5}_{\rm Out} W_{12}^3 W_{12}^4 W_{12}^5\Big]$$

Agrees with order-by-order expansion of BMS equation

$$\partial_L G_{12}(L) = \int \frac{d\Omega_j}{4\pi} W_{12}^j\left[\Theta_{\rm in}^{n\bar{n}}(j) G_{1j}(L) G_{j2}(L) - G_{12}(L)\right]$$

given in Schwartz, Zhu ’14

LL resummation

Expand RG equation order by order

$$\mathcal{S}_2^{(1)} = L\left[\boldsymbol{R}_2 + \boldsymbol{V}_2\right]$$
$$= -\left(\frac{\alpha_s}{\pi} N_C L\right) \int_\Omega \mathbf{3}_{\text{Out}} W_{12}^3,$$

similar to parton shower

$$\mathcal{S}_2^{(2)} = L^2\left[\boldsymbol{R}_2\left(\boldsymbol{R}_3 + \boldsymbol{V}_3\right) + \boldsymbol{V}_2\left(\boldsymbol{R}_2 + \boldsymbol{V}_2\right)\right]$$
$$= \frac{1}{2!}\left(\frac{\alpha_s}{\pi} N_c L\right)^2 \int_\Omega \Big[-\mathbf{3}_{\text{In}} \mathbf{4}_{\text{Out}} \left(P_{12}^{34} - W_{12}^3 W_{12}^4\right) + \mathbf{3}_{\text{Out}} \mathbf{4}_{\text{Out}} W_{12}^3 W_{12}^4 \Big],$$

$$\mathcal{S}_2^{(3)} = L^3\left[\boldsymbol{R}_2\left[\boldsymbol{R}_3(\boldsymbol{R}_4 + \boldsymbol{V}_4) + \boldsymbol{V}_3(\boldsymbol{R}_3 + \boldsymbol{V}_3)\right] + \boldsymbol{V}_2\left[\boldsymbol{R}_2(\boldsymbol{R}_3 + \boldsymbol{V}_3) + \boldsymbol{V}_2(\boldsymbol{R}_2 + \boldsymbol{V}_2)\right]\right]$$
$$= \frac{1}{3!}\left(\frac{\alpha_s}{\pi} N_c L\right)^3 \int_\Omega \Big[\mathbf{3}_{\text{In}} \mathbf{4}_{\text{Out}} \mathbf{5}_{\text{Out}} \left[P_{12}^{34}\left(W_{13}^5 + W_{32}^5 + W_{12}^5\right) - 2 W_{12}^3 W_{12}^4 W_{12}^5\right]$$
$$- \mathbf{3}_{\text{In}} \mathbf{4}_{\text{In}} \mathbf{5}_{\text{Out}} W_{ab}^1 \left[\left(P_{13}^{45} - W_{13}^4 W_{13}^5\right) + \left(P_{32}^{45} - W_{32}^4 W_{32}^5\right) - \left(P_{12}^{34} - W_{12}^4 W_{12}^5\right)\right]$$
$$- \mathbf{3}_{\text{Out}} \mathbf{4}_{\text{Out}} \mathbf{5}_{\text{Out}} W_{12}^3 W_{12}^4 W_{12}^5 \Big]$$

Agrees with order-by-order expansion of BMS equation

$$\partial_L G_{12}(L) = \int \frac{d\Omega_j}{4\pi} W_{12}^j \left[\Theta_{\text{in}}^{n\bar{n}}(j) G_{1j}(L) G_{j2}(L) - G_{12}(L)\right]$$

given in Schwartz, Zhu ’14

Conclusion

- We have derived a factorization formula for a NG observable: cone-jet process

$$\sigma = \sum_m \langle \mathcal{H}_m \otimes \mathcal{S}_m \rangle$$

$$\widetilde{\sigma} = \sigma_0 \, H \, \widetilde{S} \left[\sum_{m=1}^{\infty} \left\langle \mathcal{J}_m \otimes \widetilde{\mathcal{U}}_m \right\rangle \right]^2$$

- In both case we have checked the factorization up to NNLO and reproduce full QCD results

- All the scales are separated ⟶ RG evolution can be used to resum all large logarithms

- We develop numerical techniques to solve the associated RG equations at leading logarithmic level (NLL, NNLL,...)

- Numerous possible applications: jet cross sections, jet substructure, jet veto,......

Thank you

Extra Slides

$$S_{\mathrm{NG}} = -\frac{\pi^2}{24}L^2 + \frac{\zeta_3}{12}L^3 + \frac{\pi^4}{34560}L^4 + \left(-\frac{\pi^2\zeta_3}{360} + \frac{17\zeta_5}{480}\right)L^5 + \cdots$$

Resummation

Large logarithms in the soft function

$$\boldsymbol{\mathcal{S}}_l(\{\underline{n}\}, Q\beta, \delta, \mu_h) = \sum_{m \geq l} \boldsymbol{U}^S_{lm}(\{\underline{n}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \mu_s)$$

with the formal evolution matrix

$$\boldsymbol{U}^S(\{\underline{n}\}, \delta, \mu_s, \mu_h) = \mathbf{P} \exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu} \boldsymbol{\Gamma}^H(\{\underline{n}\}, \delta, \mu)\right]$$

Therefore the resumed cross section

$$\sigma(\beta, \delta) = \sum_{l=2}^{\infty} \langle \boldsymbol{\mathcal{H}}_l(\{\underline{n}\}, Q, \delta, \mu_h) \otimes \sum_{m \geq l} \boldsymbol{U}^S_{lm}(\{\underline{n}\}, \delta, \mu_s, \mu_h) \,\hat{\otimes}\, \boldsymbol{\mathcal{S}}_m(\{\underline{n}\}, Q\beta, \delta, \mu_s)$$

At Leading-Log level,

$$\boldsymbol{\mathcal{S}}^T = (1, 1, \cdots, 1) \qquad \mathcal{H} = (\sigma_0, 0, \cdots, 0) \qquad \boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{V}_2 & \boldsymbol{R}_2 & 0 & 0 & \ldots \\ 0 & \boldsymbol{V}_3 & \boldsymbol{R}_3 & 0 & \ldots \\ 0 & 0 & \boldsymbol{V}_4 & \boldsymbol{R}_4 & \ldots \\ 0 & 0 & 0 & \boldsymbol{V}_5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

LL resummation

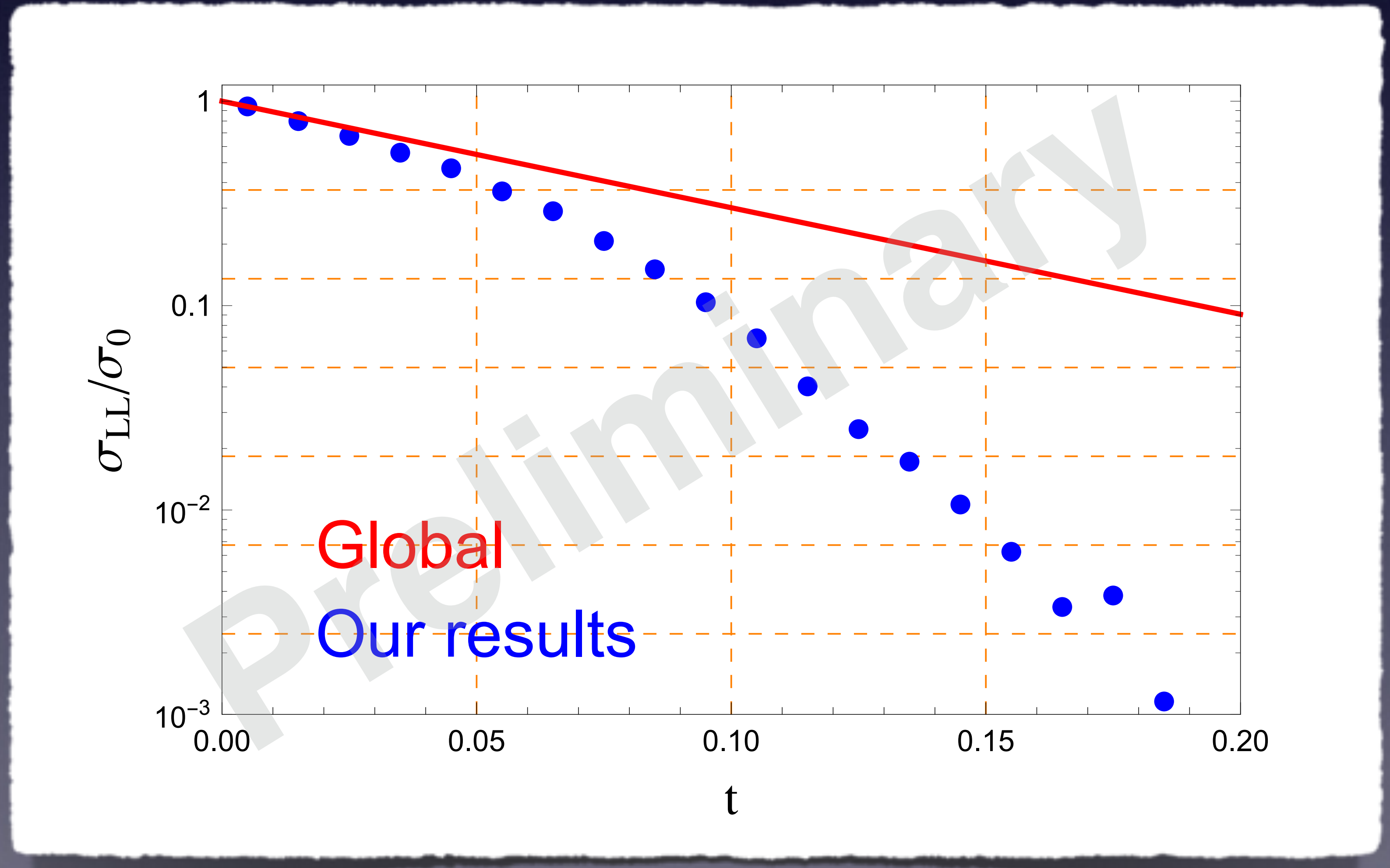
LL evolution equation: $\frac{d}{dt}\mathcal{H}_n(t) = \mathcal{H}_n(t)\mathcal{V}_n + \mathcal{H}_{n-1}(t)\mathcal{R}_{n-1}(t)$

$$t = \int_{\alpha(\mu_s)}^{\alpha(\mu_h)} \frac{d\alpha}{\beta(\alpha)} \frac{\alpha}{4\pi}$$

Solution: $\mathcal{H}_n(t) = \int_0^t dt'\, \mathcal{H}_{n-1}(t')\mathcal{R}_{n-1}(t')e^{-(t'-t)\mathcal{V}_n}$

Resummed cross section:

$$\sigma_{\rm LL} = \sum_{n=2}^{\infty} \mathcal{H}_n(t_s) \otimes \mathcal{S}_n(t_s)$$



Factorization

$$\sigma = \sum_m \langle \mathcal{H}_m \otimes \mathcal{S}_m \rangle$$

$$\boldsymbol{\mathcal{H}}_{k+l} = \mathcal{H}_2 \cdot \boldsymbol{\mathcal{J}}_k \cdot \boldsymbol{\mathcal{J}}_l \,,$$

$$\widetilde{\boldsymbol{\mathcal{S}}}_{k+l} = \widetilde{\mathcal{S}} \cdot \widetilde{\boldsymbol{\mathcal{U}}}_k \cdot \overline{\widetilde{\boldsymbol{\mathcal{U}}}}_l \,,$$

Factorization

$$\sigma = \sum_m \langle \mathcal{H}_m \otimes \mathcal{S}_m \rangle$$

$$\boldsymbol{\mathcal{H}}_{k+l} = \mathcal{H}_2 \cdot \boldsymbol{\mathcal{J}}_k \cdot \boldsymbol{\mathcal{J}}_l \,,$$

$$\widetilde{\boldsymbol{\mathcal{S}}}_{k+l} = \widetilde{\mathcal{S}} \cdot \widetilde{\boldsymbol{\mathcal{U}}}_k \cdot \overline{\widetilde{\boldsymbol{\mathcal{U}}}}_l \,,$$

$$\widetilde{\sigma}(\tau, \delta) = \sigma_0 \, H(Q) \, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \Big\langle \boldsymbol{\mathcal{J}}_m(Q\delta) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_m(Q\delta\tau) \Big\rangle \right]^2$$

Factorization

$$\sigma = \sum_m \langle \mathcal{H}_m \otimes \mathcal{S}_m \rangle$$

$$\boldsymbol{\mathcal{H}}_{k+l} = \mathcal{H}_2 \cdot \boldsymbol{\mathcal{J}}_k \cdot \boldsymbol{\mathcal{J}}_l \, ,$$

$$\widetilde{\boldsymbol{\mathcal{S}}}_{k+l} = \widetilde{\mathcal{S}} \cdot \widetilde{\boldsymbol{\mathcal{U}}}_k \cdot \overline{\widetilde{\boldsymbol{\mathcal{U}}}}_l \, ,$$

color trace

integration over angles

$$\widetilde{\sigma}(\tau,\delta) = \sigma_0 \, H(Q) \, \widetilde{S}(Q\tau) \left[\sum_{m=1}^{\infty} \left\langle \boldsymbol{\mathcal{J}}_m(Q\delta) \otimes \widetilde{\boldsymbol{\mathcal{U}}}_m(Q\delta\tau) \right\rangle \right]^2$$

hard function

soft function

jet function with m patrons at fixed direction

coft function with m Wilson lines

Comparison with approach of Caron-Huot

(1501.03754)

- Caron-Huot defines **colour density matrix:**

$$\sigma[U] = \sum_n \int d\Pi_n \, [A_n^{a_1 \cdots a_n}(\{p_i\})]^* \, U^{a_1 b_1}(\theta_1) \cdots U^{a_n b_n}(\theta_n) \, \left[A_n^{b_1 \cdots b_n}(\{p_i\})\right]$$

- Here unitary matrices $U(\theta)$ are used to track the contributions from different particle multiplicities.

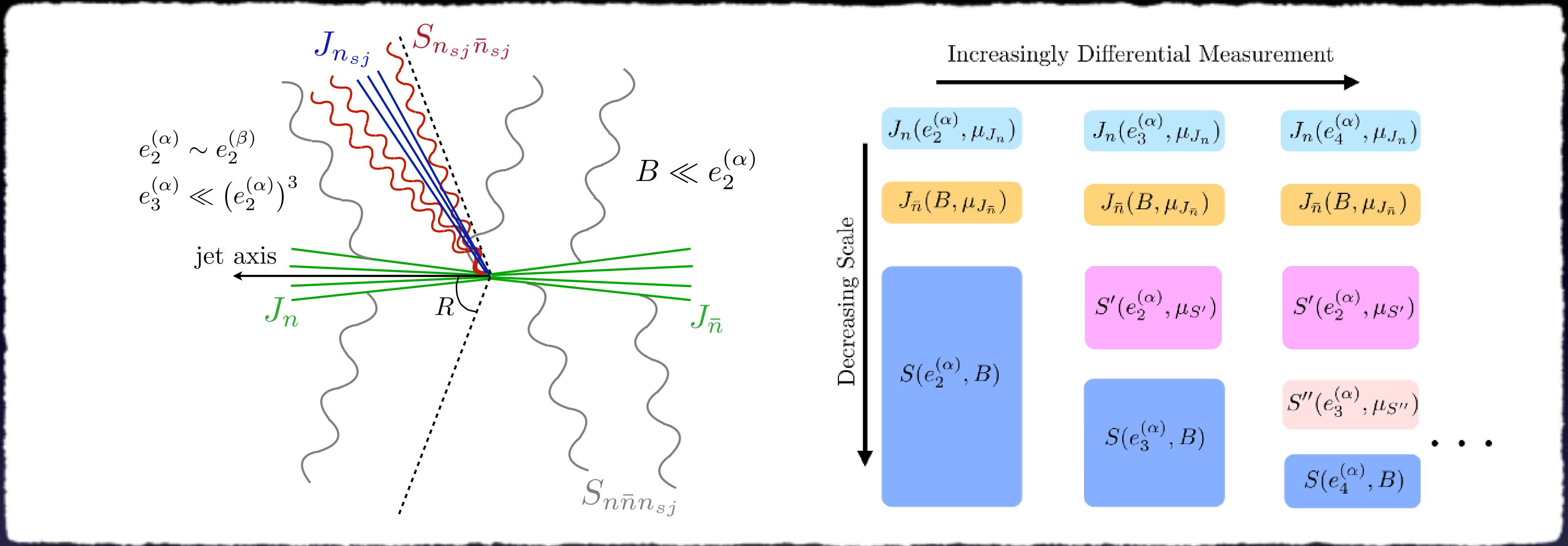
$$\left[\mu \frac{d}{d\mu} + \beta \frac{d}{d\alpha_s}\right] \sigma^{\rm ren}[U;\mu] = K(U, \delta/\delta U, \alpha_s(\mu), \epsilon)\sigma^{\rm ren}[U;\mu]$$

- The one-loop expression for K are in one-to-one correspondence to our anomalous dimensions, and the LL resumed results are the same as ours.

- Beyond LL accuracy, the relation is less immediate. Caron-Huot doesn't distinguish hard and soft partons but multiplies every parton by a matrix $U(\theta)$, and also doesn't include the Wilson line structure which is a important feature of our formula.

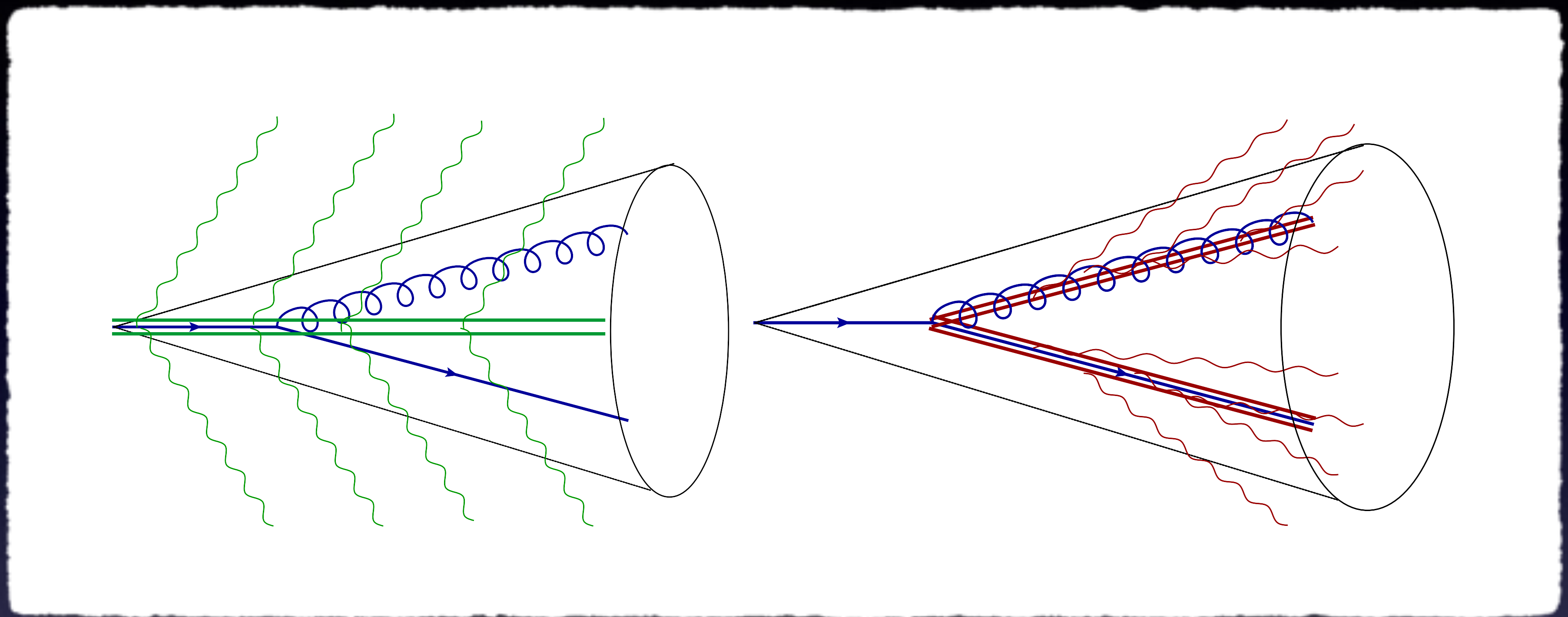
Comparison with LMN approach

(Larkoski, Moult and Neill, 1501.04596)



- LMN perform differential measurements to isolate regions where soft subjets give rise to NGLs. Resummation of the GLs associated with subjet observables resums part of the NGLs.
- We derive factorization theorem directly for NG observables: Resummation of NGLs with RG. Soft Wilson lines along energetic particles instead of soft subjets.
- LMN method involves a tower of effective theories with more and more d.o.f;
- We work with a single theory with only two d.o.f: hard and soft. NGLs get factorized into hard and soft logs.

Coft factorization



For cone-jet processes with narrow cones, small angle soft radiation became relevant

- collinear and soft ("coft")
- resolves individual collinear partons: operators with multiple Wilson lines

Method of region expansion

To isolate the different contributions, one expand the amplitudes as well as the phase space constrains in each momentum region.

- Generic soft mode has O(1) angle: after expansion, it is always outside the jet
- Collinear mode has large energy. Can never go outside the jet
- Coft mode can be inside or outside, but its contribution to momentum inside the jet is negligible.

Expansion is performed on the integrand level: the full result is obtained after combining the contributions from the different regions.

Comparison to BMS

Explicit form of the one-loop anomalous dimensions:

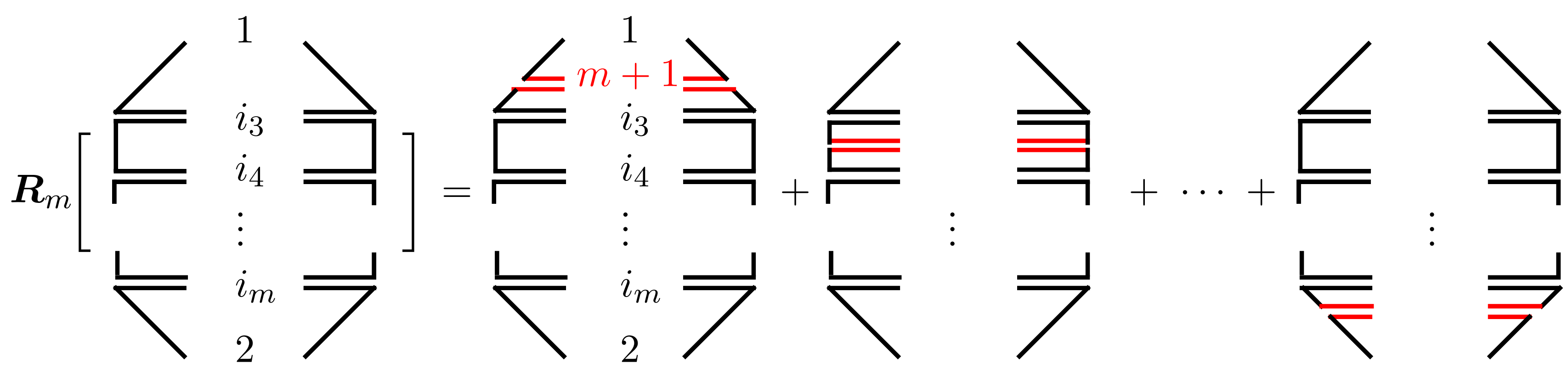
$$\boldsymbol{V}_m = \boldsymbol{\Gamma}^{(1)}_{m,m} = -\sum_{(ij)} \frac{1}{2}(\boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R}\cdot\boldsymbol{T}_{j,R}) \int \frac{d\Omega(n_k)}{4\pi} W^k_{ij} \left[\Theta_{\rm in}(k) + \Theta_{\rm out}(k)\right] ,$$

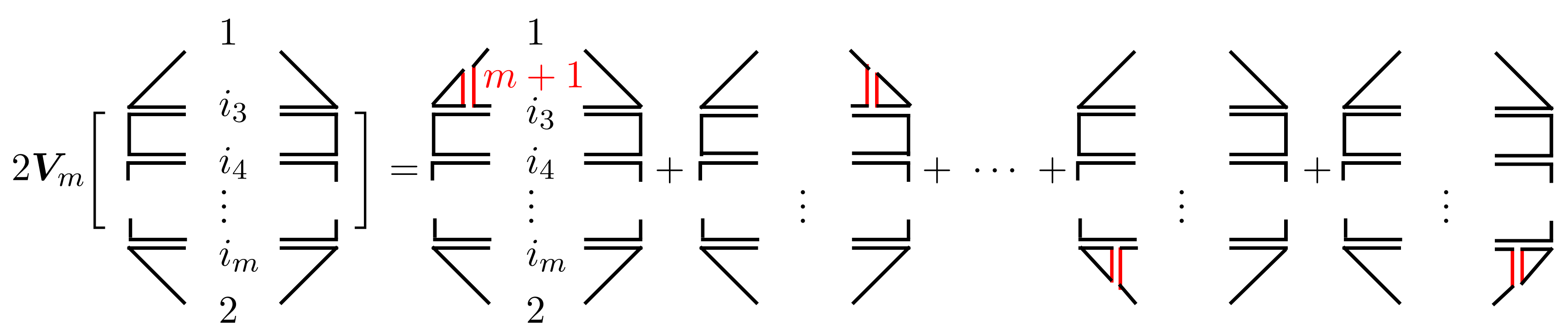
$$\boldsymbol{R}_m = \boldsymbol{\Gamma}^{(1)}_{m,m+1} = \sum_{(ij)} \boldsymbol{T}_{i,L}\cdot\boldsymbol{T}_{j,R} \int \frac{d\Omega(n_k)}{4\pi} W^k_{ij} \Theta_{\rm in}(k)$$

In the large Nc colour structure become trivial

- V_m simply gives a factor of Nc at every loop
- R_m only acts in between neighbouring patrons

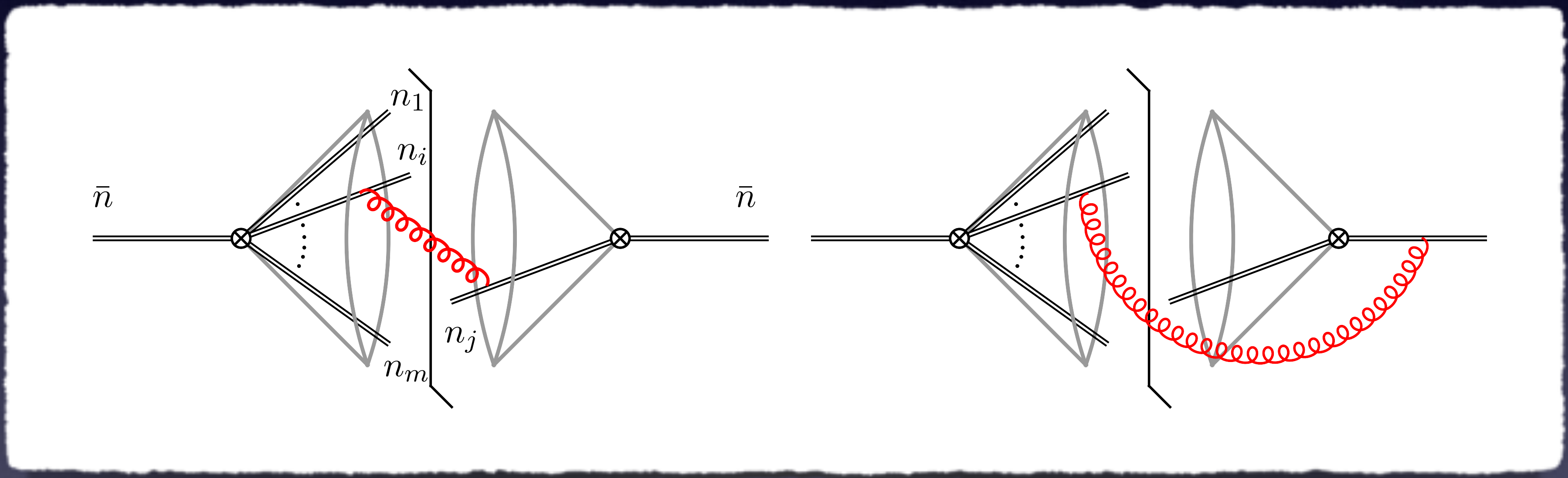
Action of R_m and V_m





One-loop renormalization for the narrow-angle jet process

$$\frac{1}{2}\mathcal{H}^{(1)}\cdot\mathbf{1}+\frac{1}{2}\widetilde{\mathcal{S}}^{(1)}\cdot\mathbf{1}+\boldsymbol{z}^{(1)}_{m,m}+\boldsymbol{z}^{(1)}_{m,m+1}+\widetilde{\boldsymbol{\mathcal{U}}}^{(1)}_{m}=\text{fin.}$$



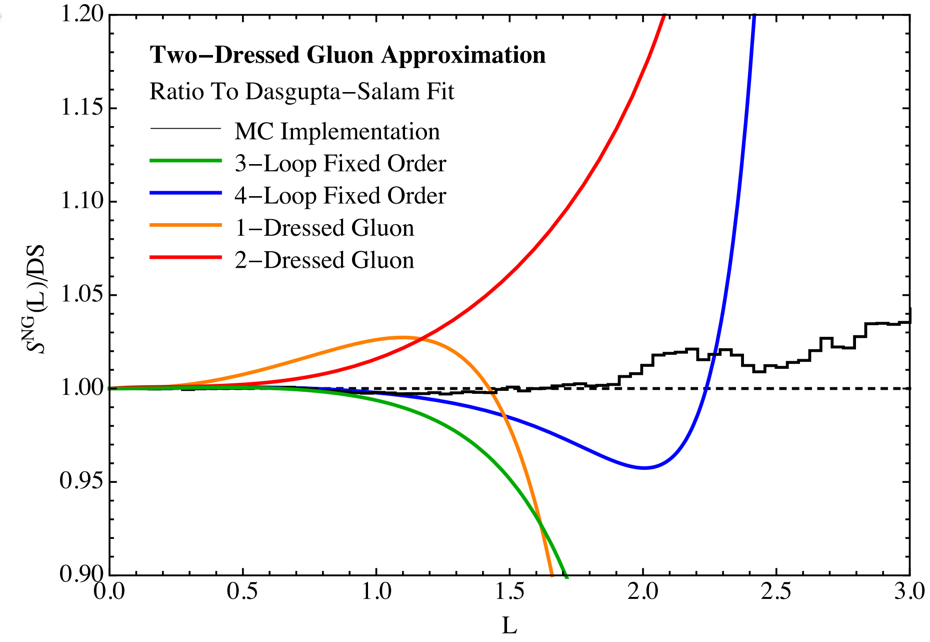
$$\widetilde{\mathcal{U}}^{(1)}_m(\{\underline{n}\},\epsilon)=-\frac{1}{\epsilon}\sum_{(ij)}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left[\ln\left(1-\hat{\theta}_i^2\right)+\ln\left(1-\hat{\theta}_j^2\right)-\ln\left(1-2\cos\phi_j\hat{\theta}_i\hat{\theta}_j+\hat{\theta}_i^2\hat{\theta}_j^2\right)\right]$$

$$-\frac{2}{\epsilon}\sum_{i=1}^{l}\boldsymbol{T}_0\cdot\boldsymbol{T}_i\ln\left(1-\hat{\theta}_i^2\right)+\boldsymbol{T}_0\cdot\boldsymbol{T}_0\left(-\frac{2}{\epsilon^2}+\frac{4\,L_{Q\tau\delta}}{\epsilon}\right)$$

Dressed-gluon expansion

- Even at LL accuracy, terms with arbitrary many subjets contribute. Not clear by which parameter higher-order terms suppressed.

	L^1	L^2	L^3	L^4
One-Dressed	0	$-\frac{\pi^2}{24}$	$\frac{\zeta(3)}{6}$	$-\frac{\pi^4}{720}$
Two-Dressed	0	0	$-\frac{\zeta(3)}{12}$	$\frac{\pi^4}{480}(1 \pm 0.05)$
Sum	0	$-\frac{\pi^2}{24}$	$\frac{\zeta(3)}{12}$	$\frac{\pi^4}{1440}(1 \pm 0.2)$
Exact	0	$-\frac{\pi^2}{24}$	$\frac{\zeta(3)}{12}$	$\frac{\pi^4}{34560}$



- Our RG resummation method is standard (but the RG is complicated!). Clear which ingredients are needed for a given log accuracy.

Open questions in LMN approach

- The problems with traditional global factorization theorems become visible only at NNLO
  - Have evaluated all ingredients to this accuracy and verified that we reproduce the full QCD result. Would be worthwhile to do the same in their approach
- One expects that a factorization theorem for a jet cross section with additional measurements is at least as complicated as the factorization theorem we obtain: Multi-Wilson-line operators in LMN approach?
- We find that the operators with different multiplicities of energetic particles mix under renormalization. This effect should be present in some form in their approach.

The Strategy of Region

The Strategy of Region

A simple self-energy one-loop integral in 2-dimension at zero external momentum

$$I = \int_0^\infty d\,k \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln \frac{M}{m}}{M^2-m^2}$$

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How to obtain the expansion by expanding the integrand before carrying out the integral ?

The Strategy of Region

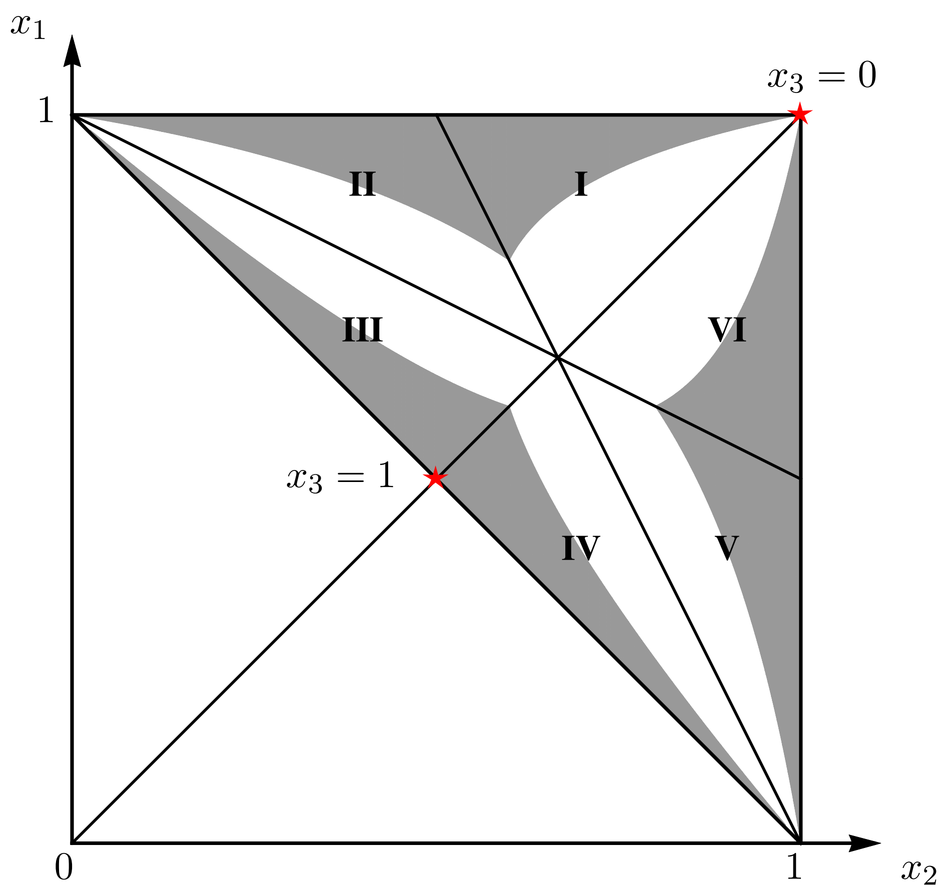
Naive expansion IR divergen integrals

$$\frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{k}{k^2(k^2+M^2)}\left(1-\frac{m^2}{k^2}+\frac{m^4}{k^2}+\cdots\right)$$

The series expansion is valid only for $k \gg m$, while the integration includes a region $k \sim m$.

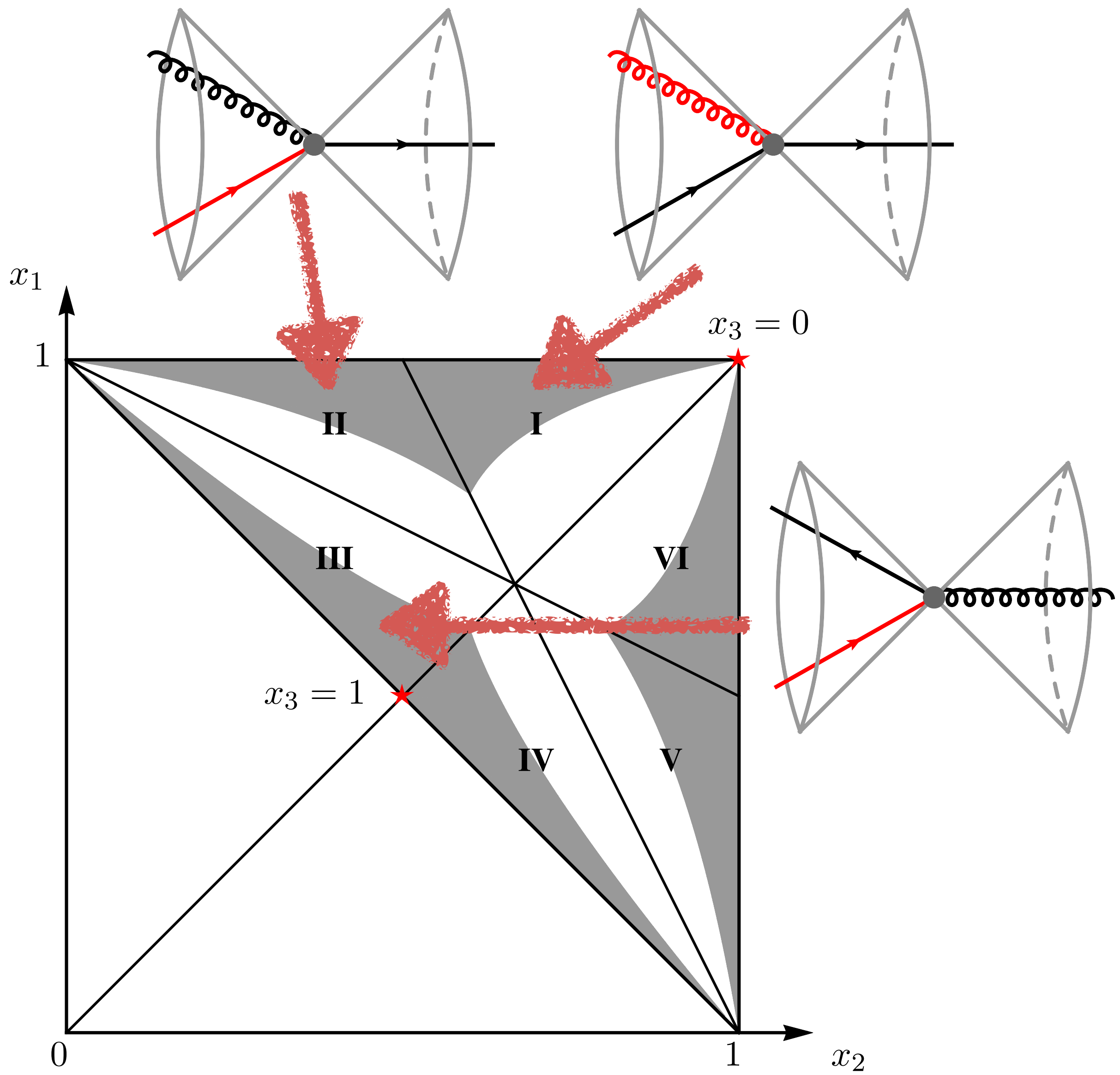
Introduce a new scale $m \ll \Lambda \ll M$ to separate the two momentum regions

Hard Function $\mathcal{H}_3$



$$x_i = 2E_i/Q$$

Hard Function $\mathcal{H}_3$



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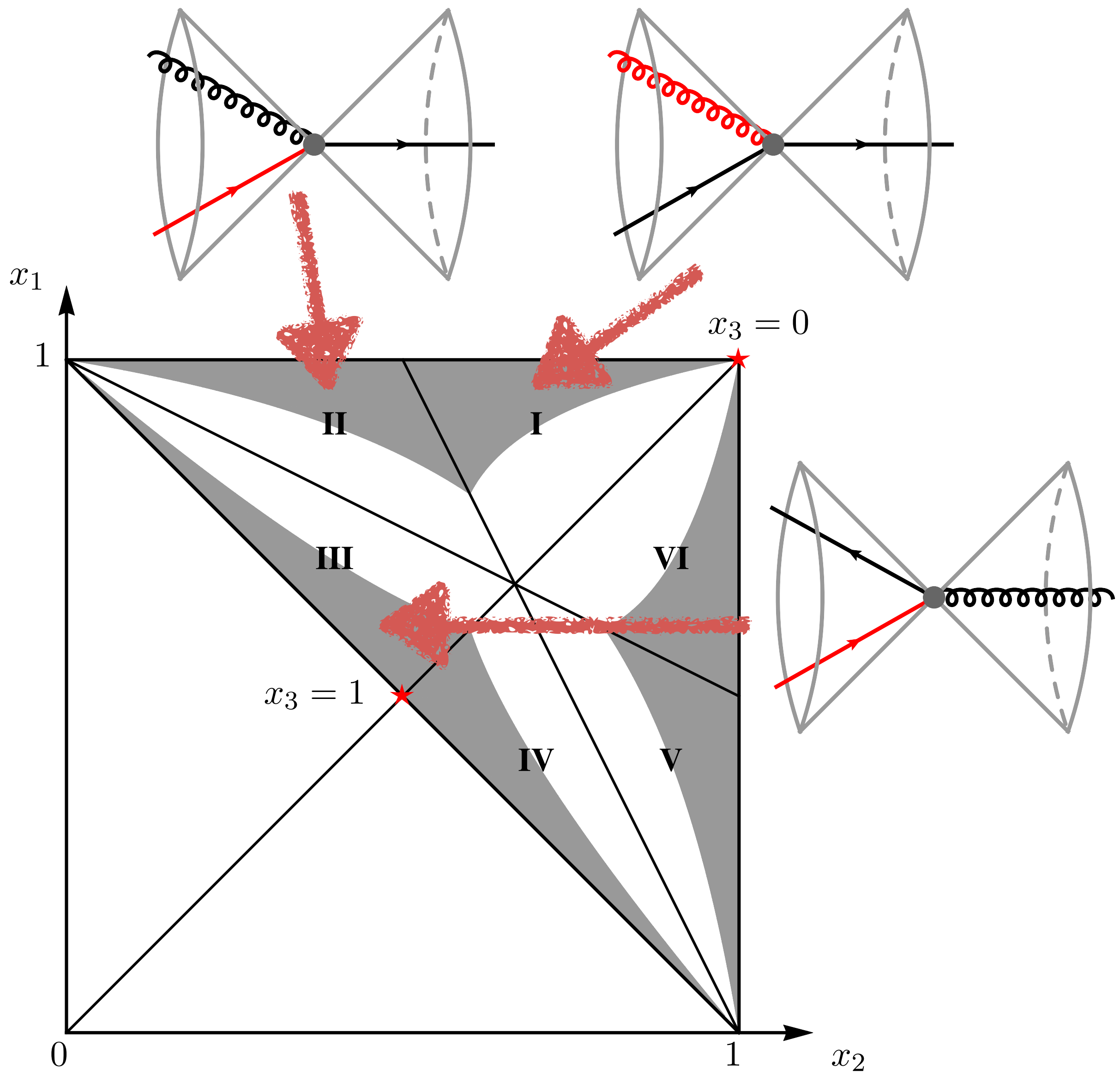

$x_i = 2E_i/Q$

region I:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} u^{-1-2\epsilon}\, v^{-1-\epsilon}\, h_3^{\mathbf{I}}(u, v, \delta, \epsilon)$$

region II:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} v^{-1-\epsilon}\, h_3^{\mathbf{II}}(u, v, \delta, \epsilon)$$

region III:

$$C_F \left(\frac{\mu}{Q}\right)^{\epsilon} h_3^{\mathbf{III}}(u, v, \delta, \epsilon)$$

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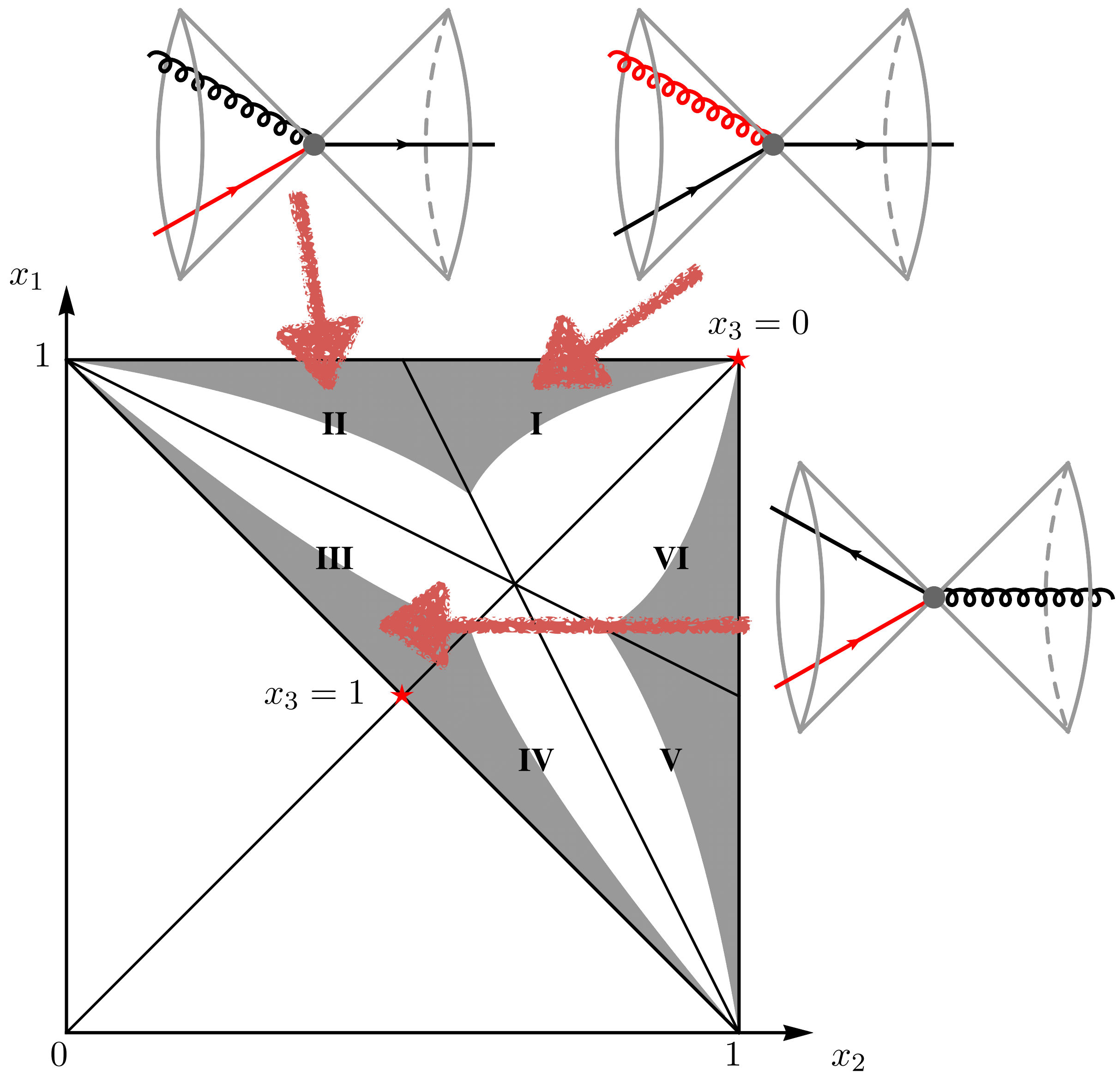

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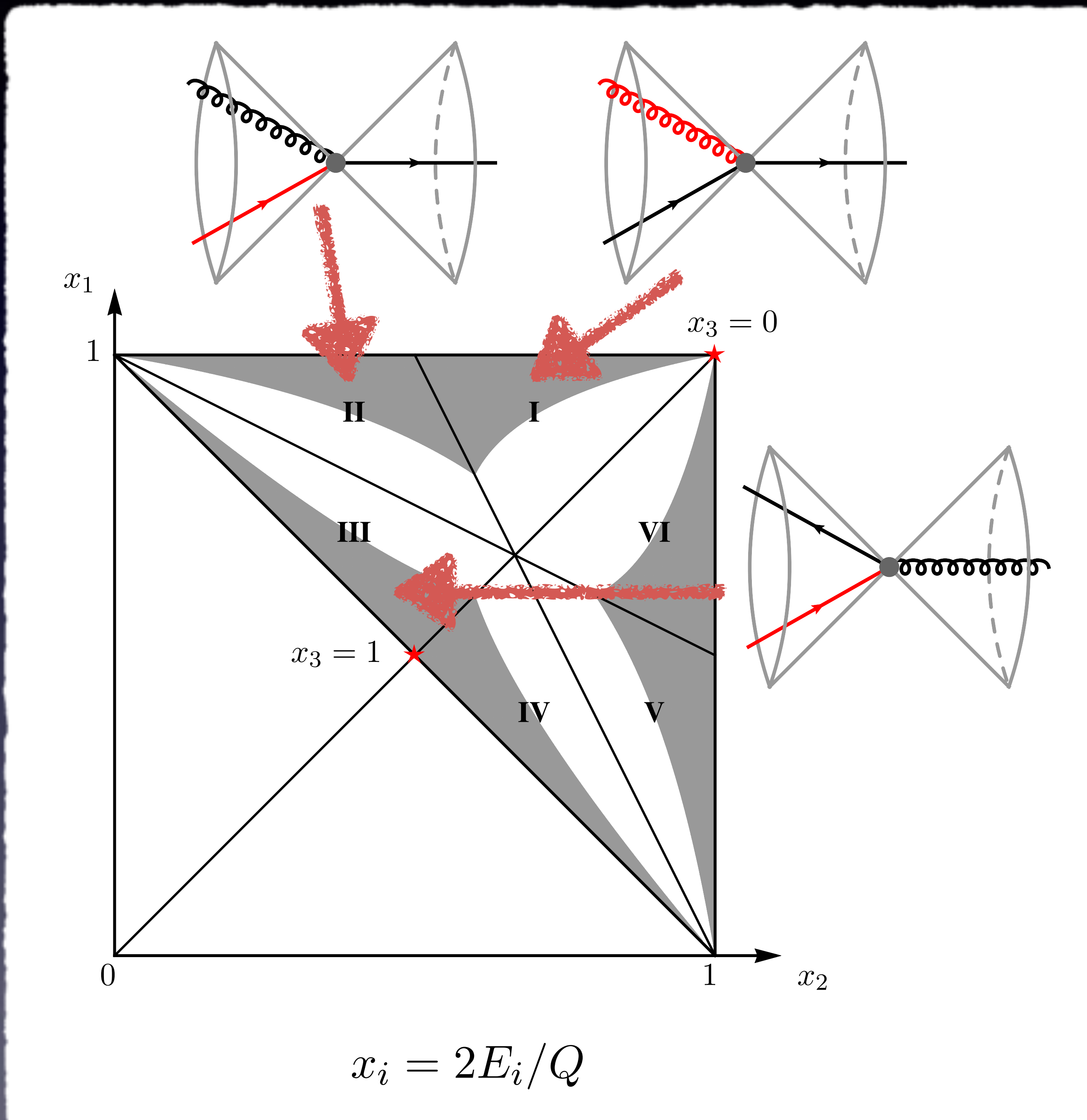
$$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \mathbf{1} \rangle$$

NLO

Hard Function $\mathcal{H}_3$



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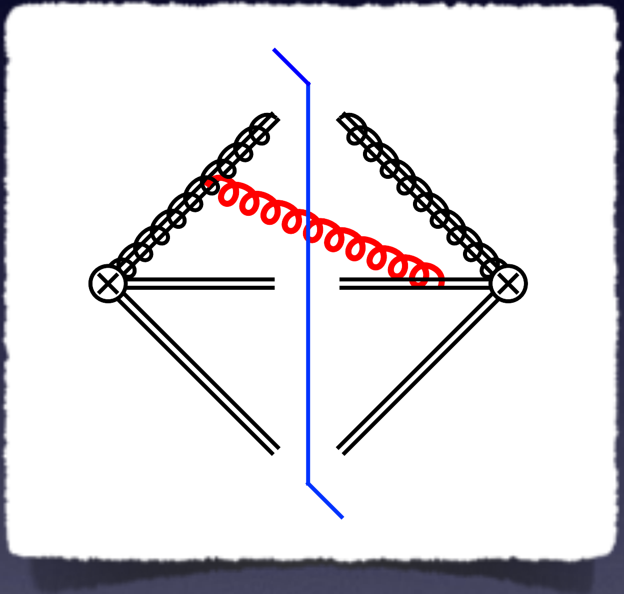
$$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \mathbf{1} \rangle \qquad \langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \boldsymbol{\mathcal{S}}_3^{(1)} \rangle$$

NLO NNLO

Soft Function $\mathcal{S}_3$

$$\boldsymbol{\mathcal{S}}_3^{(1)}(\{\underline{n}\},\epsilon) = \frac{4}{\epsilon} S_\epsilon \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi} \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_k \cdot n_j} \Theta_{\text{out}}(n_k)$$

$(i,j) = (1,2), (1,3), (2,3)$



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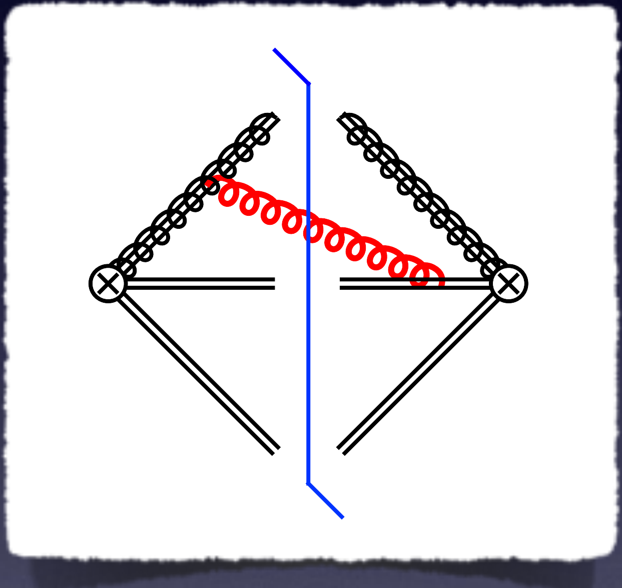

$$\left\langle \boldsymbol{\mathcal{S}}_3^{(1)} \right\rangle = \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \left[C_F \, s_3^F(u,v,\delta,\epsilon) + C_A \, s_3^A(u,v,\delta,\epsilon)\right]$$

$$s_3^A = \frac{1}{\epsilon} A_{\rm I}^{[-1]}(u,v,\delta) + A_{\rm I}^{[0]}(u,v,\delta) + \epsilon \, A_{\rm I}^{[1]}(u,v,\delta)$$

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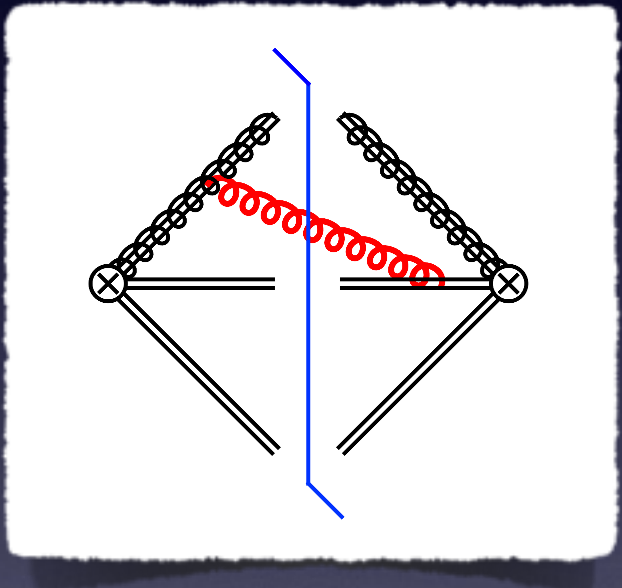

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$$\langle \boldsymbol{\mathcal{H}}_3^{(1)} \otimes \boldsymbol{\mathcal{S}}_3^{(1)} \rangle = C_\epsilon \left(\frac{\mu}{Q}\right)^{2\epsilon} \left(\frac{\mu}{Q\beta}\right)^{2\epsilon} \delta^{-2\epsilon} \left[C_F^2 \, M_F(\delta,\epsilon) + C_F \, C_A \, M_A(\delta,\epsilon)\right]$$

$$M_A(\delta,\epsilon) = \frac{1}{\epsilon^2}\left[-8\,\mathrm{Li}_2(\delta^4) + \frac{4\pi^2}{3}\right] + \frac{2M_A^{[1]}(\delta)}{\epsilon}$$

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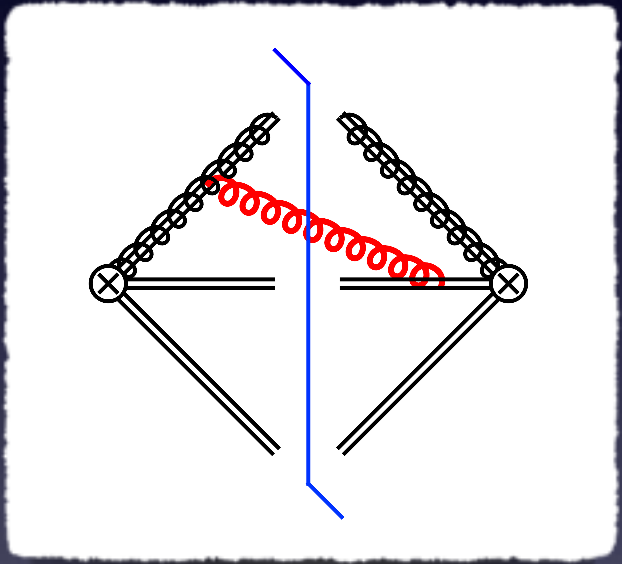

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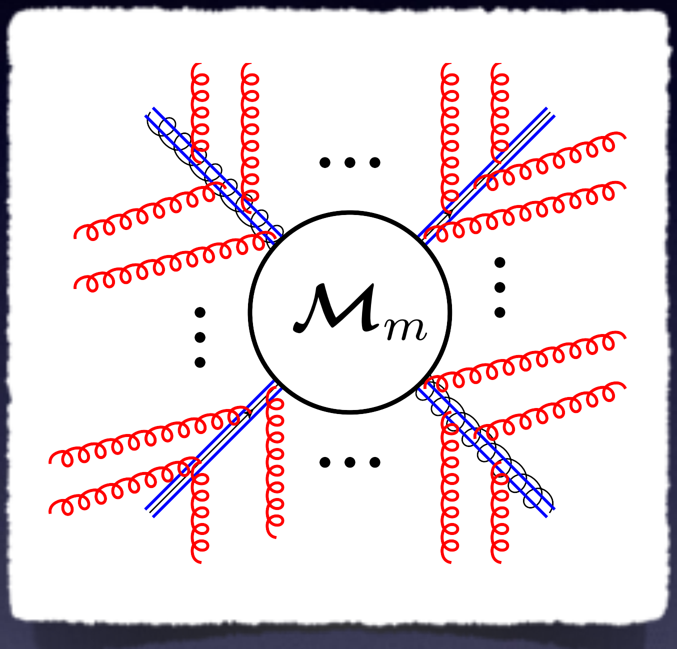
NGLs !!!

Factorization

- The operator for the emission from an amplitude with m hard partons



$$S_1(n_1)\, S_2(n_2)\, \ldots\, S_m(n_m)|\mathcal{M}_m(\{\underline{p}\})\rangle$$

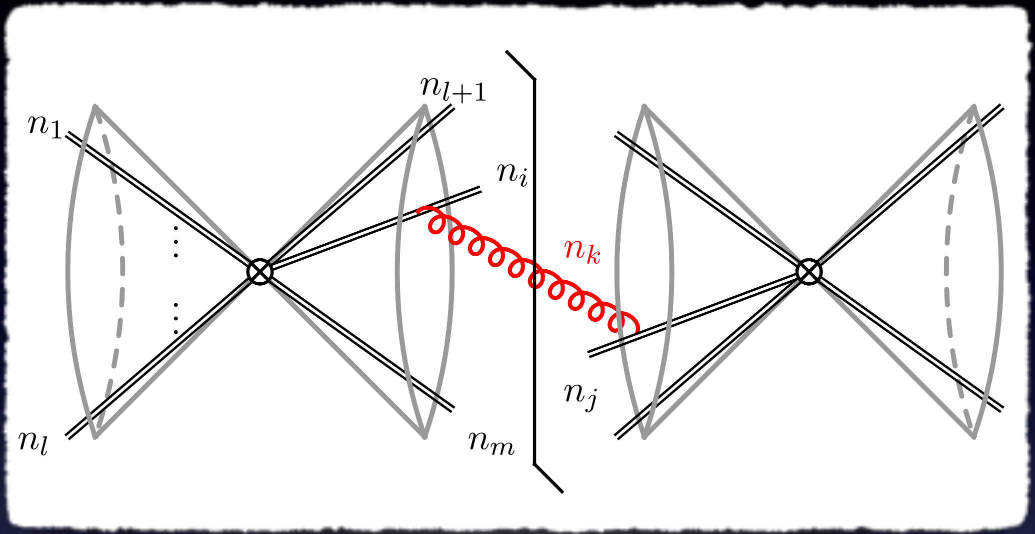
- We define the squared matrix element of this operator as

$$\mathcal{S}_m(\{\underline{n}\}, Q\beta, \delta) = \sum_X\!\!\!\!\!\!\int \langle 0|S_1^\dagger(n_1)\, \ldots\, S_m^\dagger(n_m)|X_s\rangle\langle X_s|S_1(n_1)\, \ldots\, S_m(n_m)|0\rangle\, \theta\,(Q\beta - 2E_{\text{out}})$$

Renormalization

one-loop soft function

$$\boldsymbol{\mathcal{S}}_m(\{\underline{n}\},\epsilon) = \boldsymbol{1} + \frac{2}{\epsilon}\frac{\alpha_s}{4\pi}\sum_{(ij)} \boldsymbol{T}_i\cdot\boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi}\, W_{ij}^k\, \Theta_{\text{out}}^{n\bar{n}}(n_k)$$



dipole radiator:

$$W_{ij}^k = \frac{n_i\cdot n_j}{n_i\cdot n_k\, n_j\cdot n_k}$$

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No coll. div.

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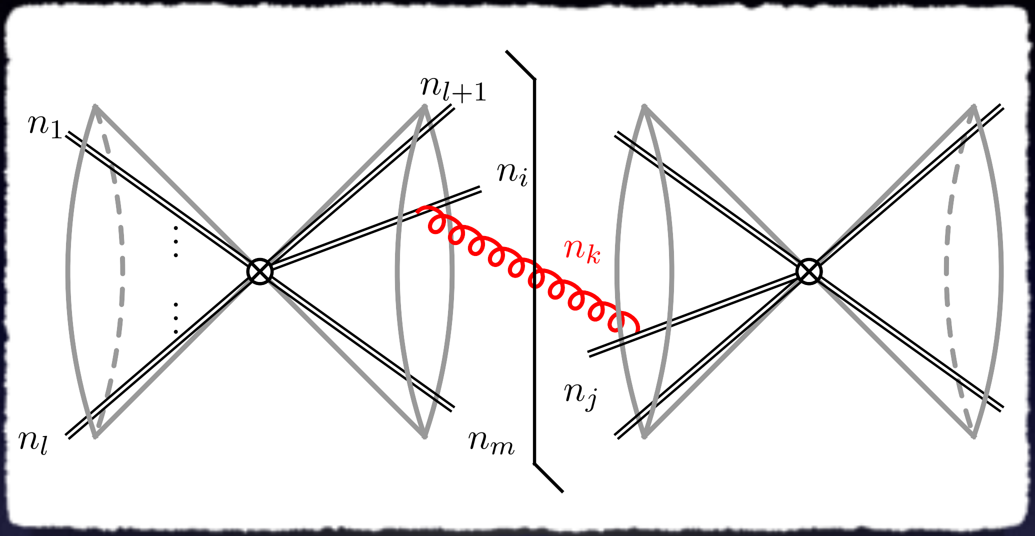

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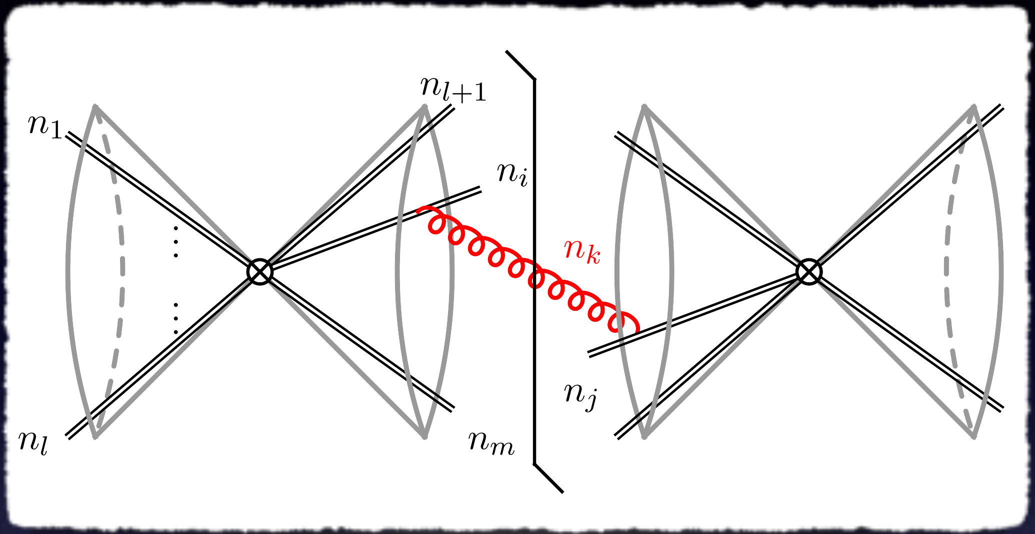


dipole radiator: $$W_{ij}^k = \frac{n_i \cdot n_j}{n_i \cdot n_k \, n_j \cdot n_k}$$

$z_{m,m}$: div. of virtual correction to m-legs amplitude

$z_{m,m+1}$: div. from additional radiation

$$V_m \sim -\int d^d k W_{ij}^k \left[\Theta_{\text{in}}(n_k) + \Theta_{\text{out}}(n_k)\right] \qquad R_m \sim \int d^d k W_{ij}^k \Theta_{\text{in}}(n_k)$$

$$V_m + R_m = -\frac{2}{\epsilon}\frac{\alpha_s}{4\pi}\sum_{(ij)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \int \frac{d\Omega(n_k)}{4\pi} W_{ij}^k \Theta_{\text{out}}^{n\bar{n}}(n_k)$$

NNLO singular terms

- Up to NNLO,

$$\sigma(\beta,\delta) = \sigma_0 \langle \mathcal{H}_2 \mathcal{S}_2 + \mathcal{H}_3 \otimes \mathcal{S}_3 + \mathcal{H}_4 \otimes \mathbf{1} \rangle .$$

- the hard function $\mathcal{H}_m$ starts from $\mathcal{O}(\alpha_s^{m-2})$

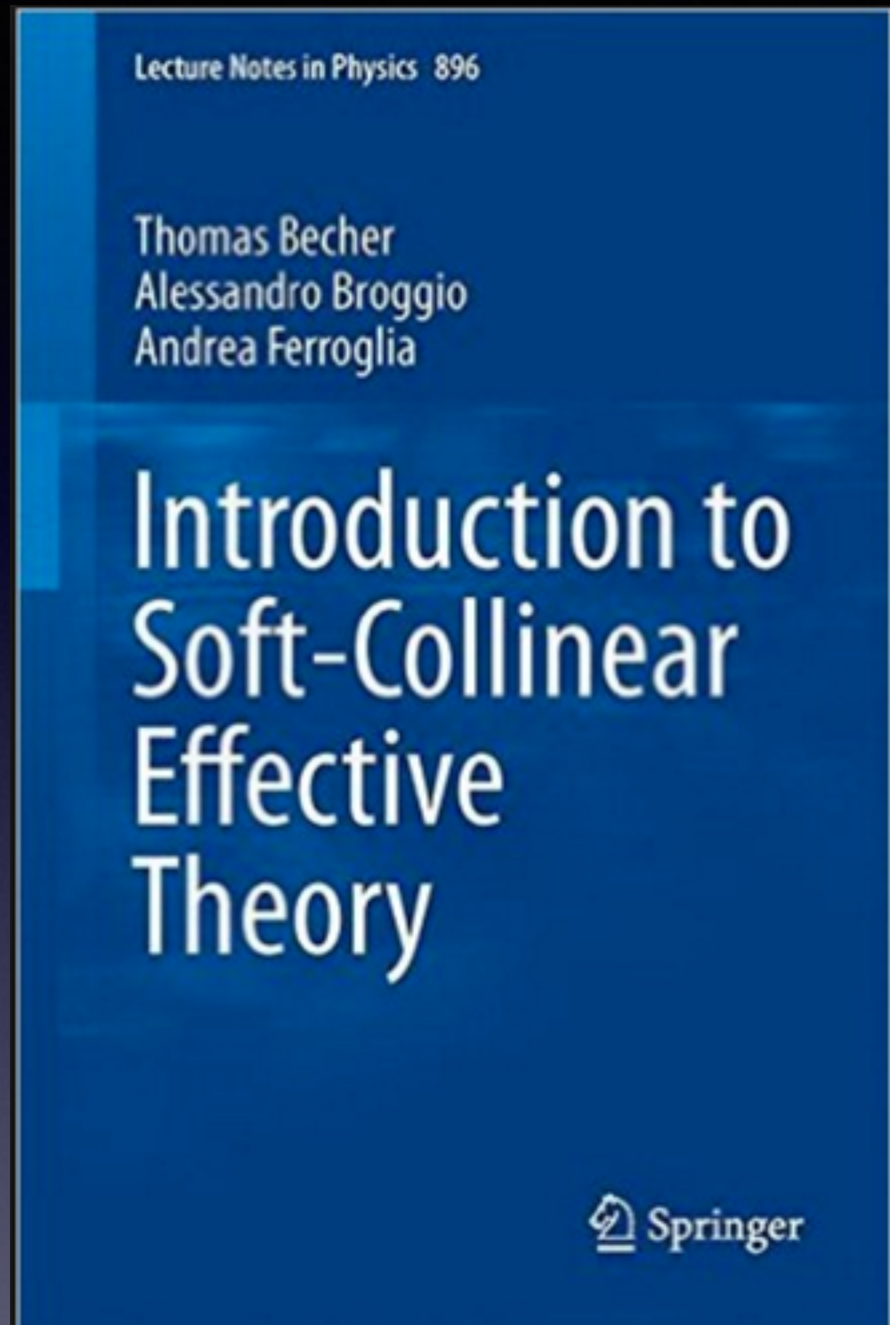
$$\mathcal{H}_2 \sim 1 \qquad \mathcal{H}_3 \sim \alpha_s \qquad \mathcal{H}_4 \sim \alpha_s^2$$

- two-loop $\mathcal{S}_2$ (Kelley, Schwartz, Schabinger & Zhu '11)

$$\mathcal{S}_2(Q\beta,\epsilon) = \int_0^{Q\beta/2} d\,\lambda \int_0^{+\infty} d\,k_L \int_0^{+\infty} d\,k_R \, \boxed{S_R(k_L,k_R,\lambda,\mu)}$$

- Combining all the bare ingredients we obtain a finite result

$$\frac{\sigma(\beta,\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} A(\beta,\delta) + \left(\frac{\alpha_s}{2\pi}\right)^2 B(\beta,\delta)$$



arXiv: 1410.1892

See also Iain Steward's edX EFT online course.
(https://www.edx.org/course/effective-field-theory-mitx-8-eftx)

Coft Scale

We emphasize that the coft modes have very low virtuality $p_t^2 = \Lambda_t^2 = (Q\beta\delta)^2$, much lower than the virtuality of the collinear and soft modes. The presence of this low physical scale might have important implications for the relevance of non-perturbative effects. These are suppressed by the ratio $\Lambda_{\rm QCD}/\Lambda_t$, where $\Lambda_{\rm QCD} \sim 0.5\,{\rm GeV}$ is a scale associated with strong QCD dynamics. Non-perturbative corrections to jet processes can thus be much larger than the naive expectation $\Lambda_{\rm QCD}/Q$. For example, for a jet opening angle $\alpha = 10^\circ$ ($\delta \approx 0.09$) and 5% of the collision energy outside the jets ($\beta = 0.1$), one obtains $\Lambda_t \approx 1\,{\rm GeV}$ for $Q = 100\,{\rm GeV}$. It would be interesting to explore phenomenological consequences of this low-scale physics.

Resummed predictions are automatically provided by standard MC:

- (+) Much more flexible, since they can give a fully exclusive description of the final state
- (+) Make possible to include hadronization effects
- (−) Difficult matching with fixed order
- (−) Logarithmic accuracy often unclear
- (−) Difficult to estimate uncertainties

Analytical resummations provide the most advanced theoretical accuracies available at present

- (+) Up to NNLL in some cases (threshold, q_T , EEC)
- (+) Easy matching with fixed order
- (+) Easier to estimate uncertainties
- (−) Have to be worked out for each observable (but progress in automatization is being made)