Spectrum and electromagnetic transitions of heavy quarkonium

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Outline

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Motivation

1. Heavy quarkonium is considered to be an excellent laboratory to study Quantum Chromodynamics (QCD) at low energies.
2. The EM decays of a hadron are sensitive to its inner structure, so it is crucial for us to determine the quantum numbers of the newly $c\bar{c}$ and $b\bar{b}$ states.
3. LHC, Belle and BESIII have demonstrated the ability to observe and measure the properties of heavy quarkonuim.
4. We expect that our theoretical predictions will be helpful for experiment to search for the missing $c\bar{c}$ and $b\bar{b}$ mesons.

The flavor wave function

For a quarkonium $Q\bar{Q}$ state, its flavor wave function should have a determined C-parity. For a $C=+1$ state, its flavor wave function is

$$\Phi_S = \frac{1}{\sqrt{2}}(Q\bar{Q} + \bar{Q}Q),$$

while for a $C=-1$ state its flavor function is

$$\Phi_A = \frac{1}{\sqrt{2}}(Q\bar{Q} - \bar{Q}Q).$$

The spin wave function

The usual spin wave functions are adopted. For the spin $S = 0$ state, it is

$$\chi_0^0 \;=\; \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow),$$

and for the spin $S = 1$ states, the wave functions are

$$\chi_1^1 = \uparrow\uparrow, \quad \chi_{-1}^1 = \downarrow\downarrow, \quad \chi_0^1 = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow).$$

The space wave function

In this work, the space wave function of a quarkonium state is adopted by the nonrelativistic wave function,i.e.,

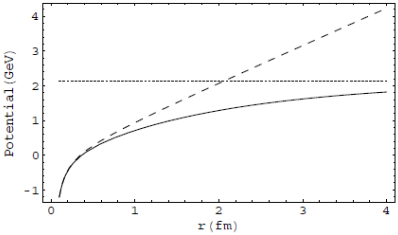
$$\psi^{n}_{Lm}(r) = R_{nL}(r)Y_{Lm}(\theta, \varphi)$$

To obtain the mass and wavefunction for a charmonium state, we need solve the radial Schrodinger equation:
$(C.H.Cai\ et.al\ HEP\,\&\,NP\ \ 27, 1005)$

$$\frac{du(r)}{dr^2} + 2\mu_R[E - V_{c\overline{c}}(r) - \frac{L(L+1)}{2\mu_R r^2}]u(r) = 0,$$

Then, the mass of a charmonium state is obtained by

$$M_{c\overline{c}}(r) = 2m_c + E.$$



$$V_{c\bar{c}}(r) = V(r) + H_{SS} + H_{SL} + H_T,$$

$$V(r) = V_c(r) + V_s(r), \quad V_c(r) = -\frac{4}{3}\frac{\alpha_s}{r},$$

$$V_s(r) = \begin{cases} br & \text{linear potential} \\ \frac{b}{\mu}(1 - e^{-\mu r}) & \text{screened potential.} \end{cases}$$

GI: S. Godfrey et al. PRD72, 054026; SNR: K.T. Chao et al. PRD79,094004.

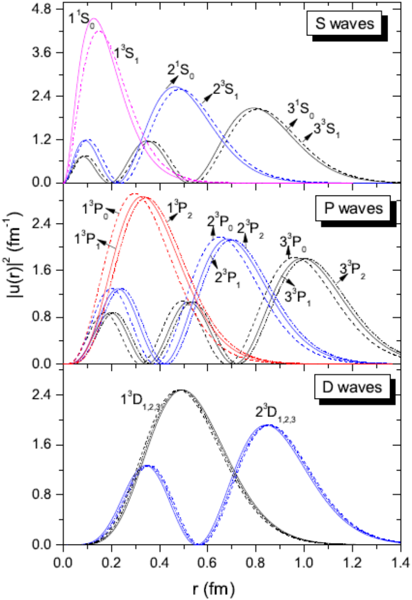


FIG. 1: (Color online) Predicted radial probability density $|u(r)|^2$ for S-, P- and D-wave bottomonium states.

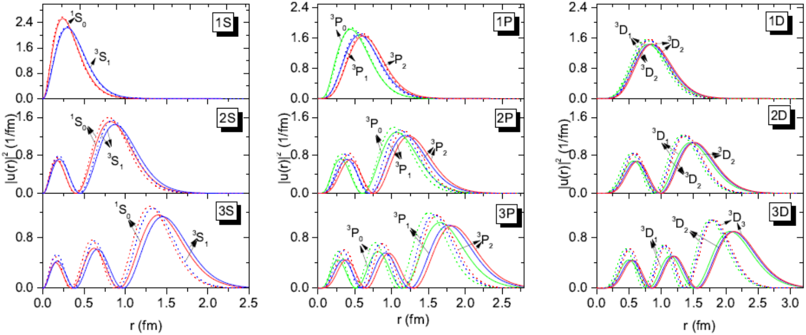


FIG. 1: (Color online) Predicted radial probability density $|u(r)|^2$ for S-, P- and D-wave bottomonium states up to $n = 3$ shell. The dotted and solid curves stand for the results obtained from the linear and screened potential models, respectively.

The total wave function

The total wave function coupling by spin-, flavor- and space wave function of heavy quarkonium is as follow:

$$|J, J_z\rangle = \sum_{J_z=m+S_z} C^{J\ l\ S}_{J_z\ m\ S_z} \Psi^n_{lm} \chi^S_{S_z} \Phi$$

If it is spin singlet and space wave functions' coupling, CG coefficient is 1, there are

$$|J, J_z\rangle = \sum_{J_z=m} C^{J\ l\ 0}_{J_z\ m\ 0} \Psi^n_{lm} \chi^0_0 \Phi = \Psi^n_{lm} \chi^0_0 \Phi$$

► Spectrum of $c\bar{c}$: *(X.H. Zhong&Q. Zhao PRD78 014029)*

$n^{2S+1}L_J$	name	J^{PC}	$M_{\exp}$	$M_{\mathrm{NR/GI/SNR}}$	LP	SP	wave function
1^3S_1	J/ψ	1^{--}	3097	3090/3098/3097	3097	3097	$\psi^0_{00}\chi^1_{S_z}\Phi_A$
1^1S_0	$\eta_c(1S)$	0^{-+}	2984	2982/2975/2979	2983	2984	$\psi^0_{00}\chi^0\Phi_S$
2^3S_1	$\psi(2S)$	1^{--}	3686	3672/3676/3673	3679	3679	$\psi^1_{00}\chi^1_{S_z}\Phi_A$
2^1S_0	$\eta_c(2S)$	0^{-+}	3639	3630/3623/3623	3635	3637	$\psi^1_{00}\chi^0\Phi_S$
3^3S_1	$\psi(3S)$	1^{--}	4040	4072/4100/4022	4078	4030	$\psi^2_{00}\chi^1_{S_z}\Phi_A$
3^1S_0	$\eta_c(3S)$	0^{-+}	3940?	4043/4064/3991	4048	4004	$\psi^2_{00}\chi^0\Phi_S$
1^3P_2	$\chi_{c2}(1P)$	2^{++}	3556	3556/3550/3554	3552	3553	$\psi^0_{1m}\chi^1_{S_z}\Phi_S$
1^3P_1	$\chi_{c1}(1P)$	1^{++}	3511	3505/3510/3510	3516	3521	$\psi^0_{1m}\chi^1_{S_z}\Phi_S$
1^3P_0	$\chi_{c0}(1P)$	0^{++}	3415	3424/3445/3433	3415	3415	$\psi^0_{1m}\chi^1_{S_z}\Phi_S$
1^1P_1	$h_c(1P)$	1^{+-}	3525	3516/3517/3519	3522	3526	$\psi^0_{1m}\chi^0\Phi_A$
2^3P_2	$\chi_{c2}(2P)$	2^{++}	3927	3972/3979/3937	3967	3937	$\psi^1_{1m}\chi^1_{S_z}\Phi_S$
2^3P_1	$\chi_{c1}(2P)$	1^{++}	3872?	3925/3953/3901	3937	3914	$\psi^1_{1m}\chi^1_{S_z}\Phi_S$
2^3P_0	$\chi_{c0}(2P)$	0^{++}	3918?	3852/3916/3842	3869	3848	$\psi^1_{1m}\chi^1_{S_z}\Phi_S$
2^1P_1	$h_c(2P)$	1^{+-}		3934/3956/3908	3940	3916	$\psi^1_{1m}\chi^0\Phi_A$
3^3P_2	$\chi_{c2}(3P)$	2^{++}	4350?	4317/4337/4208	4310	4211	$\psi^2_{1m}\chi^1_{S_z}\Phi_S$
3^3P_1	$\chi_{c1}(3P)$	1^{++}		4271/4317/4178	4284	4192	$\psi^2_{1m}\chi^1_{S_z}\Phi_S$
3^3P_0	$\chi_{c0}(3P)$	0^{++}		4202/4292/4131	4230	4146	$\psi^2_{1m}\chi^1_{S_z}\Phi_S$
3^1P_1	$h_c(3P)$	1^{+-}		4279/4318/4184	4286	4193	$\psi^2_{1m}\chi^0\Phi_A$
1^3D_3	$\psi_3(1D)$	3^{--}		3806/3849/3799	3811	3808	$\psi^0_{2m}\chi^1_{S_z}\Phi_A$
1^3D_2	$\psi_2(1D)$	2^{--}	3823	3800/3838/3798	3807	3807	$\psi^0_{2m}\chi^1_{S_z}\Phi_A$
1^3D_1	$\psi_1(1D)$	1^{--}	3778	3785/3819/3787	3787	3792	$\psi^0_{2m}\chi^1_{S_z}\Phi_A$
1^1D_2	$\eta_{c2}(1D)$	2^{-+}		3799/3837/3796	3806	3805	$\psi^0_{2m}\chi^0\Phi_S$
2^3D_3	$\psi_3(2D)$	3^{--}		4167/4217/4103	4172	4112	$\psi^1_{2m}\chi^1_{S_z}\Phi_A$
2^3D_2	$\psi_2(2D)$	2^{--}		4158/4208/4100	4165	4109	$\psi^1_{2m}\chi^1_{S_z}\Phi_A$
2^3D_1	$\psi_1(2D)$	1^{--}	4191	4142/4194/4089	4144	4095	$\psi^1_{2m}\chi^1_{S_z}\Phi_A$
2^1D_2	$\eta_{c2}(2D)$	2^{-+}		4158/4208/4099	4164	4108	$\psi^1_{2m}\chi^0\Phi_S$

▸ Spectrum of $b\bar{b}$: (M_{R}:S. Godfrey et al. arXiv:1507.00024)

$n^{2S+1}L_J$	name	J^{PC}	M_{exp}	M_{SNR}	M_{R}	M_{NR}	Ours
1^3S_1	$\Upsilon(1S)$	1^{--}	9460	9460	9465	9502	9460
1^1S_0	$\eta_b(1S)$	0^{-+}	9398	9389	9402	9455	9390
2^3S_1	$\Upsilon(2S)$	1^{--}	10023	10016	10003	10015	10015
2^1S_0	$\eta_b(2S)$	0^{-+}	9999	9987	9976	9990	9990
3^3S_1	$\Upsilon(3S)$	1^{--}	10355	10351	10354	10349	10343
3^1S_0	$\eta_b(3S)$	0^{-+}		10330	10336	10330	10326
1^3P_2	$\chi_{b2}(1P)$	2^{++}	9912	9918	9897	9886	9921
1^3P_1	$\chi_{b1}(1P)$	1^{++}	9893	9897	9876	9874	9903
1^3P_0	$\chi_{b0}(1P)$	0^{++}	9859	9865	9847	9855	9864
1^1P_1	$h_b(1P)$	1^{+-}	9899	9903	9882	9879	9909
2^3P_2	$\chi_{b2}(2P)$	2^{++}	10269	10269	10261	10246	10264
2^3P_1	$\chi_{b1}(2P)$	1^{++}	10255	10251	10246	10236	10249
2^3P_0	$\chi_{b0}(2P)$	0^{++}	10233	10226	10226	10221	10220
2^1P_1	$h_b(2P)$	1^{+-}	10260	10256	10250	10240	10254
3^3P_2	$\chi_{b2}(3P)$	2^{++}		10540	10550	10521	10528
3^3P_1	$\chi_{b1}(3P)$	1^{++}	10516	10524	10538	10513	10515
3^3P_0	$\chi_{b0}(3P)$	0^{++}		10502	10522	10500	10490
3^1P_1	$h_b(3P)$	1^{+-}		10529	10541	10516	10519
1^3D_3	$\Upsilon_3(1D)$	3^{--}		10156	10155	10127	10157
1^3D_2	$\Upsilon_2(1D)$	2^{--}	10164	10151	10147	10122	10153
1^3D_1	$\Upsilon_1(1D)$	1^{--}		10145	10138	10117	10146
1^1D_2	$\eta_{b2}(1D)$	2^{-+}		10152	10148	10123	10153
2^3D_3	$\Upsilon_3(2D)$	3^{--}		10442	10455	10422	10436
2^3D_2	$\Upsilon_2(2D)$	2^{--}		10438	10449	10418	10432
2^3D_1	$\Upsilon_1(2D)$	1^{--}		10432	10441	10414	10425
2^1D_2	$\eta_{b2}(2D)$	2^{-+}		10439	10450	10419	10432

The Hamiltonian of EM coupling

$$H_e = -\sum_j e_j \bar{\psi}_j \gamma_\mu^j A^\mu(\mathbf{k}, \mathbf{r}) \psi_j,$$

where ψ_j stands for the j-th quark field in a hadron. The photon has three momentum $\mathbf{k}$, and the constituent quark ψ_j carries a c-harge e_j.

Amptitude of EM transitions

$$\boldsymbol{\alpha}_j \equiv i[\hat{H}, \mathbf{r}_j],$$

$$\begin{aligned}
\langle N_f \mid H_e \mid N_i \rangle &= \langle N_f \mid \sum_j e_j \boldsymbol{\alpha}_j \cdot \boldsymbol{\epsilon} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \mid N_i \rangle \\
&= i\langle N_f \mid [\widehat{H}, \sum_j e_j \mathbf{r}_j \cdot \boldsymbol{\epsilon} e^{-i\mathbf{k}\cdot\mathbf{r}_j}] \mid N_i \rangle \\
&\quad + i\langle N_f \mid \sum_j e_j \mathbf{r}_j \cdot \boldsymbol{\epsilon}\ \boldsymbol{\alpha}_j \cdot \mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \mid N_i \rangle \\
&= -i(E_i - E_f - \omega_\gamma)\langle N_f \mid g_e \mid N_i \rangle \\
&\quad - i\omega_\gamma \langle N_f \mid h_e \mid N_i \rangle
\end{aligned}$$

$$h_e = \sum_j e_j \mathbf{r}_j \cdot \boldsymbol{\epsilon}(1 - \boldsymbol{\alpha}_j \cdot \hat{\mathbf{k}}) e^{-i\mathbf{k}\cdot\mathbf{r}_j},$$

and

$$g_e = \sum_j e_j \mathbf{r}_j \cdot \boldsymbol{\epsilon} e^{-i\mathbf{k}\cdot\mathbf{r}_j}.$$

To match the NR wave functions of hadrons, we should adopt the NR form of h_e in the calculations. ($Z.P.\ Li\ et.al\ PRC 52, 1648$)

$$h_e = \sum_j [e_j \mathbf{r}_j \cdot \boldsymbol{\epsilon} - \frac{e_j}{2m_j} \boldsymbol{\sigma}_j \cdot (\boldsymbol{\epsilon} \times \hat{\mathbf{k}})] e^{-i\mathbf{k}\cdot\mathbf{r}_j}.$$

The multipole expansion of EM transition

To easily work out the EM transition matrix elements, we use the multipole expansion of the plane wave

$$e^{-i\mathbf{k}\cdot\mathbf{r}_j} = e^{-ikz_j} = \sum_l \sqrt{4\pi(2l+1)}(-i)^l j_l(kr_j) Y_{l0}(\Omega),$$

where $j_l(x)$ is the Bessel function.

$$\begin{aligned}\mathcal{A}^{\mathrm{E}l} &= \sqrt{\frac{\omega_\gamma}{2}}\Big\langle f\Big|\sum_j (-i)^l B_l e_j r_j j_{l+1}(kr_j) Y_{l1}(\Omega)\Big|i\Big\rangle \\ &\quad + \sqrt{\frac{\omega_\gamma}{2}}\Big\langle f\Big|\sum_j (-i)^l B_l e_j r_j j_{l-1}(kr_j) Y_{l1}(\Omega)\Big|i\Big\rangle,\end{aligned}$$

where $B_l \equiv \sqrt{\frac{2\pi l(l+1)}{2l+1}}$.

$$\mathcal{A}^{\mathrm{M}l} = \sqrt{\frac{\omega_\gamma}{2}}\Big\langle f\Big|\sum_j (-i)^l C_l \frac{e_j\sigma_+}{2m_j} j_{l-1}(kr_j) Y_{l-1\ 0}(\Omega)\Big|i\Big\rangle,$$

where $C_l \equiv i\sqrt{8\pi(2l-1)}$, $\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y)$.

Possible EM multipole contributions to a EM transition between two $c\bar{c}$ or $b\bar{b}$ states:

process	multipole contribution
$n^3S_1 \longleftrightarrow m^1S_0$	M1
$n^3P_J \longleftrightarrow m^3S_1$	E1, M2
$n^1P_1 \longleftrightarrow m^1S_0$	E1
$n^3D_J \longleftrightarrow m^3P_J$	E1, E3, M2, M4
$n^1D_1 \longleftrightarrow m^1P_1$	E1, E3
$n^3P_J \longleftrightarrow m^1P_1$	M1, M3

Width of EM transition

$$\Gamma = \frac{|\mathbf{k}|^2}{\pi} \frac{2}{2J_i + 1} \frac{M_f}{M_i} \sum_{J_{fz}, J_{iz}} |\mathcal{A}_{J_{fz}, J_{iz}}|^2,$$

where J_i is the total angular momenta of the initial mesons , J_{fz} and J_{iz} are the components of the total angular momentum along the z axis of initial and final mesons,respectively.

Results and discussion

([1] : *D. Ebert et al. PRD67, 014027*;[2] : *J. Segovia et al. PRD93, 074027*)

Table: Radiative transitions between $b\bar{b}$ states (E1 dominant).

Initial state	Final state	E_γ (MeV) ours	$\Gamma_{\rm E1}$ (keV) [1]	SNR$_0$	SNR$_1$	GI	[2]	$\Gamma_{\rm EM}$ (keV) Ours	Exp.
$\Upsilon(2^3S_1)$	$\chi_{b2}(1^3P_2)$	110	2.46	2.62	2.46	1.88	2.08	2.62	2.29 ± 0.20
	$\chi_{b1}(1^3P_1)$	129	2.45	2.54	2.08	1.63	1.84	2.17	2.21 ± 0.19
	$\chi_{b0}(1^3P_0)$	163	1.62	1.67	1.11	0.91	1.09	1.09	1.22 ± 0.11
$\eta_b(2^1S_0)$	$h_b(1^1P_1)$	99	3.09	6.10	5.57	2.48	2.85	3.41	
$\Upsilon(3^3S_1)$	$\chi_{b2}(2^3P_2)$	86	2.67	3.23	3.04	2.30	2.56	3.16	2.66 ± 0.27
	$\chi_{b1}(2^3P_1)$	100	2.41	2.96	2.44	1.91	2.13	2.61	2.56 ± 0.26
	$\chi_{b0}(2^3P_0)$	121	1.49	1.83	1.23	1.03	1.21	1.21	1.20 ± 0.12
	$\chi_{b2}(1^3P_2)$	434	0.097	0.25	1.26	0.45	0.083	0.14	0.20 ± 0.03
	$\chi_{b1}(1^3P_1)$	452	0.067	0.17	0.14	0.05	0.16	0.0005	0.018 ± 0.010
	$\chi_{b0}(1^3P_0)$	484	0.027	0.07	0.05	0.01	0.15	0.097	0.055 ± 0.010
$\eta_b(3^1S_0)$	$h_b(2^1P_1)$	77	2.78	11.0	10.1	2.96	2.60	2.36	
	$h_b(1^1P_1)$	429	0.348	1.24	5.68	1.30	0.0084	0.67	
$\chi_{b2}(2^3P_2)$	$\Upsilon(1^3D_3)$	97	2.35	3.33	3.13	1.5	2.06	2.51	
	$\Upsilon(1^3D_2)$	104	0.449	0.66	0.58	0.3	0.35	0.42	
	$\Upsilon(1^3D_1)$	113	0.035	0.05	0.04	0.03	0.021	0.026	
	$\Upsilon(2^3S_1)$	243	16.7	18.8	14.2	14.3	17.50	15.7	15.1 ± 5.6
	$\Upsilon(1^3S_1)$	777	8.02	13.0	12.5	8.4	11.38	13.6	9.8 ± 2.3
$\chi_{b1}(2^3P_1)$	$\Upsilon(1^3D_2)$	91	1.56	2.31	2.26	1.2	1.26	0.50	
	$\Upsilon(1^3D_1)$	100	0.615	0.92	0.84	0.5	0.41	0.56	
	$\Upsilon(2^3S_1)$	229	14.7	15.9	13.8	13.3	15.89	15.7	19.4 ± 5.0
	$\Upsilon(1^3S_1)$	764	7.49	12.4	8.56	5.5	9.13	11.7	8.9 ± 2.2

Table: Partial widths of the radiative transitions between the established $c\bar{c}$ states.

Initial	Final	E_γ (MeV)	Γ_{E1} (keV)					Γ_{EM} (keV)		
state	state	Ours	[1]	NR/GI	$\mathrm{SNR}_{0/1}$	LP	SP	LP	SP	Exp.
$\psi(2S)$	$\chi_{c2}(1P)$	128	18.2	38/24	43/34	36	44	38	45	25.2 ± 2.9
	$\chi_{c1}(1P)$	171	22.9	54/29	62/36	45	48	42	45	25.5 ± 2.8
	$\chi_{c0}(1P)$	261	26.3	63/26	74/25	27	26	22	22	26.3 ± 2.6
$\chi_{c2}(1P)$	J/ψ	429	327	424/313	473/309	327	338	284	292	371 ± 34
$\chi_{c1}(1P)$		390	265	314/239	354/244	269	278	306	319	285 ± 14
$\chi_{c0}(1P)$		303	121	152/114	167/117	141	146	172	179	133 ± 8
$h_c(1P)$	$\eta_c(1S)$	499	560	498/352	764/323	361	373	361	373	357 ± 280
$\psi_1(1D)$	$\chi_{c2}(1P)$	215	6.9	4.9/3.3	5.8/4.6	5.7	6.0	13.2	14.5	< 17.4
	$\chi_{c1}(1P)$	258	135	125/77	150/93	115	119	151	160	81 ± 27
	$\chi_{c0}(1P)$	346	355	403/213	486/197	243	232	272	261	202 ± 42
$\psi_2(1D)$	$\chi_{c2}(1P)$	258	59	64/66	70/55	79	83	93	98	
	$\chi_{c1}(1P)$	299	215	307/268	342/208	285	296	294	306	
$\psi_1(2D)$	$\chi_{c2}(1P)$	587		0.79/0.027		52	50	60	67	
	$\chi_{c1}(1P)$	625		14/3.4		25	42	46	79	
	$\chi_{c0}(1P)$	704		27/35		120	149	150	189	
	$\chi_{c2}(2P)$	256		5.9/6.3		24	33	48	59	

★ $X(3872)$

• $R^{\rm exp}_{\psi'\gamma/\psi\gamma} \simeq 3.4 \pm 1.4$ obtained by the BaBar Collaboration
B. Aubert et al. PRL 102, 132001 (2009);

• Recently, the LHCb Collaboration's measurement is 2.46 ± 0.93
R. Aaij et al. NPB 886, 665 (2014);

According to our prediction $R_{\psi'\gamma/\psi\gamma} \approx 1.3$, we can not exclude the $X(3872)$ as a candidate of $\chi_{c1}(2P)$.

Table: Partial widths for the radiative transitions of unestablished D-wave $c\bar{c}$ states.

Initial state	Final state	E_γ (MeV) LP/SP	Γ_{E1} (keV) [1]	NR/GI	SNR$_{0/1}$	LP	SP	Γ_{EM} (keV) LP	SP
$\psi_3(1D)$	$\chi_{c2}(1P)$	264/264	156	272 /296	284/223	377	349	351	365
$\eta_{c2}(1D)$	$h_c(1P)$	284/284	245	339 /344	575/375	363	378	363	378
$\psi_3(2D)$	$\chi_{c2}(1P)$	571/518		29 / 16		88	81	79	72
$\psi_2(2D)$	$\chi_{c2}(1P)$	564/516		7.1 /0.62		30	24	29	24
	$\chi_{c1}(1P)$	603/554		26 / 23		62	63	75	73
$\eta_{c2}(2D)$	$h_c(1P)$	590/542		40 / 25		112	101	112	101
$\psi_3(2D)$	$\chi_{c2}(2P)$	231/178		239 /272		458	256	429	243
$\psi_2(2D)$	$\chi_{c2}(2P)$	226/178		52 / 65		103	58	119	65
	$\chi_{c1}(2P)$	222/204		298 /225		225	190	229	193
$\psi_1(2D)$	$\chi_{c1}(2P)$	281/281		168 /114		253	280	341	384
	$\chi_{c0}(2P)$	312/329		483 /191		299	321	332	360
$\eta_{c2}(2D)$	$h_c(2P)$	256/203		336 /296		443	273	443	273

Summary

- According to mass splitting between $\chi_{c2}(2P)$ and $\chi_{c0}(2P)$, i.e., $\triangle m \approx 90 MeV$, which does not support the $X(3915)$ assigned as the $\chi_{c0}(2P)$ state.

- Assigning $X(3823)$ as $\psi_2(1D)$, our results are in agreement with the measurements of Belle and BESIII

- Assigning the $X(3872)$ resonance as the $\chi_{c0}(2P)$, we predict the ratio $\frac{\Gamma(X(3872)\to\psi(2S)\gamma)}{\Gamma(X(3872)\to J/\psi\gamma)} \approx 1.3$, which is close to the lower limit of the measurements from the BaBar and LHCb.

- $\psi(4040)\to\chi_{cJ}(2P)$ and $\psi(4160)\to\chi_{cJ}(2P)$ might provide us a good chance to look for the missing $\chi_{c0}(2P)$ and $\chi_{c1}(2P)$.

- $2D\to 2P\to 2S\to 1P$, and $2D\to 2P\to 1S$ might be useful for establishing the missing $\psi_2(2D)$,$\psi_3(2D)$)and $\eta_{c2}(2D)$ states in experiments

Thanks for your attention !