Quick Review on Superstrings

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DISCLAIMER

This is just a quick review, and focused on the introductory part. Many important topics are not covered (some of them are briefly mentioned), for example

- conformal field theory
  (OPE, bosonization, vertex op, modular invariance, etc.)

- string duality (S,T,U-duality, open-closed duality)

- D-branes

- M-theory

- AdS/CFT, etc.
REFERENCES (books)

- Green-Schwarz-Witten “Superstring theory” vol. 1, 2
- Polchinski “String Theory” vol. 1, 2
- Becker-Becker-Schwarz “String Theory and M-theory”
- Dine “Supersymmetry and String Theory”
- Ibáñez-Uranga “String Theory and Particle Physics”
- Blumenhagen-Lust-Theisen “Basic Concepts of String Theory”
- West “Introduction to Strings and Branes”, etc.

As well as these books, there are many good lecture notes in arXiv.
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PART 1: First quantization of bosonic sting and its spectrum

PART 2: First quantization of fermionic sting

- RNS formalism
- GSO projection and spacetime supersymmetry
- The five superstring theories

PART 3: Low energy SUGRA and CY compactification

- supergravity as effective field theory of superstring
- Calabi-Yau compactification
1. First quantization of bosonic string

1-1. bosonic string

Action of a point particle: \(-\text{(mass)} \times \text{(length of worldline)}\)

\[ S_{\text{p.p.}} = -m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \]

Action of a string: \(-\text{(string tension)} \times \text{(area of worldsheet)}\)

\[ S_{\text{NG}} = -\frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{- \det \left( \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \right)} \]

(Nambu-Goto action, 南部-後藤作用)
The Nambu-Goto action contains the square root, which is a bit hard to study. It is convenient to rewrite the action by introducing new variables: *worldsheet metric* $h_{\alpha\beta}(\tau, \sigma)$.

\[ S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^\alpha\beta \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \]

Note that $h_{\alpha\beta}(\tau, \sigma)$ is not dynamical and its equation of motion gives

\[ T_{\alpha\beta} \equiv \frac{2}{\alpha'} \left( \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\mu}{\partial \sigma^\beta} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \frac{\partial X^\mu}{\partial \sigma^\gamma} \frac{\partial X^\mu}{\partial \sigma^\delta} \right) = 0 \]

By substituting the above into the Polyakov action, it is reduced to the Nambu-Goto action.
Local symmetries of Polyakov action

1. diffeomorphism $\tau \rightarrow \tau'(\tau, \sigma), \quad \sigma \rightarrow \sigma'(\tau, \sigma)$

2. Weyl transformation $h_{\alpha\beta}(\tau, \sigma) \rightarrow \Lambda(\tau, \sigma) h_{\alpha\beta}(\tau, \sigma)$

Using these symmetries, we can gauge-fix $h_{\alpha\beta}(\sigma)$ as

$$h_{\alpha\beta}(\tau, \sigma) = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This name comes from the fact that the gauge-fixed action

$$S_{\text{g.f.}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \eta^{\alpha\beta} \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}$$

is still invariant under the conformal symmetry.
Conformal symmetry

\[ \sigma^+ \rightarrow f^+(\sigma^+), \quad \sigma^- \rightarrow f^-(\sigma^-), \quad \sigma^\pm = \tau \pm \sigma \]

Here \( f^\pm \) are arbitrary functions.

Remark

The generators of the conformal transformation is \( T_{\alpha\beta} \) itself. In the conformal gauge, the constraint \( T_{\alpha\beta} = 0 \) becomes

\[ T_{++} = -\frac{1}{\alpha'} \partial_+ X^\mu \partial_+ X_\mu = 0, \quad T_{--} = -\frac{1}{\alpha'} \partial_- X^\mu \partial_- X_\mu = 0 \]

In string theory, conformal symmetry plays a role of the gauge symmetry, since it is a residual symmetry of (diffeo.) \( \times \) (Weyl) after the gauge fixing.
Equation of motion for gauge-fixed action

\[
\left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^\mu = \frac{\partial^2}{\partial \sigma^+ \partial \sigma^-} X^\mu = 0
\]

Solution = (left mover) + (right mover)

\[
X^\mu(\tau, \sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)
\]

Boundary condition \((-\infty < \tau < \infty, 0 \leq \sigma \leq l)\)

\[
\left[ \frac{\partial X^\mu}{\partial \sigma} \delta X^\mu \right]_{\sigma=0}^{\sigma=l} \equiv \frac{\partial X^\mu}{\partial \sigma} \delta X^\mu \bigg|_{\sigma=l} - \frac{\partial X^\mu}{\partial \sigma} \delta X^\mu \bigg|_{\sigma=0} = 0
\]

Here we have assumed that b.c. for temporal direction is trivial.
1. periodic b.c. $\rightarrow$ closed string

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + l)$$

2. Neumann b.c. $\rightarrow$ open string

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = \frac{\partial X^\mu}{\partial \sigma}(\tau, l) = 0$$

3. Dirichlet b.c. $\rightarrow$ open string attached to D-brane

$$\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, l) = 0$$
(Fourier) mode expansion of closed string \((l = 2\pi)\)

\[
X^\mu = x^\mu + \alpha' p^\mu \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)} \right)
\]

Here

\(x^\mu, p^\mu\) : center of mass position and momentum (zero modes)

\(\alpha_n^\mu, \tilde{\alpha}_n^\mu\) : amplitudes for oscillation modes

Mode expansion of open string with Neumann b.c. \((l = \pi)\)

\[
X^\mu = x^\mu + 2\alpha' p^\mu \tau + i \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma
\]

(Dirichlet b.c. : \(p^\mu = 0, \cos \rightarrow \sin\)
Doubling trick

We can extend the domain of $\sigma$ from $0 \leq \sigma \leq \pi$ to $0 \leq \sigma \leq 2\pi$ as

$$X^\mu(\tau, \sigma) = \begin{cases} X^\mu(\tau, \sigma) & \text{for } 0 \leq \sigma \leq \pi \\ X^\mu(\tau, 2\pi - \sigma) & \text{for } \pi \leq \sigma \leq 2\pi \end{cases}$$

The extended $X^\mu$’s satisfy the closed string boundary condition. However they contain only a single mover, of course. This is called the doubling trick.
Quantization for constrained system

1. Old covariant quantization (Gupta-Bleuler formalism)

2. Light-cone (or non-covariant) quantization

3. BRST quantization

Here we adopt the old covariant quantization.
Canonical commutation relations

\[
\left[ X^\mu(\tau, \sigma), P^\nu(\tau, \sigma') \right] = i\eta^{\mu\nu}\delta(\sigma - \sigma'), \quad P^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \partial_\tau X^\mu \\
\left[ X^\mu(\tau, \sigma), X^\nu(\tau, \sigma') \right] = \left[ P^\mu(\tau, \sigma), P^\nu(\tau, \sigma') \right] = 0
\]

Commutation relation for Fourier modes

\[
[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}
\]

Here

\[
\alpha^\mu_{n>0}, \tilde{\alpha}^\mu_{n>0} : \text{annihilation op.} \quad \alpha^\mu_{n<0}, \tilde{\alpha}^\mu_{n<0} : \text{creation op.}
\]

\[
\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu \quad \text{(closed)} \quad \alpha^\mu_0 = \sqrt{2\alpha'} p^\mu \quad \text{(open)}
\]
Fock vacuum for left mover

\[ p^\mu |0; 0\rangle = 0, \quad \alpha^\mu_n |0; 0\rangle = 0 \quad \text{for all} \quad n > 0 \]

Fock space \( \{ |\{N_{\mu,n}\}; k\rangle\} \)

\[ |\{N_{\mu,n}\}; k\rangle = \prod_{\mu=0}^{D-1} \prod_{n=1}^{\infty} (\alpha^\mu_{-n})^{N_{\mu,n}} \exp(ik_\mu x^\mu) |0; 0\rangle \]

Remark

The Fock space contains the states with negative norm, i.e. ghosts.

example: \[ |\alpha^0_{-1}|0; k\rangle|^2 = -\langle 0; k|0; k\rangle = -1 \]
Conformal symmetry and physical state condition

Mode expansion of generators

\[ T_{++} = - \sum_{n} L_{n} e^{-in\sigma^+}, \quad T_{--} = - \sum_{n} \tilde{L}_{n} e^{-in\sigma^-} \]

\[ L_{n} = \frac{1}{2} \sum_{m} : \alpha_{n-m}^{\mu} \alpha_{m\mu} : , \quad \tilde{L}_{n} = \frac{1}{2} \sum_{m} : \tilde{\alpha}_{n-m}^{\mu} \tilde{\alpha}_{m\mu} : \]

The symbol \( : \) denotes normal ordering. In fact, only \( L_0 \) and \( \tilde{L}_0 \) are affected by the ordering as

\[ L_0 = \frac{1}{2} \alpha_0^{\mu} \alpha_{0\mu} + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu} , \quad \tilde{L}_0 = \frac{1}{2} \tilde{\alpha}_0^{\mu} \tilde{\alpha}_{0\mu} + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_{n\mu} \]
Virasoro algebra

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \]

The constant \( c \) is called central charge, explicit calculation gives

\[ c = \eta_{\mu\nu}\eta^{\mu\nu} = D \quad \text{(dimension of spacetime)} \]

Physical state condition

\[ (L_0 - a)|\text{phys}\rangle = L_n|\text{phys}\rangle = 0 \quad \text{for } n > 0 \]

The intercept \( a \) represents the normal ordering ambiguity of \( L_0 \).
Explicit calculation of central extension

\[ [L_m, L_n] = \frac{1}{4} \sum_{k,l} \left[ \alpha_{m-k} \cdot \alpha_k, \alpha_{n-l} \cdot \alpha_l \right] \]

\[ = \frac{1}{4} \sum_{k,l} \left[ k\alpha_{m-k} \cdot \alpha_{n-l}\delta_{k+l,0} + k\alpha_{m-k} \cdot \alpha_l\delta_{k+n-l,0} \right. \]

\[ + (m - k)\alpha_{n-l} \cdot \alpha_k\delta_{m-k+l,0} + (m - k)\alpha_l \cdot \alpha_k\delta_{m-k+n-l,0} \right] \]

\[ = \frac{1}{2} \sum_{k} \left[ k\alpha_{m-k} \cdot \alpha_{n+k} + (m - k)\alpha_{m+n-k} \cdot \alpha_k \right] \]

Now it is clear that the ordering ambiguity may happen only when \( m + n = 0 \). Otherwise it gives \((m - n)L_{m+n}\) by shifting the label of the first sum by \( k \rightarrow k - n \).
\( m + n = 0 \) case

\[
[L_m, L_{-m}] = \frac{1}{2} \sum_k \left[ k \alpha_{m-k} \cdot \alpha_{-m+k} + (m - k) \alpha_{-k} \cdot \alpha_k \right]
\]

\[= \frac{1}{2} \sum_k k : \alpha_{m-k} \cdot \alpha_{-m+k} : + \frac{1}{2} \sum_{k \leq m-1} k \left[ \alpha_{m-k}^\mu, \alpha_{-m+k, \mu} \right]
\]

\[+ \frac{1}{2} \sum_k (m - k) : \alpha_{-k} \cdot \alpha_k : + \frac{1}{2} \sum_{k \leq -1} (m - k) \left[ \alpha_{-k}^\mu, \alpha_{k \mu} \right]
\]

\[= 2mL_0 + \frac{\delta_{\mu}^\mu}{2} \left[ \sum_{k \leq m-1} - \sum_{k \leq -1} \right] k(m - k)
\]

\[= 2mL_0 + \frac{D}{12}(m - 1)m(m + 1) \quad \therefore c = D.
\]
Spurious states

\[ |\chi\rangle = \sum \left( L_{-n} |\ast\rangle \right) \text{ satisfying physical state condition } (n > 0) \]

properties

- orthogonal to any physical states
  \[ \langle \chi | \text{phys} \rangle = \sum \langle \ast | L_n | \text{phys} \rangle = 0 \quad (n > 0) \]

- vanishing norm
  \[ \langle \chi | \chi \rangle = \sum \langle \ast | L_{n>0} | \chi \rangle = 0 \]

We require the equivalence relation for physical states

\[ |\text{phys}\rangle \sim |\text{phys}'\rangle \text{ if } |\text{phys}'\rangle = |\text{phys}\rangle + |\chi\rangle, \]

which preserves the inner product.
No-ghost theorem

The ghosts (negative norm states) can be removed from the physical Hilbert space (not only tree, but also loop level), if and only if

\[ D = 26 \quad \text{(critical dimension)}, \quad a = 1 \]

Comment

The above can also be derived from other quantization scheme.

- **light-cone quantization**
  
  Lorentz algebra is closed if and only if \( D = 26 \) and \( a = 1 \).

- **BRST quantization**

  BRST charge \( Q \) must be nilpotent: \( Q^2 = 0 \), which holds if and only if \( D = 26 \) and \( a = 1 \).
Low-lying spectrum of open string

\[(L_0 - 1)|\text{phys}\rangle = 0: \text{on-shell condition}\]

\[L_0 = \alpha' p^2 + N, \quad N = \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{n\mu} \quad \Rightarrow \quad m^2 = \frac{1}{\alpha'}(N - 1)\]

Examples

- \(|\text{phys}\rangle = |0; k\rangle, \quad m^2 = -\frac{1}{\alpha'}: \text{tachyon}\)

- \(|\text{phys}\rangle = \epsilon_\mu \alpha_{-1}^\mu |0; k\rangle, \quad m^2 = 0: \text{massless vector}\)

All other states have positive \((\text{mass})^2 \sim (\alpha')^{-1}\) and decouple from low energy physics.
Gauge invariance of massless vector state

\[ L_n |\text{phys}\rangle = 0, \quad |\text{phys}\rangle \sim |\text{phys}\rangle + L_{-n}|^*\rangle \quad \text{for } n > 0 \]

For massless vector state \( |\text{phys}\rangle = \epsilon_\mu \alpha_{-1}^\mu |0; k\rangle \), the physical state conditions are reduced to

\[ k_\mu \epsilon^{\mu} = 0, \quad \epsilon_\mu \sim \epsilon_\mu + \lambda k_\mu \]

which correspond to the gauge invariance of massless vector field

\[ \partial_\mu A^\mu = 0, \quad A_\mu \sim A_\mu + \partial_\mu \lambda \]

The conditions imply that \( \epsilon_\mu \) has transverse polarization only. In particular, the ghost state \( \alpha_{-1}^0 |0; k\rangle \) is removed.
Low-lying spectrum of closed string

On-shell condition and level-matching condition

\[
(L_0 - 1)|\text{phys}\rangle = (\tilde{L}_0 - 1)|\text{phys}\rangle = 0
\]

\[
\Rightarrow (L_0 + \tilde{L}_0 - 2)|\text{phys}\rangle = (L_0 - \tilde{L}_0)|\text{phys}\rangle = 0
\]

On-shell condition

\[
L_0 + \tilde{L}_0 = \alpha' p^2 + N + \tilde{N} \quad \Rightarrow \quad m^2 = \frac{1}{\alpha'}(N + \tilde{N} - 2)
\]

Level-matching condition (invariance under $\sigma$-translation)

\[
(L_0 - \tilde{L}_0)|\text{phys}\rangle = 0 \quad \Rightarrow \quad N = \tilde{N} \text{ for physical states}
\]
Examples

- $|\text{phys}\rangle = |0, 0; k\rangle$, $m^2 = -\frac{2}{\alpha'}$: tachyon

- $|\text{phys}\rangle = \zeta_{\mu\nu} \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0, 0; k\rangle$, $m^2 = 0$: massless tensor

For the massless tensor, physical state condition is reduced to

$$k^\mu \zeta_{\mu\nu} = k^\nu \zeta_{\mu\nu} = 0, \quad \zeta_{\mu\nu} \sim \zeta_{\mu\nu} + k_\mu \lambda_\nu + k_\nu \lambda'_\mu$$

For the symmetric part $\zeta_{\mu\nu} = \zeta_{\nu\mu}$, the second condition becomes

$$\zeta_{\mu\nu} \sim \zeta_{\mu\nu} + k_\mu \lambda_\nu + k_\nu \lambda_\mu \quad \Rightarrow \quad h_{\mu\nu} \sim h_{\mu\nu} + \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu$$

This is the gauge symmetry of graviton! (at linearized level)
The physical state condition implies that the transverse modes $\zeta_{ij}$ of the polarization tensor are only physical, which can be decomposed as

- **graviton**  \( \zeta_{ij} = \zeta_{ji}, \quad \delta_{ij} \zeta^{ij} = 0, \)

- **B-field**  \( \zeta_{ij} = -\zeta_{ji} \)

- **dilaton**  \( \zeta_{ij} = \zeta \delta_{ij} \)

All states other than tachyon or massless have positive \((\text{mass})^2 \sim (\alpha')^{-1}\) and decouple from low energy physics.
2. First quantization of fermionic string

2-1. RNS formalism

Bosonic string does not contain fermions in the spectrum. There are two or more ways to extend the theory to include fermions:

- Ramond-Neveu-Schwarz (RNS) fermionic string
- Green-Schwarz (GS) superstring
- Pure-spinor formalism [Berkovits], etc.

Here we adopt RNS formalism.
Worldsheet action (supersymmetric extension of Polyakov action)

[Brink-Di Vecchia-Howe], [Dezer-Zumino]

\[ S = -\frac{1}{8\pi} \int d^2\sigma \left[ \frac{2}{\alpha'} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + 2i \bar{\Psi}\mu^\rho \rho^\alpha \partial_\alpha \Psi_\mu 
- i \sqrt{\frac{2}{\alpha'}} \bar{\chi}_\alpha \rho^\beta \rho^\alpha \Psi^\mu \partial_\beta X_\mu - \frac{1}{4} \bar{\chi}_\alpha \rho^\beta \rho^\alpha \Psi^\mu \bar{\chi}_\beta \Psi_\mu \right] \]

Here

\[ \Psi^\mu : \text{worldsheet Majorana spinor and spacetime vector} \]

\[ e^a_\alpha : \text{2D vielbein(zweibein)}, \quad \eta_{ab} e^a_\alpha e^b_\beta = h_{\alpha\beta}, \quad e = \det e^a_\alpha \]

\[ \chi_\alpha : \text{2D gravitino}, \quad \rho^a : \text{2D gamma matrices}, \quad \rho^\alpha = e^a_\alpha \rho^a \]
Local symmetries

1. diffeomorphism

2. Weyl transformation

\[(\text{weight: } [X^\mu] = 0, \ [\psi^\mu] = -1/2, \ [e_a^\alpha] = 1, \ [\chi^\alpha] = 1/2)\]

3. local supersymmetry (set \(\alpha' = 2\))

\[
\begin{align*}
\delta X^\mu &= \bar{\epsilon} \Psi^\mu, \\
\delta \Psi^\mu &= -i \rho^\alpha \epsilon (\partial_\alpha X^\mu - \bar{\Psi}^\mu \chi_\alpha), \\
\delta e_a^\alpha &= -2i \bar{\epsilon} \rho^a \chi_\alpha, \\
\delta \chi_\alpha &= \nabla_\alpha \epsilon
\end{align*}
\]

4. super Weyl transformation

\[
\begin{align*}
\delta X^\mu &= \delta \Psi^\mu = \delta e_a^\alpha = 0, \\
\delta \chi_\alpha &= i \rho_\alpha \eta
\end{align*}
\]
Using those symmetries, we can choose the gauge

\[ e^a_\alpha = \delta^a_\alpha, \quad \chi_\alpha = 0 \quad \text{conformal gauge} \]

But we still have to concern the constraints

\[
T_{\alpha\beta} = \frac{2}{\alpha'} \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2} \bar{\Psi}^\mu \rho_{(\alpha} \partial_{\beta)} \Psi_\mu - \text{(trace)} = 0,
\]

\[
J_\alpha = \frac{1}{2} \rho^\beta \rho_\alpha \Psi^\mu \partial_\beta X_\mu = 0
\]
gauge-fixed action

\[ S_{g.f.} = -\frac{1}{4\pi} \int d^2\sigma \left[ \frac{1}{\alpha'} \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + i \bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu \right] \]

worldsheet supersymmetry \((\alpha' = 2)\)

\[ \delta X^\mu = \bar{\epsilon} \Psi^\mu, \quad \delta \Psi^\mu = -i \rho^\alpha \partial_\alpha X^\mu \epsilon \]

equation of motion

\[ \partial_+ \partial_- X^\mu = \partial_- \psi^\mu = \partial_+ \tilde{\psi}^\mu = 0, \quad \Psi^\mu = \begin{pmatrix} \psi^\mu \\ \tilde{\psi}^\mu \end{pmatrix} \]

boundary condition

\[ \left[ \psi^\mu \delta \psi_\mu - \tilde{\psi}^\mu \delta \tilde{\psi}_\mu \right]_{\sigma=0}^{\sigma=l} = 0 \]
NS and R boundary conditions

Closed string (for $\psi^\mu$, similar for $\tilde{\psi}^\mu$)

$$\psi^\mu(\tau, \sigma + 2\pi) = \begin{cases} 
\psi^\mu(\tau, \sigma) & \text{R sector} \\
-\psi^\mu(\tau, \sigma) & \text{NS sector}
\end{cases}$$

Open string

$$\psi^\mu(\tau, 0) = \pm \tilde{\psi}^\mu(\tau, 0) \quad (+): \text{R sector}, \ (-): \text{NS sector}$$

$$\psi^\mu(\tau, \pi) = \tilde{\psi}^\mu(\tau, \pi)$$

doubling trick for $\psi^\mu$

$$\psi^\mu(\tau, \sigma) = \begin{cases} 
\psi^\mu(\tau, \sigma) & \text{for } 0 \leq \sigma \leq \pi \\
\tilde{\psi}^\mu(\tau, 2\pi - \sigma) & \text{for } \pi \leq \sigma \leq 2\pi
\end{cases}$$
Mode expansion

\[
\psi^\mu(\tau, \sigma) = \begin{cases} 
\sum_{r \in \mathbb{Z}+1/2} \psi_r^\mu \exp[-ir(\tau + \sigma)] & \text{NS sector} \\
\sum_{n \in \mathbb{Z}} \psi_n^\mu \exp[-in(\tau + \sigma)] & \text{R sector}
\end{cases}
\]

\[\updiamond\text{ Quantization}\]

\[
\{\psi^\mu(\tau, \sigma), \psi^\nu(\tau, \sigma')\} = 2\pi \eta^{\mu\nu} \delta(\sigma - \sigma')
\]

\[
\Rightarrow \quad \{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu} \delta_{r+s,0}, \quad \{\psi^\mu_m, \psi^\nu_n\} = \eta^{\mu\nu} \delta_{m+n,0}
\]

Here

\[\psi^\mu_{r>0}, \psi^\mu_{n>0}\ : \ \text{annihilation op.} \quad \psi^\mu_{r<0}, \psi^\mu_{n<0}\ : \ \text{creation op.} \]

\[\psi^\mu_0\ : \ \text{zero modes (exists only in R sector)}\]
NS Fock states \((N_{\mu,r}^{(f)} = 0, 1\) for fixed \(\mu\) and \(r\))

\[
D-1 \prod_{\mu=0}^{D-1} \prod_{r>0} (\psi_{\mu}^{\mu} r_{\mu}^{(f)} N_{\mu,r}^{(f)} |0\rangle_{NS}, \text{ where } \psi_{\mu}^{\mu} |0\rangle_{NS} = 0 \text{ for all } r > 0
\]

R Fock vacuum (nonzero modes are similar to NS sector)

1. Take a vacuum state s.t. \(\psi_{\mu}^{\mu} |0\rangle_{R} = 0\) for all \(n > 0\).

2. Act \(\psi_{\mu}^{\mu}\)'s to the vacuum, e.g. \(\psi_{\mu}^{\mu} |0\rangle_{R}, \psi_{\mu}^{\mu} \psi_{\mu}^{\nu} |0\rangle_{R}, \text{ etc.}\)

3. These states satisfy the vacuum condition \(\Rightarrow\) also vacuum states

4. These vacuum states are the representation of the Clifford algebra

\[
\{\psi_{\mu}^{\mu}, \psi_{\nu}^{\nu}\} = \eta^{\mu\nu} \Rightarrow \text{ spacetime spinor}
\]
Clifford algebra and spinor representation

\[ \Gamma^\mu \equiv \sqrt{2}\psi_0^\mu \implies \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \quad \text{gamma matrices} \]

creation-annilation ops. (suppose \( D = 2k + 2, I = 0, 1, \ldots, k \))

\[ \gamma_0^\pm = \frac{1}{2}(\pm \Gamma^0 + \Gamma^1), \quad \gamma_I^\pm = \frac{1}{2}(\Gamma^{2I} \pm \Gamma^{2I+1}) \quad \text{for} \quad I \neq 0 \]

algebra

\[ \{\gamma_+^I, \gamma_+^J\} = \{\gamma_-^I, \gamma_-^J\} = 0, \quad \{\gamma_+^I, \gamma_-^J\} = \delta^{IJ} \]

Clifford vacuum \( |0\rangle \)

\[ \gamma_-^I|0\rangle = 0 \quad \text{for all} \quad I \]
spinor representation \( \text{dim.} = 2^{k+1} = 2^{(D/2)} \)

\[
|s\rangle = (\gamma^0)^{s_0+\frac{1}{2}} (\gamma^1)^{s_1+\frac{1}{2}} \cdots (\gamma^k)^{s_k+\frac{1}{2}} |0\rangle
\]

\[
s = (s_0, s_1, \ldots, s_k), \quad s_I = \pm \frac{1}{2}
\]

generators of Lorentz group (or spin group, more strictly)

\[
\Sigma^{\mu\nu} = -\frac{i}{4} [\Gamma^\mu, \Gamma^\nu] \quad \Rightarrow \quad \text{Cartan: } S_I \equiv i\delta_{I,0} \Sigma^{2I,2I+1} = \gamma^I_+ \gamma^I_- - \frac{1}{2}
\]

\(|s\rangle\): eigenstate of \(S_I\)

\[
S_I |s\rangle = s_I |s\rangle
\]

\(s\) becomes the weight of the spinor representation.
Weyl representation

\[ \Gamma_{D+1} \equiv i^{-k} \Gamma^0 \Gamma^1 \cdots \Gamma^{D-1}, \quad (\Gamma_{D+1})^2 = 1, \quad \{\Gamma_{D+1}, \Gamma^\mu\} = 0 \]

We can show that

\[ \Gamma_{D+1} = 2^{k+1} S_0 S_1 \cdots S_k \]

Then the spinor representation \( |s\rangle \) can be decomposed by the eigenvalue of \( \Gamma_{D+1} \) as

\[ |s\rangle = |A\rangle \oplus |\dot{A}\rangle \]

|\( |A\rangle \) for \( \Gamma_{D+1} = +1 \) \( \Rightarrow \) \# of \( s_I = -\frac{1}{2} \) is even. |

|\( |\dot{A}\rangle \) for \( \Gamma_{D+1} = -1 \) \( \Rightarrow \) \# of \( s_I = -\frac{1}{2} \) is odd. |
Super Virasoro generators (from $T_{\alpha\beta}$ and $J_\alpha$)

NS sector

\[
L_n = \frac{1}{2} \sum_{m} : \alpha_{n-m}^\mu \alpha_{m\mu} : + \frac{1}{4} \sum_r (2r - n) : \psi_{n-r}^\mu \psi_{r\mu} : ,
\]

\[
G_r = \sum_{m} \alpha_{m\mu}^\mu \psi_{r-m,\mu}
\]

R sector

\[
L_n = \frac{1}{2} \sum_{m} : \alpha_{n-m}^\mu \alpha_{m\mu} : + \frac{1}{4} \sum_m (2m - n) : \psi_{n-m}^\mu \psi_{m\mu} : ,
\]

\[
F_n = \sum_{m} \alpha_{m\mu}^\mu \psi_{n-m,\mu}
\]
super Virasoro algebra (NS sector)

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},
\]

\[
[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r},
\]

\[
\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}
\]

(R sector)

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0},
\]

\[
[L_m, F_n] = \left(\frac{m}{2} - n\right)F_{m+n},
\]

\[
\{F_m, F_n\} = 2L_{m+n} + \frac{c}{3}m^2\delta_{m+n,0}
\]
Explicit calculation gives

$$c = D + \frac{1}{2}D = \frac{3}{2}D$$

Physical state condition (NS sector)

$$(L_0 - a)|\text{phys}\rangle = L_{n>0}|\text{phys}\rangle = G_{r>0}|\text{phys}\rangle = 0,$$

$$|\text{phys}\rangle \sim |\text{phys}'\rangle \quad \text{if} \quad |\text{phys}'\rangle = |\text{phys}\rangle + L_{-n}|\ast\rangle + G_{-r}|\ast\rangle \quad \text{for } n, r > 0$$

(R sector)

$$L_{n\geq0}|\text{phys}\rangle = F_{n\geq0}|\text{phys}\rangle = 0,$$

$$|\text{phys}\rangle \sim |\text{phys}'\rangle \quad \text{if} \quad |\text{phys}'\rangle = |\text{phys}\rangle + L_{-n}|\ast\rangle + G_{-n}|\ast\rangle \quad \text{for } n > 0$$
Here we choose $a = 0$ for R sector, because we have

$$L_0 = F_0^2$$

and $F_0$ has no ambiguity of the normal ordering. Furthermore this is also consistent with the result of no-ghost theorem:

$$D = 10 \quad \text{(critical dim. of superstring),} \quad a = \begin{cases} 
1/2 & \text{(NS)} \\
0 & \text{(R)}
\end{cases}$$
Low-lying spectrum of open fermionic string

**NS sector (integer spin)**

\[
m^2 = -\frac{1}{2\alpha'} |0; k \rangle \otimes |0\rangle_{\text{NS}} \quad \text{(tachyon)}
\]

\[
m^2 = 0 \quad |0; k \rangle \otimes \psi_{-\frac{1}{2}}^{\mu} |0\rangle_{\text{NS}} \quad \text{(vector)}
\]

\[
m^2 = \frac{1}{2\alpha'} \quad \alpha_{-1}^\mu |0; k \rangle \otimes |0\rangle_{\text{NS}}, \quad |0; k \rangle \otimes \psi_{-\frac{1}{2}}^{\mu} \psi_{-\frac{1}{2}}^{\nu} |0\rangle_{\text{NS}}
\]

**R sector (half-integer spin)**

\[
m^2 = 0 \quad |0; k \rangle \otimes |s\rangle_{\text{R}} \quad \text{(Majorana spinor)}
\]

\[
m^2 = \frac{1}{\alpha'} \quad \alpha_{-1}^\mu |0; k \rangle \otimes |s\rangle_{\text{R}}, \quad |0; k \rangle \otimes \psi_{-1}^{\mu} |s\rangle_{\text{R}}
\]

The spectrum of closed fermionic string can be similarly obtained.
2-3. Gliozzi-Scherk-Olive (GSO) projection

“G-parity” operator \((-1)^F\) (\(F\): worldsheet fermion number)

\[
(-1)^F = \exp\left[i\pi \left(\sum_{r>0} \psi^\mu_{-r} \psi_{r\mu} - 1\right)\right]
\]
for NS sector

\[
(-1)^F = \Gamma_{11} \exp\left[i\pi \sum_{n>0} \psi^\mu_{-n} \psi_{n\mu}\right]
\]
for R sector

properties

\[
\{(−1)^F, \psi^\mu(τ, σ)\} = 0,
\]

\[
(−1)^F |0\rangle_{NS} = - |0\rangle_{NS}, \quad (−1)^F |s\rangle_R = \sum_{s'} (\Gamma_{11})_{ss'} |s'\rangle_R
\]
Since \((-1)^F\) anticommutes with all the modes of \(\psi^\mu\), any states are the eigenstates of \((-1)^F = \pm 1\). Projecting out one of the eigenstates is called GSO projection.

GSO projection in NS sector \((n: \text{zero or positive integer})\)

\[
m^2 = \frac{1}{\alpha'} \left(n - \frac{1}{2}\right) : \quad (-1)^F = -1 \quad \text{NS− sector}
\]

\[
m^2 = \frac{n}{\alpha'} \quad : \quad (-1)^F = +1 \quad \text{NS+ sector}
\]

GSO projection in R sector

vacuum : \((-1)^F = \Gamma_{11} = \pm 1\), one of chirality is chosen.

excited : flipping the sign by acting \(\psi_{-n}^\mu (n > 0)\)
Low-lying spectrum of open fermionic string after GSO (keeping NS+ and R+)

NS sector (Tachyon and the first massive state are eliminated.)

\[ m^2 = 0 \quad |0; k\rangle \otimes \psi^\mu_{-\frac{1}{2}}|0\rangle_{NS} \] (vector)

Physical state condition selects the transverse modes:

\[ \Rightarrow \quad |0; k\rangle \otimes \psi^i_{-\frac{1}{2}}|0\rangle_{NS} \]

This is the $8_v$ representation of the little group SO(8).

\[ SO(9, 1) \to SO(1, 1) \otimes SO(8), \quad 10 \to 1 \oplus 1 \oplus 8_v \]
R sector

\[ m^2 = 0 \quad |0; k\rangle \otimes \sum_s \zeta_s |s\rangle_R \implies |0; k\rangle \otimes \sum_A \zeta_A |A\rangle_R \]

\[ F_0 \text{ condition} \]

\[ F_0 = 0 \implies k_\mu (\Gamma^\mu) \hat{A}^A \zeta_A = 0 \quad \text{Dirac equation} \]

Let us choose \( k_\mu = (k, k, 0, \ldots, 0) \). Dirac operator becomes

\[ k_\mu \Gamma^\mu = k_0 \Gamma^0 + k_1 \Gamma^1 = -2k \Gamma^0 \left( S_0 - \frac{1}{2} \right), \]

which implies that \( s_0 = \frac{1}{2} \) and

\[ A = \left( \frac{1}{2}, a \right), \quad a = \left( \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \right) \] with even \# of \(-\frac{1}{2}\)
In terms of the decomposition of the representation, we have

\[ 16 \rightarrow \left( \frac{1}{2}, 8 \right) \oplus \left( -\frac{1}{2}, 8' \right) \]

Here we have to choose the first term in the right hand side. \( R \)-sector is similarly decomposed as

\[ 16' \rightarrow \left( \frac{1}{2}, 8' \right) \oplus \left( -\frac{1}{2}, 8 \right) \]

and we must choose \( \left( \frac{1}{2}, 8' \right) \).

(Notation)

without prime: chiral representation

with prime: anti-chiral representation
Spacetime supersymmetry

boson: \(8_v\) from \(10\) of SO(9,1) \(\text{ (vector)}\)
fermion: \(8\) from \(16\) of SO(9,1) \(\text{ (MW spinor)}\)

These spectra form 10D \(\mathcal{N} = (1, 0)\) vector multiplet and the effective action is given by the super Yang-Mills theory:

\[
S_{\text{SYM}} = \frac{2}{g^2} \int d^{10}x \; \text{Tr} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{i}{2} \bar{\Lambda} \Gamma^\mu D_\mu \Lambda \right]
\]

Here we have introduced the Chan-Paton factor to the endpoints of the open string. Note that this spacetime supersymmetry exists not only in massless level but also any massive level, that is, whole superstring theory.
2-3. The five superstring theories

⋄ Type IIA superstring (closed)

left mover: \( NS^+, R^+ \)

right mover: \( NS^+, R^- \)

This projection gives

boson: \( (NS^+, NS^+), (R^+, R^-) \)

fermion: \( (R^+, NS^+), (NS^+, R^-) \)

massless spectrum (in terms of \( \text{SO}(8) \) little group)

\( (NS^+, NS^+) \)

\[
8_v \otimes 8_v = 1 \oplus 28 \oplus 35 = \text{(dilaton)} \oplus \text{(2-form)} \oplus \text{(graviton)}
\]
\((R^+, R^-)\)

\[8 \otimes 8' = 8_v \oplus 56_T = (1\text{-form}) \oplus (3\text{-form})\]

\((R^+, \text{NS}^+)\)

\[8 \otimes 8_v = 8' \oplus 56 = \text{(dilatino)} \oplus \text{(gravitino)}\]

\((\text{NS}^+, R^-)\)

\[8_v \otimes 8' = 8 \oplus 56' = \text{(dilatino)} \oplus \text{(gravitino)}\]

This theory has \(\mathcal{N} = (1, 1)\) supersymmetry.
Type IIB superstring (closed)

left mover: \( \text{NS}+, \text{R}+ \)

right mover: \( \text{NS}+, \text{R}+ \)

This projection gives

boson: \( (\text{NS}+, \text{NS}+), (\text{R}+, \text{R}+) \)

fermion: \( (\text{R}+, \text{NS}+), (\text{NS}+, \text{R}+) \)

massless spectrum

\( (\text{NS}+, \text{NS}+) \)

\( 8_v \otimes 8_v = 1 \oplus 28 \oplus 35 = (\text{dilaton}) \oplus (2\text{-form}) \oplus (\text{graviton}) \)
\[(R+, \, R+)\]

\[8 \otimes 8 = 1 \oplus 28 \oplus 35_+ = (0\text{-form}) \oplus (2\text{-form}) \oplus (\text{self-dual 4-form})\]

\[(R+, \, \text{NS}+)\]

\[8 \otimes 8_v = 8' \oplus 56 = (\text{dilatino}) \oplus (\text{gravitino})\]

\[(\text{NS}+, \, R+)\]

\[8_v \otimes 8 = 8' \oplus 56 = (\text{dilatino}) \oplus (\text{gravitino})\]

This theory has \(\mathcal{N} = (2, 0)\) supersymmetry.
Type I superstring (closed + open, unoriented)

closed string sector = (type IIB) / (left ↔ right flip)

left ↔ right flip

\[(\alpha^\mu_n, \psi^\mu_r, \psi^\mu_n) \leftrightarrow (\tilde{\alpha}^\mu_n, \tilde{\psi}^\mu_r, \tilde{\psi}^\mu_n)\]

massless spectrum (IIB ⇒ I)

(NS+, NS+): \[1 \oplus 28 \oplus 35 \quad \Rightarrow \quad 1 \oplus 35\]

(R+, R+): \[1 \oplus 28 \oplus 35_+ \quad \Rightarrow \quad 28\]

(R+, NS+), (NS+, R+): \[(8' \oplus 56) \times 2 \quad \Rightarrow \quad 8' \oplus 56\]
open string sector

• unoriented $\rightarrow$ SO or Sp gauge group ($U(n)$ is not allowed.)

• anomaly cancelation $\rightarrow$ SO(32) gauge group

massless spectrum

$$(8_v, 496) \oplus (8, 496) : \text{SO(32) gauge boson and gaugino}$$

This theory has $\mathcal{N} = (1, 0)$ supersymmetry.
Heterotic string

left mover: fermionic string (10D)
right mover: bosonic string (26D)

We compactify 16D of the right mover and construct 10D theory.

left mover: $X^\mu_L$, $\psi^\mu$
right mover: $X^\mu_R$, $X^m_R$ ($m = 10, \ldots, 25$)

Here the fields $X^m_R$ are compactified on the torus $T_\Lambda = \mathbb{R}^{16}/\Lambda$ with the lattice $\Lambda$. 
massless spectrum

left mover: \( X_L^\mu, \psi^\mu \Rightarrow 8_v \oplus 8 \)

right mover: \( X_R^\mu, X_R^m \Rightarrow 8_v \oplus (16 \text{ scalars } + \text{ more}) \)

Actually, we can construct 480 scalars in addition to 16 from \( X_R^m \), by choosing \( \Lambda \) to be the root lattice of SO(32) or \( E_8 \times E_8 \). Those scalars form the adjoint representation of each group (Frenkel-Kac construction), where 16 scalars correspond to the Cartan subalgebra. Then the spectrum becomes

\[
\text{boson:} \quad 8_v \otimes 8_v = 1 \oplus 28 \oplus 35 \quad \text{and} \quad (8_v, 496) \\
\text{fermion:} \quad 8 \otimes 8_v = 8' + 56 \quad \text{and} \quad (8, 496)
\]
fermionic construction (focusing on low-lying states)

bosonization (one chiral scalar ↔ two MW spinor)

\[ X^m_R \iff \lambda^A \quad A = 1, 2, \ldots, 32. \]

boundary condition and intercept

\[
\begin{align*}
R : & \quad \lambda^A(\tau, \sigma + 2\pi) = +\lambda^A(\tau, \sigma), \quad a = -1 \quad \text{heavy} \to \text{ignored} \\
NS : & \quad \lambda^A(\tau, \sigma + 2\pi) = -\lambda^A(\tau, \sigma), \quad a = +1
\end{align*}
\]

low-lying states

\[
\begin{align*}
|0\rangle_{NS} (m^2 = -4/\alpha'), & \quad \lambda^A_{-\frac{1}{2}}|0\rangle_{NS} (m^2 = -2/\alpha'), \\
\lambda^A_{-\frac{1}{2}}\lambda^B_{-\frac{1}{2}}|0\rangle_{NS} (m^2 = 0)
\end{align*}
\]
For right mover, NS vacuum is chosen to be \((-1)^F|0\rangle_{\text{NS}} = +|0\rangle_{\text{NS}}\).

\[
|0\rangle_{\text{NS}} : \text{ dropout by level-matching} \\
\lambda^A_{-\frac{1}{2}}|0\rangle_{\text{NS}} : \text{ dropout by GSO (keeping } (-1)^F = +1) \\
\lambda^A_{-\frac{1}{2}}\lambda^B_{-\frac{1}{2}}|0\rangle_{\text{NS}} : \text{ remaining}
\]

Then we have \(\lambda^A_{-\frac{1}{2}}\lambda^B_{-\frac{1}{2}}|0\rangle_{\text{NS}}\) as lowest level state, which forms the adjoint representation of SO(32).

The \(E_8 \times E_8\) theory can be obtained by the following GSO projection

\[
(-1)^{F_1} = (-1)^{F_2} = 1
\]

Here \(F_1\) is the fermion number for \(\lambda^A\) \((A = 1, \ldots, 16)\) and \(F_2\) is the one for \(\lambda^{A'}\) \((A' = 17, \ldots, 32)\).
Summary of five superstring theories

1. Type I
2. Type IIA
3. Type IIB
4. Heterotic SO(32)
5. Heterotic $E_8 \times E_8$

If we do not need supersymmetry, we can construct other theories by different GSO projections, for example

Type 0A, Type 0B, SO(16)×SO(16) heterotic, etc.
3. Low energy SUGRA and CY compactification

3-1. supergravity (SUGRA)

◊ 11D SUGRA (← M-theory) [Cremmer-Julia-Scherk]

Field contents and their on-shell degrees of freedom ($44 + 84 = 128$)

- 11D metric $G_{MN}$: ($\#$ of d.o.f.) $= \frac{11 \times (11 + 1)}{2} - 11 - 11 = 44$

- 3-form $A_{MNP}$: ($\#$ of d.o.f.) $= 11 - 2 C_3 = 84$

- gravitino $\chi_M$: ($\#$ of d.o.f.) $= \frac{1}{2} \times 32 \times (11 - 1 - 2) = 128$
Action for bosonic fields (kinetic $+$ Chern-Simons)

$$S_{11D} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4$$

Here

$$A_3 = \frac{1}{3!} A_{MNP} dx^M \wedge dx^N \wedge dx^P, \quad F_4 = dA_3,$$

$$\int d^Dx \sqrt{-G} |F_n|^2 = \int F_n \wedge ^* F_n$$

supersymmetry

$$\delta e^m_M = \frac{1}{4} \bar{\epsilon} \Gamma^m \chi_M, \quad \delta A_{MNP} = \frac{3}{4} \bar{\epsilon} \Gamma_{[MNP\chi]},$$

$$\delta \chi_M = D_M(\hat{\omega})\epsilon - \frac{1}{288} (\Gamma_M^{NPQR} - 8\delta_M^{NP} \Gamma^{PQR}) \hat{F}_{NPQR} \epsilon$$
Type IIA SUGRA (← circle compactification of 11D SUGRA)

Field contents and their on-shell degrees of freedom

bosons: dilaton $\phi$ (1) + NS-NS 2-form $B_2$ (28) + metric $G_{\mu\nu}$ (35) 
+ R-R 1-form $C_1$ (8) + R-R 3-form $C_3$ (56)

fermions: dilatino(8) × 2 + gravitino(56) × 2

Action for bosonic fields (string frame)

$$S_{\text{IIA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left( R + 4(\partial_M \phi)^2 - \frac{1}{2} |H_3|^2 \right)$$

$$- \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left( |F_2|^2 + |\tilde{F}_4|^2 \right) - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4$$
Type IIB SUGRA (← T-dual of IIA SUGRA)

Field contents and their on-shell degrees of freedom

bosons: dilaton $\phi$ (1) + NS-NS 2-form $B_2$ (28) + metric $G_{MN}$ (35) 
+ R-R 0-form $C_0$ (1) + R-R 2-form $C_2$ (28) 
+ self-dual R-R 4-form $C_4$ (35)

fermions: dilatino $(8) \times 2 + \text{gravitino} (56) \times 2$

Action for bosonic fields (string frame)

\[
S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left( R + 4(\partial_M \phi)^2 - \frac{1}{2} |H_3|^2 \right) \\
- \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left( |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) - \frac{1}{4\kappa_{10}^2} \int H_3 \wedge F_3 \wedge C_4
\]
Type I SUGRA

Field contents and their on-shell degrees of freedom

bosons: dilaton $\phi$ (1) + 2-form $C_2$ (28) + metric $G_{\mu\nu}$ (35) + SO(32) gauge field $a_1$ ($8 \times 496$)

fermions: dilatino (8) + gravitino (56) + SO(32) gaugino ($8 \times 496$)

Action for bosonic fields (string frame)

$$S_I = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[ e^{-2\phi} \left( R + 4(\partial_M \phi)^2 \right) - \frac{1}{2} |\tilde{F}_3|^2 \right]$$

$$- \frac{1}{2g_{10}^2} \int d^{10}x \sqrt{-G} e^{-\phi} \text{Tr}_v |f_2|^2$$
Heterotic SUGRA

Field contents and their on-shell degrees of freedom

bosons: dilaton \( \phi \) (1) + 2-form \( B_2 \) (28) + metric \( G_{\mu\nu} \) (35)
+ SO(32) or \( E_8 \times E_8 \) gauge field \( a_1 \) (8 \( \times \) 496)

fermions: dilatino(8) + gravitino(56)
+ SO(32) or \( E_8 \times E_8 \) gaugino (8 \( \times \) 496)

Action for bosonic fields (string frame)

\[
S_H = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left[ R + 4(\partial_M \phi)^2 - \frac{1}{2} |\tilde{H}_3|^2 - \frac{\kappa_{10}^2}{g_{10}^2} \text{Tr}_v |f_2|^2 \right]
\]
3-2. Calabi-Yau (CY) compactification

◊ **Parallel spinor** (⇒ unbroken supersymmetry)

$$\delta \chi_\mu = D_\mu(\omega) \epsilon + \cdots = D_\mu(\omega) \epsilon = 0$$

(suppose the fields other than metric = 0)

compactification from 10D to 4D

$$\text{SO}(9, 1) \rightarrow \text{SO}(3, 1) \otimes \text{SO}(6) : \quad 16 \rightarrow (2, 4) \oplus (\bar{2}, \bar{4})$$

$$\epsilon = \epsilon_L \otimes \epsilon_+ + \epsilon_R \otimes \epsilon_- , \quad \epsilon_R = \epsilon_L^*, \quad \epsilon_- = \epsilon_+^*$$

Suppose 4D is flat:

$$D_\mu(\omega) \epsilon = 0 \quad \Rightarrow \quad \partial_m \epsilon_L = 0 , \quad D_i(\omega) \epsilon_+ = 0$$
necessary condition for existence of solution

\[ [D_i, D_j] \epsilon_+ = \frac{1}{4} R_{ijkl} \Gamma^{kl} \epsilon_+ = 0 \]

mutiplying $\Gamma^j$ with use of Bianchi identity:

\[ R_{ijkl} \Gamma^j \Gamma^{kl} \epsilon_+ = 2R_{ij} \Gamma^j \epsilon_+ = 0 \implies R_{ij} = 0 \text{ Ricci-flat} \]

sufficient condition: reduction of holonomy $R_{ijkl} \in SU(3) \subset SO(6)$

\[ SO(6) \rightarrow SU(3) : \quad 4 \rightarrow 1 \oplus 3 \]

The above 1 gives the parallel spinor, which represents the unbroken $\mathcal{N} = 1$ supersymmetry in 4D.
geometric interpretation: SU(3) invariant bilinear forms

\[ J_{ij} = -i \epsilon_+^\dagger \Gamma_{ij} \epsilon_+ \quad (2\text{-form}), \quad \Omega_{ijk} = -i \epsilon_+^T \Gamma_{ijk} \epsilon_+ \quad (3\text{-form}) \]

properties

\[ J^2 = -1, \quad dJ = d\Omega = 0, \quad J \wedge \Omega = 0, \quad \text{etc.} \]

The above implies that \( J \) is the Kähler form and \( \Omega \) is the (unique) (3.0)-form. The manifolds satisfying these properties are called Calabi-Yau manifolds. CY manifolds are shown to have Ricci-flat metric (Calabi conjecture and Yau’s proof).
Moduli of Calabi-Yau manifolds (6D)
deformation of metric keeping Calabi-Yau condition

\[ g + \delta g : \text{Kähler and Ricci-flat} \]

two kinds of deformation

- keeping complex structure: \( \delta g = \delta g_{a\bar{b}} \)
- changing complex structure: \( \delta g = \delta g_{ab} = (\delta g_{\bar{a}\bar{b}})^* \)

differential forms

\[ \omega_{(1,1)} = \delta g_{a\bar{b}} dz^a \wedge d\bar{z}^b \]
\[ \omega_{(2,1)} = \Omega_{abc} \delta g_{\bar{c}\bar{d}} \bar{g}^{\bar{d}c} dz^a \wedge d\bar{z}^c \wedge d\bar{z}^\bar{c} \]
• $\omega_{(1,1)}$: Kähler moduli

• $\omega_{(2,1)}$: complex structure moduli

Note that they are the harmonic forms on the Calabi-Yau manifolds. Then these forms express the massless degrees of freedom after the compactification.