Lecture 1

Moduli of heterotic string compactifications

Idea of these lectures:
Review recent progress in understanding of moduli space for heterotic string compactifications on smooth manifolds with including O(dx) corrections

Motivation:
• Why should string phenomenologists worry about moduli & moduli spaces?

1. Stability of solutions
2. Predictability of 4D model — moduli dependence of 4D kinetic terms, couplings etc.
3. Moduli in inflationary models; stability issues again
4. Cosmological constant/AdS solutions

• Why heterotic?

Very good starting point for particle physics phenomenology:

(MS)SM models: several constructions (not all fulfilling all phenomenology)

Statistical studies possible

Ex. Orbifold models
Calabi-Yau models

In both cases, get particle physics from gauge bundle with gauge group in \( E_8 \times E_8 \)

Drawback: moduli stabilisation
see however P-RS-2 1009.3931 (orbifold)
References CY model building of MS(SM)/GUTs

Non-Abelian
- Bouchard-Donagi
- Braun et al
- Anderson et al

Line bundles
- Anderson et al

Ref's orbifold
- Dixon, Harvey, Vafa, Witten '85
- Ibanez, Quevedo, Giros, Martinec, Kobayashi, Nilles, Vaudrevange, Zavala, Parameswaran, ...
Setting the stage

Heterotic, low-energy, 10D
\[ \downarrow_{\text{SO}(32)} \]
Bosonic: metric $g$, dilaton $\Phi$, B-field $B$, $E_8 \times E_8$ gauge field $A$
Fermions: gravitino $\Psi_M$, dilatino $\lambda$, gaugino $X$
$N=1$ SOSY in 10D

Action
\[ S = \int e^{-2 \phi} \left[ \ast R - 4 \ast d \Phi^2 + \frac{1}{2} |H|^2 + \frac{\alpha'}{4} \left( \text{tr} |F|^2 - \text{tr} |R|^2 \right) \right] \]
\[ R \text{ Ricci scalar} \]
\[ R \text{ curv. wrt connection} \Theta \text{ on} T \Sigma \text{ bundle} \]
\[ \text{supplemented by BI for flux} \]
\[ d_A F = 0 \quad (F = dA + A \wedge A = d_A A) \]

\[ dH = \frac{\alpha'}{4} \left( \text{tr} (F \wedge F) - \text{tr} (R \wedge R) \right) \]
Green-Schwarz anomaly cancellation
\[ H = dB + \frac{\alpha'}{4} \left( \omega_{\text{cs}}^A - \omega_{\text{cs}}^E \right) \]
\[ \omega_{\text{cs}}^A = \text{tr} (A \wedge dA + A \wedge A + A \wedge A \wedge 4 + 4) \]

SOSY variations of fermions
\[ \delta \Psi_M = \left( \nabla_M + \frac{1}{2} H_{MPN} \Gamma_{UP} \right) \epsilon \]
\[ \delta \lambda = (d \Phi + \frac{1}{2} H) \epsilon \]
\[ \delta X = F_{MN} \Gamma^{MN} \epsilon \]

\[ \Gamma = \nabla_M \Gamma^{MN} \text{ etc.} \]
\[ 10D \text{ Maj.-Weyl spinor} \Rightarrow 16 \text{ supercharges} \]

Ansatz 2: $M_{10} = M_4 \times M_6$, $M_6$ any OP, $m,d$
\[ M_4 \text{ max sym} 4D \]
\[ \nabla_M \Phi = 0, \quad F_{\mu \nu}, H_{\mu \nu \rho} = 0 \]
\[ \Gamma_{\mu \nu} \text{ 4D redef.} \]
Lorentz group decmp: \(so(1,9) = so(1,3) \times so(6)\)  

\[ 16 \times 8^\frac{1}{2} \text{ chiral} \]

6D spinors of \(\#\) chiral

plug Ansatz into SUSY eqs to determine geometry of \(M_6\)

\[ 0 = \delta \Psi \Psi \in 4D \Rightarrow 0 = [\nabla A, \nabla B] \eta = R_{\mu \nu \sigma \rho} \nabla^{\sigma \rho} \eta \]

max sym 4D

\[ R_{\mu \nu \sigma \rho} = \frac{1}{12} (g_{\mu \nu} g_{\rho \sigma} - g_{\mu \sigma} g_{\rho \nu}) \]

\[ \Rightarrow r = 0 \text{ Minkowski} \]

- external dilatino trivial

\[ \Theta = \delta \Psi = (\nabla \Phi + \frac{1}{2} H_{\mu \nu \rho} \gamma^{\mu \nu \rho}) \eta = 0 \]

\[ 0 = (\nabla \Phi + \frac{1}{2} H_{\mu \nu \rho} \gamma^{\mu \nu \rho}) \eta = 0 \]

The internal gravitino eq says: \(M_6\) must admit a nowhere-vanishing spinor which is covariantly constant with respect to \(\nabla^+\), a connection

\[ \nabla^+ \eta = \nabla \eta + \frac{1}{8} H_{\mu \nu \rho} \gamma^{\mu \nu \rho} \]

Such a spinor reduces the holonomy of the connection \(\nabla^+\) to \(SU(3)\). Why?

Holonomy group: how do spinors, vectors transform around \(M_6\) \(\Rightarrow so(6)\) hold group closed loops in the space \(\eta\) cov. constant \(\Rightarrow\) cannot change

\(SO(6) \cong SU(4)\) as Lie alg.

6D spinors \(\eta\) as \(4, \bar{4}\) of \(SU(4)\)

Can always choose \(\eta = \begin{pmatrix} 0 \\ g \end{pmatrix}, g \in SU(4)\)

like...
The elements $g_{ij}$ of SU(4) that preserve this $\eta$ are of the form

$$U_4 = \begin{pmatrix} U_3 & 0 \\ \frac{1}{U_3} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad U_3 \in SU(3)$$

Conclude: 4D $\mathcal{N}=1$ SUSY requires $\text{Hol}(\mathcal{V}^+) = SU(3)$

**Simple example**: Set $H = 0 \Rightarrow \mathcal{V}^+ = \mathcal{V}^{Le}$

$\text{Hol}(\mathcal{V}^{Le}) = SU(3)$

In this case we simply say that the manifold has $SU(3)$ holonomy.

Of course, recall from Sunday that this is a CY, but in order to prove that we must also prove:

1) complex
2) Kähler

Before this, note that gauge fields are even

$H = 0 \Rightarrow$ constant $\phi$

Rewrite conditions in terms of differential forms:

On a 6D manifold, a nowhere vanishing spinor

$\Rightarrow$ exist 2-form & 3-form

\begin{align*}
\omega_{mn} &= \bar{\gamma}_{mn} \eta \\
\xi_{mnp} &= \bar{\gamma}_{mnp} \eta
\end{align*}

Real 2-form

Decomposable 3-form

That satisfy

$$\omega \wedge \omega = 0 \quad \omega \wedge \omega \wedge \omega = \frac{3!}{4!} \bar{\omega} \wedge \bar{\omega}$$

Prove using Frenet identities

2-form: $15 = 8 + 3 + 3 + 1$

Remark: no 1-form since vector rep of $SU(4) \to 3 + \bar{3}$ of $SU(3)$ no singlet!
The internal gravitino & dilatino variations then give

\[ d(e^{-2\phi} \omega) = 0 \]  
\[ d(e^{2\phi} \omega \wedge \omega) = 0 \]  
\[ e^{2\phi} d(e^{-2\phi} \omega) = * H \]  
(1)  
(2)  
(3)

(1): \( \Omega = e^{-2\phi} \omega \) is a decomposable closed 3-form  
\[ \Rightarrow \text{integrable complex structure } \mathcal{I} \]  
\( \Omega, \mathcal{I} \) are (3,0)-forms w.r.t. \( \mathcal{I} \)  
\( \Omega \) "not top form"  
\( \mathcal{I} \) "not volume form"

(2), (3): If \( H \neq 0 \) then \( d\omega \neq 0 \) not Kahler "Hermitian form"  
\( H = 0 \), \( d\omega = 0 \) Kahler "Kahler form"

Hence, have shown \( H = 0 \Leftrightarrow M_6 \cong CY \)  
["Candelas, Hocking, Strominger, Wilk185]  

\[ d(e^{-2\phi} \omega \wedge \omega) \]  
\( H \neq 0 \): \( M_6 \) is conformally balanced  
["Kleban 86", "Hull 86", "Strominger 86"]

**Exercises**

Use integrability condition \( [\nabla m, \nabla n] \eta = 0 \)  

\[ R_{mn} = 0 \] for \( H = 0 \).  

Hint: use  
\[ R_{mnpq} + R_{mpqn} + R_{mqnp} = 0 \]  

What can you deduce when \( H \neq 0 \)?

Remark: SUSY, torsion classes  
[Example mfd's ]
CY manifolds: $\text{CY}_3 \sim 1/2$ billion

Kähler mfd with $c_1(TX) = 0$ always allow a Ricci flat metric of SU(3) holonomy

Conf. balanced opt. mfd:
(Chen: uniqueness, Yau: existence)

Complex hermitian mfd w. conf balanced metric & hat (3,0)-form / Hull - Strominger manifolds:

not so many

- Non-Kähler elliptic fibres over $K3$ (Goldstein-Pukhlikov-Fu-Yau-Becker-Yan-Sethi...)
- $S^2 \times S^3 \times \mathbb{C}^2 \ (\text{Lu-Tian, Fu-Li-Yau})$
- Some in the Fujiki class (bi-mero to Kähler mfds)

Why so few $1/2$ billion?

Non-Kähler $\Rightarrow$ cannot be constructed using alg geo.

However, we'll see that at in het comp.

$\mathcal{O}(x)$ con often deform CY $\Rightarrow$ non Kähler.
Tools we'll need

Spinors & spin connection

Vector bundles: \((E, B, \pi, F)\)

* Fiber bundle is a space \(E\) with base space \(B\), a projection \(\pi: E \to B\) and fibers \(F\).

For any local open set \(U(B)\), \(\pi^{-1}(U) \cong U \times F\) not even for fibration.

* Vector bundle: as above but \(F\) a vector space.

Example: \(B = S^1\), \(F = \mathbb{R}\) cylinder or Möbius strip this is a real line bundle, with 1D real vector space.

In retrograded sense complex line bundles, tensor products as well as more intricate bundles.

* Tangent bundle: bundle of all tangent spaces.

* Gauge bundles

A geometric way to describe gauge fields.
Allows gauge field configurations with non-trivial transition functions.
Magnetic monopole!
Gaugino eq.

\[ F^{mn} \gamma^{mn} \eta = 0 \]

\[ \Rightarrow F^{mn} \eta^* \gamma_{mn} \eta = 0 \quad F^{mn} \omega_{mn} = 0 \]

\[ \Rightarrow F \wedge \omega = 0 \]

\[ \Rightarrow \quad F^{mn} \eta^T (F \eta) \gamma_{mn} \eta = 0 \quad \Rightarrow \quad F^{mn} \Omega_{\rho mn} = 0 \]

\[ \Rightarrow \quad F \wedge \Omega = 0 = F \wedge \Omega \]

\[ F^{(2)} = 0 = F^{(2,0)} \]

\[ \text{The gauge fields are described by a vector bundle } V, \text{ with connection } A \text{ whose curvature is holomorphic } \Omega(2), \text{ and satisfies the Hermitian Yang Mills eq. (1) + (2).} \]

Technical problem:

\[ \omega_{mn} \text{ is related to unknown metric on } M_6, \]

which is unknown for CY's and we have very few examples of Hull-Strominger in fields that with known metric

Luckily there is a nice theorem theorem [Li-Yau; Donaldson-Ohlhusch-Yau]

\[ A \text{ HYM connection exists if the hol. bundle } (V, A) \text{ is (poly) stable bundle} \]

Checking whether a bundle is (poly) stable is hard, but we have examples only on CY's?
**Lecture 2**

- Anomaly cancellation condition/ H-flux \( BI \)

\[
dH = \frac{\alpha'}{4} \left( \text{tr}(F \wedge F) - \text{tr}(R \wedge R) \right)
\]

- Topological constraint \( \int_C \text{tr}(F \wedge F) = \int_C \text{tr}(R \wedge R) \neq 0 \)

where \( C \) is some 4-cycle in the CY

and we have used \( \int_C dH = \int_C H = 0 \) for a 4-cycle

Must have a non-Abelian gauge field over (a bundle over) the internal space

- \( H = 0 \) \( \Rightarrow \ \text{tr}(F \wedge F) - \text{tr}(R \wedge R) = 0 \) needed

\( C \)

\( R \) is the curvature for the BI connection on \( TX \)

\( \Rightarrow \ \text{spin connection} \ \omega \) as gauge field for the holonomy group

To satisfy \( BI \) w. \( H = 0 \), must turn on a gauge field \( A \) for an SU(3) subgroup \( H' \) of \( E_8 \times E_8 \), so we can equate \( A \) with \( \omega \)

Sorting out some subtleties with the brane

have then embedded the SU(3) holonomy group connection in \( E_8 \times E_8 \)

"Standard embedding"
H ≠ 0 gives more freedom —
we then have a E8 non-CY compactification (at least at order e')
Can choose other vector bundles with
that satisfy the topological constraint
but that gives non-zero H
ex: line bundles, monads

Remark: The gauge group of the 4D theory
is given by the commutant of the H in E8
Idea: "Embed one of the E8's, the other gives
a hidden sector" (also important - see tomorrow!)

- E8 ⊃ SU(3) × E6 standard embedding
  ⇒ E6 GUT
  Non-standard embedding: can get other
  4D gauge groups

- Break E8 GUT ⇒ standard model worry
  Wilson lines

A comment about connections:
the TN in heterotic should be treated carefully

On CY: I preferred connection \( \nabla^{lc} \)
Can show SUSY + H-fielded BI ⇒ EOMs

Non-CY: \( \nabla^+ = \nabla^{lc} \pm \frac{1}{8} H_{mnp} \phi^m \phi^n \)
\( \nabla^+ \) appears in gravitino eq.
\( \nabla^- \) in action!
Also, only if \( \nabla^+ \phi^2 \) on instanton does
SUSY + H-fielded BI ⇒ EOMs
Summary

Have rewritten the SUSY variations as geometric constraint on any manifold \((X, \omega, \Psi)\) and vector bundle \((V)\)

\[
\begin{align*}
\text{d}(e^{-2\phi} \omega \wedge \omega) &= 0 \\
\text{d}(e^{-2\phi} \Psi) &= 0 \\
H &= -\frac{1}{2} (\partial - \bar{\partial}) \omega = dB + \frac{\alpha'}{4} (\omega_A^A - \omega_G^G) \\
F \wedge \omega \wedge \omega &= 0 \\
F \wedge \Psi &= 0 = F \wedge \bar{\Psi} \\
R \wedge \omega \wedge \omega &= 0 \\
R \wedge \Psi &= 0 = R \wedge \bar{\Psi}
\end{align*}
\]

This is the system whose moduli we're after.

Short-hand: \((X, V, H)\) satisfying these eqs is a "heterotic structure"
Geometric def's (recap from Hiroaki Nakajima's talk)

1) $\partial_{\bar{t}}$ Complex structure defined by real top form $\Omega$

$$d(\partial_{\bar{t}} \Omega) = 0 \iff \begin{cases} dK_{\bar{t}} \wedge \Omega + \partial X_{\bar{t}} = 0 & \Lambda^{3,1}(X) \\ -\partial X_{\bar{t}} = 0 & \Lambda^{2,2}(X) \end{cases}$$

Diffeomorphisms: (need to quotient out to get moduli)

$$L_{\bar{t}}(\Omega) = d(\bar{t} \Omega) - \bar{t} d\Omega = d(\bar{t} \Omega)$$

Note $\bar{t} \Omega \in \Lambda^{2,0}$

$$\partial_{\text{trivial}} \Omega = \bar{\partial}(\bar{t} \Omega) + \partial (\bar{t} \Omega) = K_{\bar{t}} \Omega + \partial \Lambda^{2,0}$$

so prop to $\Omega$

$$\Rightarrow K_{\bar{t}} \text{ non-trivial } \iff K_{\bar{t}} \text{ constant } \Rightarrow \partial X_{\bar{t}} = 0$$

$$\Rightarrow \boxed{X_{\text{trivial}} = \partial \Lambda^{2,0}}$$

$$X_{\bar{t}} \in \frac{\{ \partial \text{-closed}\ (2,1)\-forms\}}{\{ \bar{\partial} \text{-exact}\ (2,1)\-forms\}} = H_{\bar{\partial}}^{(2,1)}(X)$$

We'll have use for a reformulation

$$X_{\bar{t}} = \frac{1}{2} \Delta_{\bar{t}}^a \wedge \Omega_{abc} d\bar{z}^b$$

$$\Delta_{\bar{t}}^a \in H_{\bar{\partial}}^{(0,1)}(X, TX)$$
2) Deformation of Hermitian structure, \( \i.e. \omega \); that

\[ d\omega = 0 \]

\[ d\partial_\omega = 0 \]

Diffeomorphisms:

\[ d\nu \omega = d(d\omega) - d\nu \omega = d(e^{2\phi} (v \omega \wedge \omega)) \]

Note: this is not a genuine 3-form, and it lacks a primitive piece \( \Lambda_3^{\text{prim}} \).

\( \Rightarrow \) space of dif's not necessarily finite.

\( \& \) not a cohomology.

We'll return to this when discussing anomaly cancellation condition.

3) Dif's of \((X, V)\) w.r.t. holomorphic structure

\[ \text{[Abraham '75, Anderson et al. 10, 11, 13]} \]

The holomorphicity constraint \( F^{(0,2)} = 0 \Rightarrow F \Omega = F \bar{\Omega} = 0 \) couples variations: if the complex structure changes the bundle must adjust to stay holomorphic.

\[ 0 = \partial_\Omega (F \wedge \Omega) = \partial_\Omega F \wedge \Omega + F \wedge \partial_\Omega \]

If \( \partial_\Omega \Omega = 0 \) (fixed c.s.):

\[ 0 = (\partial_\Omega F)_{(a,b)} = (d_A \partial_\Omega A)_{(a,b)} = \partial_\Omega \alpha_t, \quad \alpha_t = (\partial_\Omega A)_{(a,b)} \]

Trivial difs of \( A \) are gauge 4:

\[ \alpha_{\text{triv}} = \partial_\Omega \lambda, \quad \lambda \in \Lambda^0 (X, \text{End} V) \]

\[ \Rightarrow \quad \alpha_t \in H^1_{\partial_\Omega} (X, \text{End} V) \]
Now, let's vary also the complex structure

\[ F \wedge \chi_t = F \wedge \partial_t \Omega = - \partial_A \chi_t \wedge \Omega \]

contract with $\overline{\Omega}$ \( F \wedge \partial_t \overline{\Omega} \)

\[ \Rightarrow \quad \partial_A \chi_t = \Delta^m \wedge F_{mn} \, dx^n \]

**Constraint on $\Delta^m$**!

**Define the Atiyah map**:

\[ F : \Lambda^{(0,q)}(X, TX) \rightarrow \Lambda^{(0,q+1)}(X, \text{End } V) \]

\[ \Delta \rightarrow F(\Delta) = (-1)^q \Delta^m \wedge F_{mn} \, dx^n \]

The constraint on $\Delta^m$ is then

\[ \Delta \in \ker F \subseteq H_{\partial_A}^{(0,1)}(X, TX) \]

Note that $F$ is a map in cohomology:

\[ F(\partial \Delta) = \partial_A (F(\Delta)) = 0 \]

(use BI $\partial_A F = 0$ to prove this)

- closed forms map to closed forms
- exact — "" exact — ""

Thus, the 1st order moduli space of the holomorphic structure of $(X, V)$ is

\[ H_{\partial_A}^{(0,1)}(X, \text{End } V) \otimes \ker F \subseteq H_{\partial_A}^{(0,1)}(X, \text{End } V) \otimes H_{\partial}^{(0,1)}(X, TX) \]

"Atiyah class stabilization"
The versal space for the holomorphic structure of \((X, V)\) can also be described via a new bundle \(\mathcal{Q}\):

\[
0 \to \text{End}V \to \mathcal{Q} \overset{\pi}{\to} TX \to 0
\]

Note: short exact sequence so \(\text{End}V \subseteq \mathcal{Q} \& \mathcal{Q}\)
projects onto \(TX\). If map \(\pi\) surjective then \(\mathcal{Q} = \text{End}V \otimes TX\) and \((\mathcal{Q}, TX, \text{End}V, \pi)\) is a fibration.

We can define a holomorphic structure on \(\mathcal{Q}\), if \(x = \begin{bmatrix} \alpha \\ \Delta \end{bmatrix} \in \mathcal{Q}\) then

\[
\overline{\partial}_Q = \begin{bmatrix} \overline{\partial}_A \\ 0 \end{bmatrix} \begin{bmatrix} \Delta \\ \partial \end{bmatrix}
\]

Note that \(\overline{\partial}_Q^2 \iff \mathcal{F}(\overline{\partial}_Q) + \overline{\partial}_A \mathcal{F}(\Delta) = 0\).

Using \(\overline{\partial}_Q\) can rewrite

\[
TM = H^{(q,1)}(X; \text{End}V) \oplus \ker \mathcal{F} = H^{(q,1)}(X, \mathcal{Q})
\]

Thus, another way of arriving at this result would be to write down an extension bundle \(\mathcal{Q}\) as above. The long exact sequence in cohomology associated to the SES above then gives the desired infinitesimal moduli space.
4) Deformation of $\omega \wedge F = 0$

By OUY & LY terms: Stability of the bundle is preserved by 1st order def's
\[ \Rightarrow \text{no constraint on moduli} \]

CY: in 4D EFT get D-terms (Anderson et al. 11/)

5) Subtlety for Hull-Strominger case:

There are two connections with torsion

\[ \nabla^\pm_m = \nabla^+_m = \frac{i}{8} H_{mnp} f^{np} \]

SUSY & BI $\Rightarrow$ EOMs
\[ \{ R(\nabla^+) \wedge \psi = 0 \}
\[ R(\nabla^-) \wedge \omega = 0 \]

for non-Kähler

Need to vary these eq's too.

$\Rightarrow$ extra "moduli" for $V$ (can remove by field redef [d'Aléon-Svanes:14])

$\Rightarrow$ extra extension bundle $E_a$

$\Rightarrow$ extra Atiyah map on cpl of mod. sp.

$0 \to \text{End} V \otimes \text{End} (TX) \to \hat{A}_a \to \pi_1 X \to 0$

$\Delta^a \in \ker ( R + F )$
6) Last eq. B.I for \( H = J(d\omega) \)

Can construct an extension bundle

\[
0 \rightarrow T^*X \rightarrow \hat{\mathcal{Q}} \rightarrow \hat{\Theta} \rightarrow 0
\]

with operator

\[
\hat{\mathcal{D}} = \begin{bmatrix}
\delta & H \\
0 & \delta Q
\end{bmatrix}
\]

\[
\hat{\mathcal{D}}^2 = 0 \iff d\mathcal{H}(J(d\omega)) = \frac{\alpha_1}{4} \left( \text{tr}(F\Lambda F) - \text{tr}(R\Omega R) \right)
\]

The (analogue of the) Atiyah map takes

\[
x = \begin{pmatrix}
\kappa \\
\alpha \\
\Delta n
\end{pmatrix}
\]

\[
\text{ext. spin}
\]

to \( \mathcal{T}^{(1,0)}X \)-valued form

\[
\mathcal{H} : \Omega^{(0,q)}(X, \hat{\Theta}) \rightarrow \Omega^{(0,q+1)}(X, \mathcal{T}^{(1,0)}X)
\]

\[
\mathcal{H}(x) = i \left( -1 \right)^q \Delta \wedge (d\omega)_{\rho \mu \nu} dx^\rho dx^\mu dx^\nu - \frac{\alpha_1}{4} \left( \text{tr}(\Lambda F) - \text{tr}(R\Lambda R) \right)
\]
Result

holomorphic structure on $(X,V,H)$ $\leftrightarrow$ infinitesimal deformations of hol. structure on $\hat{X}$

infinitesimal moduli of hut structure $\leftrightarrow$ infinitesimal deformations of hut structure on $\hat{X}$

References for deformation:
- Candiles - de la Ossa (gene CY geometry)
- Beeker - Tseng - Yau '06 (Hull-Strominger geometry)
- Anderson - Gray - Lukas - Ovrut '10, 11
- Anderson - Gray - Sharpe '14, de la Ossa - Swann '14
- García-Fernández - Rubio - Traut '15
- Borghgra - Hekmati '13

generalized geometry approach