

Lecture 1

(1)

Moduli of heterotic string compactifications

Refs: GSW 2 ch 12, 14-16, BBS ch 9, 10.4

Idea of these lectures:

Review recent progress on understanding of moduli space for het. string compactifications on smooth manifolds with including $\mathcal{O}(x')$ corrections

Motivation: • Why should string phenomenologists worry about moduli & moduli spaces?

1. a) Stability of solutions b) 5th forces & observations
 2. Predictability of ^{eg} 4D model - moduli dependence
of 4D kinetic terms, couplings etc.
 3. Moduli in inflationary models; stability issues again
 4. Cosmological constant / dS solutions
sys \leftrightarrow moduli
-

• Why heterotic?

Very good starting point for particle physics phenomenology:

(MS)SM models : several constructions

(not all fulfilling all phys. constraints)

Statistical studies

possible

Ex. - Orbifold models

Dixon, Harvey, Vafa, Witten '85, Ibanez, Quevedo

- Calabi-Yau models*

Nilles, Vaeravange, Gross, Martinec, Kobayashi, ...

Paranuraman-Ramos Sanchez-Zavala 1009.3931

In both cases, get particle physics from gauge bundle with gauge group in $E_8 \times E_8$

Q1:

Drawback: moduli stabilisation

see however P-RS-2 1009.3931 (orbifold)

hep-th/9603074

non Abelian

~~Notes~~

Decap that information

References CY model building of MS(SM)/GUTs

Non - Abelian	Bouchard - Donagi	0512149
	Braun et al	0501070
	Anderson et al	0911.1569
	Braun et al	1112.1097
Line bundles	Anderson et al	1106.4804 1202.1757

Ref's orbifold

Dixon Harvey Vafa Witten '85
Ibanez, Quevedo, Gross, Martinec, Kobayashi,
Nilles, Vandevrière, Zavala, Parameswaran, ...

Setting the stage

Heterotic, low-energy, 10 D

$SO(32)$

Bosonic: metric g , dilaton ϕ , B-field B , $E_8 \times E_8$ gauge field A

Formions: gravitino Ψ_M , dilatino λ , gaugino χ

$N=1$ SUSY in 10 D

$$\text{Action } S = \int e^{-2\theta} [*R - 4|d\phi|^2 + \frac{1}{2}|H|^2 + \frac{\alpha'}{4} (\text{tr}|F|^2 - \text{tr}|R|^2)]$$

R Ricci scalar

R curv. wrt connection θ Hull connection if SUSY

supplemented by BI for flux

$$d_A F = 0 \quad (F = dA + A \wedge A = d_A A)$$

$$dH = \frac{\alpha'}{4} (\text{tr}(F \wedge F) - \text{tr}(R \wedge R))$$

Green-Schwarz anomaly cancellation

$$H = dB + \frac{\alpha'}{4} (\omega_{CS}^A - \omega_{CS}^\Theta)$$

$$\omega_{CS}^A = \text{tr}(A \wedge dA + A \wedge A \wedge A)$$

SUSY variations of fermions

~~$\delta \Psi_M = [\nabla_M + \frac{1}{2} H_{MNP} \Gamma^{NP}] \epsilon$~~

$$\delta \lambda = (\not{\partial} \phi + \frac{1}{2} H) \epsilon$$

$$\delta \chi = F_{MN} \Gamma^{MN} \epsilon$$

$$\not{\partial} = \nabla_M \Gamma^M \text{ etc}$$

ϵ 10 D Majorana-Weyl spinor \Rightarrow 16 supercharges

Ansatz: $M_{10} = M_4 \times M_6$

M_6 any cp. mf'd

M_4 max sym 4D

$$\nabla_\mu \phi = 0, F_{\mu\nu}, H_{\mu\nu\rho} = 0$$

$\Gamma_{\mu,\nu}$ 4D indices

$\overline{M}_{M,N}$ 10D
indices]

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Lorentz group decomps: $SO(1,9) = SO(1,3) \times SO(6)$

$$16 = (2,4) + (\bar{2},\bar{4})$$

1 chir + 1 antichir in 4D $\Rightarrow N=1$

Plug Ansatz into SUSY eq's to determine geometry of M_6

- $0 = \delta \Psi_\mu = \nabla_\mu \epsilon_{4D} \Rightarrow 0 = [\nabla_\mu, \nabla_\nu] \eta = R_{\mu\nu\rho\sigma} \Gamma^{\rho\sigma} \eta$

max sym 4D

$$R_{\mu\nu\rho\sigma} = \frac{r}{12} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$

$$\Rightarrow r = 0 \quad \text{Minkowski}$$

- external dilatino trivial

- $0 = \delta \Psi_m = (\nabla_m + \frac{1}{8} H_{mnp} \gamma^{np}) \eta = 0$

- $0 = (\nabla \phi + \frac{1}{2} H_{mnp} \gamma^{mp}) \eta = 0$

~~External dilatino eq determines ϕ in terms of H~~
 The internal gravitino eq says: M_6 must admit a nowhere-vanishing spinor which is covariantly constant with respect to the connection

$$\nabla_m^+ = \nabla_m + \frac{1}{8} H_{mnp} \gamma^{np}$$

Such a spinor reduces the holonomy of the connection

∇^+ to $SU(3)$. Why?

Holonomy group: how do spinors, vectors transform around closed loops in the space
 $M_6 \Rightarrow SO(6)$ not group η cov. constant \Rightarrow cannot change

$SO(6) \cong SU(4)$ as Lie alg

6D spinors η as $4, \bar{4}$ of $SU(4)$

Can always choose
 \uparrow
 by $SU(4)$ of

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad g\eta = \eta \not\rightarrow g \in SU(4) \text{ of } g$$

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The elements g^4 of $SU(4)$ that preserve two η are of the form

$$U_4 = \begin{pmatrix} U_3 & 0 \\ \hline 0 & 1 \end{pmatrix} \text{ where } U_3 \in SU(3)$$

Conclude 4D $N=1$ SUSY requires $\text{Hol}(\nabla^+) = SU(3)$

Simple example: Set $H = 0 \Rightarrow \nabla^+ = \nabla^{LC}$
 $\text{Hol}(\nabla^{LC}) = SU(3)$

In this case we simply say that the manifold has $SU(3)$ holonomy.

Of course, recall from Sunday that this is a CY, but in order to prove that must also prove
 1) complex
 2) Kähler

Before this, note that gradations of gravis
 $H = 0 \Rightarrow$ constant ϕ

Rewrite conditions in terms of differential forms:

On a 6D mfd, a nowhere vanishing spinor
 $\Rightarrow \exists$ 2-form & 3-form

such that $w_{mn} = \bar{\eta} \gamma_{mn} \eta$ real 2-form

$\Psi_{mnp} = \bar{\eta}^T \gamma_{mnp} \eta$ decomposable cpt 3-form

that satisfy

$$\omega_1 \Omega = 0 \quad \omega_1 w_1 w = \frac{3i}{4} \Psi_1 \bar{\Psi}$$

prove using Fierz identities

Remark: no 1-form since ^{dec of} vector rep⁶ of $SU(4) \Rightarrow 3 + \bar{3}$ of $SU(3)$ no singlet!

$$2\text{-form: } 15 = 8 + 3 + \bar{3} + 1$$

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The internal gravitino & dilatino variations then give

$$d(e^{-2\phi} \Psi) = 0 \quad (1)$$

$$d(e^{-2\phi} \omega \wedge \omega) = 0 \quad (2)$$

$$e^{2\phi} d(e^{-2\phi} \omega) = *H \quad (3)$$

(1) : $\Omega = e^{-2\phi} \Psi$ is a decomposable closed 3-form
 \Rightarrow integrable complex structure \mathbb{E}

Ω, Ψ are $(3,0)$ -forms w.r.t \mathbb{I} Ω "hol top form"
 (so) "hol volume form"

(2), (3) If $H \neq 0$ $d\omega \neq 0$ not Kähler "Hermitian form"
 $H = 0$ $d\omega = 0$ Kähler "Kähler form"

Hence, have shown $H=0 \Leftrightarrow M_6$ is CY

[Candelas, Horowitz, Strominger, Witten '85]

W

$$H \neq 0 : d(e^{-2\phi} \omega \wedge \omega)$$

M_6 is conformally balanced

(Kähler ~~not~~)

[Hull '86, Strominger '86.]

Exercises

Use integrability condition $[\nabla_m, \nabla_n] \eta = 0$
 to show that $R_{mn} = 0$ for $H = 0$.

Hint: use $\gamma^m \gamma^{kl} = \gamma^{ml} + g^{mk} \gamma^{l} - g^{ml} \gamma^k$
 $R_{mnpq} + R_{mpqn} + R_{mqnp} = 0$

What can you deduce when $H \neq 0$?

Remark: SUSY, torsion classes

Example mfds \rightarrow

CY manifolds: $CY_3 \sim 1/2$ billion

Kähler mfd with $c_1(TX)=0$ always allow a Ricci flat metric of $SO(3)$ holonomy

Conf. balanced cpt mfd: (Calabi: unique, Yau: existence)

Complex hermitian mfd w. conf balanced metric &
hol $(3,0)$ -form / Hull-Strominger manifolds:
not so many

• Non-Kähler elliptic fibr over K3 (Goldstein-Pavshkin -
Fu-Yau
Becker²-Yau-Sethi...)

• $\# S^3 \times S^3$ $k \geq 2$ (Lu-Tran, Fu-Li-Yau)

• some in the Fuyuki class (birnero to Kähler mfd)

Why about $1/2$ billion?

Non-Kähler \Rightarrow cannot be constructed using
alg geo.

However, we'll see that ~~all~~ on net comp.

$O(\epsilon)$ correction $\xrightarrow{\text{deform}} \text{CY} \rightarrow$ non-Kähler.

Tools we'll need

ch 12 of GSW 2

Spinors & spin connection

Vector bundles:

- Fiber bundle $\overset{(E,B,\pi,F)}{\sim}$ a space E with base space B , a projection $\pi: E \rightarrow B$ and fibers F

For any ~~local~~ open set $U(B)$ $\pi^{-1}(U) \cong U \times F$

not true for fibration

- Vector bundle: as above but F a vector space

Example: • $B = S^1$ $F = \mathbb{R}$ cylinder or Möbius strip
this is a real line bundle, with 1D real vector space

In heterotrich ~~and~~ complex line bundles, bivector products as well as more intricate bundles

- Tangent bundle:
bundle of all tangent spaces

- Gauge bundles

A geometric way to describe gauge fields

Allows gauge field configurations with non-trivial transition functions

Magnetic monopole!

Gauge eq $F_{mn} \gamma^{mn} \eta = 0$

$$\Leftrightarrow F^{mn} \bar{\eta}^* \gamma_{mn} \eta = 0 \quad F^{mn} \omega_{mn} = 0 \quad (1)$$

$$\Rightarrow F^{mn} \eta^T \gamma_p \gamma_{mn} \eta = 0 \Leftrightarrow F^{mn} \Omega_{pmn} = 0$$

$$F \lrcorner \Omega = 0 \Leftrightarrow * (F \lrcorner \Omega) = 0 \Leftrightarrow * (F \lrcorner \Omega) = 0$$

γ matrix id
+ no. 1 forms

$$\Leftrightarrow F \lrcorner \Omega = 0 = F \lrcorner \bar{\Omega}$$

$$F^{(0,1)} = 0 = F^{(1,0)} \quad (2)$$

The gauge fields are described by a vector bundle V , with connection A whose curvature ω is holomorphic $\Omega(A)$ and satisfies the Hermitian Yang Mills eqs (1) & (2).

Technical problem:

ω_{mn} is related to unknown metric on M_0 , which is unknown for CY's and we have very few examples of Hull-Strominger manifolds that with known metric

Luckily, there is a nice theorem
Theorem [Li-Yau ; Donaldson-Uhlenbeck-Yau]

HSS

CY

The A HYM connection exists iff the hol. bundle (V, A) is a (poly) stable bundle

Checking whether a $\overset{\text{hol}}{\text{bundle}}$ is (poly) stable is hard, but we have examples
Only on CY's?

Lecture 2

- Anomaly cancellation condition/ H-flux BI

$$dH = \frac{\alpha'}{4} (\text{tr}(F \wedge F) - \text{tr}(R \wedge R))$$

~~From notes of Prof. Becker~~

- Topological constraint $\int_{C_4} \text{tr}(F \wedge F) = \int_{C_4} \text{tr}(R \wedge R) \neq 0$
where C_4 is some 4-cycle in the CY
and we have used $\int_{C_4} dH = \int_{\partial C_4} H = 0$ & 4-cycle

Must have a non-trivial gauge field on (a bundle over) the internal space

- $H=0 \Rightarrow \int_{C_4} \text{tr}(F \wedge F) - \text{tr}(R \wedge R) = 0$ needed

~~Observation~~

R is the curvature for the LC connection on TX ,
 \Rightarrow can view spin connection ω as gauge field
 for the holonomy group

To satisfy BI w. $H=0$, must turn on a
 gauge field A for an $SU(3)$ subgroup H'
 of $E_8 \times E_8$, so we can equate A with ω

Sorting out some subtleties with the trace
 have then embedded the $SU(3)$ holonomy
 group connection in $E_8 \times E_8$

"Standard embedding"

- $H \neq 0$ gives more freedom → we then have a ~~CY~~ non-CY compactification (at least at order α')
Can choose other vector bundles with that satisfy the topological constraint but ~~does~~ gives non-zero H
Ex: Line bundles, monads

Remark: The gauge group of the 4D theory

is given by the commutant of ~~the~~ H in E_8

Idea: ~~Embed~~ one of the E_8 's, the other gives "hidden sector" (also important - see tomorrow!)

- $E_8 \supset SU(3) \times E_6$ standard embedding
 $\Rightarrow E_6$ GUT

~~non~~-standard embedding: can get other 4D gauge groups

- Break ~~the~~ GUT \rightarrow standard model using Wilson lines

- A comment about connections:

The TN in heterotic should be treated carefully

- On CY: 1 preferred connection ∇^{LC}
Can show SUSY + H-field BI \Rightarrow EOMs ✓

- Non-CY: $\nabla^\pm = \nabla_m^{LC} \pm \frac{1}{8} H_{mnp} \gamma^{\mp p}$

∇^+ appears in gravitino eq,
 ∇^- action!

Also, only if ∇^- is an instanton does
SUSY + H-field BI \Rightarrow EOMs

Summary& BI forth

Have rewritten the SUSY variations as
 geometric constraint on an ^{SUSY} manifold
 (X, ω, Ψ) and ~~vector~~^{holomorphic} bundle (V)

$$d(e^{-2\phi} \omega \wedge \omega) = 0$$

$$d(e^{-2\phi} \Psi) = 0$$

$$H = -\frac{1}{2}(\partial - \bar{\delta})\omega = dB + \frac{\alpha'}{4}(\omega_{\alpha}^A - \omega_{\alpha}^{\bar{A}})$$

$$F \wedge \omega \wedge \omega = 0 \quad F \wedge \Psi = 0 = F \wedge \bar{\Psi}$$

$$R \wedge \omega \wedge \omega = 0 \quad R \wedge \Psi = 0 = R \wedge \bar{\Psi}$$

This is the system whose moduli we're after.

Short-hand: (X, V, H) satisfying these eqs
 is a "heterotic structure"

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Geometric def's (recap from Hiroaki Nakajima's talk)

1) ∂_t Complex structure defined by hol top form Ω

$$d(\partial_t \Omega) = 0 \Rightarrow \begin{cases} dK_t \wedge \Omega + \partial X_t = 0 & \Lambda^{3,1}(X) \\ \boxed{\bar{\partial} X_t = 0} & \Lambda^{2,2}(X) \end{cases}$$

Diffeomorphisms: (need to quotient out to get moduli)

$$\mathcal{L}_v(\Omega) = d(i_v \Omega) - i_v d\Omega^0 = d(i_v \Omega)$$

Note $i_v \Omega \in \Lambda^{2,0}$

$$\partial_{\text{trivial}} \Omega = \underbrace{\partial(i_v \Omega)}_{\substack{\uparrow \\ \Lambda^{3,0} + \cancel{\Lambda^{3,1}}}} + \overline{\partial}(i_v \Omega) = K_t \Omega + \bar{\partial} \pi^{(2,0)}$$

so prop to Ω

$$\Rightarrow K_t \text{ non-trivial} \Leftrightarrow K_t \text{ constant} \Rightarrow \partial X_t = 0$$

$$\Rightarrow \boxed{X_{\text{trivial}} = \bar{\partial} \pi^{(2,0)}}$$

$$X_t \in \frac{\{\bar{\partial}\text{-closed } (2,1)\text{-forms}\}}{\{\bar{\partial}\text{-exact } (2,1)\text{-forms}\}} = H_{\bar{\partial}}^{(2,1)}(X)$$

We'll have use for a reformulation

$$X_t = \frac{1}{2} \Delta_t^a \wedge \Omega_{abc} dz^{bc}$$

$$\Delta_t^a \in H_{\bar{\partial}}^{(0,1)}(X, TX)$$

[Becker, Tseng, Yam - 06]

- 2) Deformation of Hermitian structure, i.e. ω ; that preserve $d\hat{\beta} = 0$: $\hat{\beta} = (e^{-2\phi} \omega \wedge \omega)^0$
 $\Rightarrow d\partial_t \hat{\beta} = 0$

Diffeomorphisms: $\mathcal{L}_v \hat{\beta} = d(v \lrcorner \hat{\beta}) - v \lrcorner d\hat{\beta}^0 = d(e^{2\phi}(v \lrcorner \omega) \wedge \omega)$

Note: this is not a general 3-form, since it lacks a primitive piece Λ_3^{prim} : ($\Lambda_3^{\text{prim}} \lrcorner \omega = 0$)

\Rightarrow space of def's not necessarily finite.
 (& not a cohomology)

We'll return to this when discussing anomaly cancellation condition.

- 3) Def's of (X, V) w.r.t holomorphic structure
 [Atiyah '57, Anderson et al 10, 11, 13]

The holomorphicity constraint $F^{(0,2)} = 0 = F^{(2,0)} \Leftrightarrow F_1 \Omega = F_1 \bar{\Omega} = 0$
 couples variations: if the complex structure changes the bundle must adjust to stay holomorphic

$$0 = \partial_t (F_1 \Omega) = \partial_t F_1 \Omega + F_1 \wedge \partial_t \Omega$$

If $\partial_t \Omega = 0$ (fixed c.s.)

$$0 = (\partial_t F)^{(0,2)} = (d_A \partial_t A)^{(0,2)} = \bar{\partial}_A \alpha_t \quad \alpha_t = (\partial_t A)^{(0,2)}$$

Trivial defs of A are gauge if

$$\alpha_{\text{triv}} = \bar{\partial} \lambda \quad \lambda \in \Lambda^0(X, \text{End } V)$$

$$\Rightarrow \alpha_t \in H_{\bar{\partial}_A}^{(0,1)}(X, \text{End } V)$$

Now, let's vary also the complex structure

$$F \wedge \chi_t = F \wedge \bar{\partial}_A \Omega = -\bar{\partial}_A \alpha_t \wedge \Omega$$

contract with $\bar{\Omega}$ (exact)

$$\Rightarrow \bar{\partial}_A \alpha_t = \Delta_t^m \wedge F_{mn} dx^n$$

Constraint on Δ_t^m !

Define the Atiyah map:

$$F: \Lambda^{(0,q)}(X, TX) \rightarrow \Lambda^{(0,q+1)}(X, \text{End } V)$$

$$\Delta \mapsto F(\Delta) = (-1)^q \Delta^m \wedge F_{mn} dx^n$$

The constraint on Δ^m is then

$$\Delta \in \ker F \subseteq H_{\bar{\partial}_A}^{(0,1)}(X, TX)$$

Note that F is a map in cohomology:

$$F(\bar{\partial} \Delta) = \bar{\partial}_A(F(\Delta)) = 0$$

(use BI $\bar{\partial}_A F = 0$ to prove this)

- closed forms map to closed forms
- exact " " exact "

Thus, the 1st order moduli space
of the holomorphic structure of (X, V) is

$$H_{\bar{\partial}_A}^{(0,1)}(X, \text{End } V) \oplus \ker F \subseteq H_{\bar{\partial}_A}^{(0,1)}(X, \text{End } V) \oplus H_{\bar{\partial}}^{(0,1)}(X, TX)$$

"Atiyah class stabilization"

infinitesimal deformation
The moduli space for the holomorphic structure of (X, V) can also be described via a new bundle Q :

$$0 \rightarrow \text{End } V \rightarrow Q \xrightarrow{\pi} TX \rightarrow 0$$

Note: short exact sequence so $\text{End } V \subseteq Q$ & Q projects onto TX . If map π surjective then $Q = \text{End } V \oplus TX$ ~~& otherwise~~ $(Q, TX, \text{End } V, \pi)$ is a fibred

We can define a holomorphic structure on Q , if $x = \begin{bmatrix} * \\ \Delta \end{bmatrix} \in Q$ then

$$\bar{\partial}_Q = \begin{bmatrix} \bar{\partial}_A & F \\ 0 & \bar{\partial} \end{bmatrix}$$

Note that $\bar{\partial}_Q^2 \Leftrightarrow F(\bar{\partial} \Delta) + \bar{\partial}_A F(\Delta) = 0$

Using $\bar{\partial}_Q$ can rewrite

$$TM = H_{\bar{\partial}_A}^{(0,1)}(X, \text{End } V) \oplus \ker F = H_{\bar{\partial}_Q}^{(0,1)}(X, Q)$$

Thus, another way of arriving at this result would be to write down an extension bundle Q as above. The long exact sequence in cohomology associated to the SES above then gives the desired infinitesimal moduli space.

4) Deformation of $\omega \wedge \omega \wedge F = 0$

By DUY & LY thms: Stability of the bundle is preserved by 1st order diff's
 \Rightarrow no constraint on moduli

CY: in 4D EFT get D-forms [Anderson et al '11]

5) Subtlety for Hull-Strominger case:

(SO(3)-invariant, metric)
 There are two connections with torsion

$$\nabla_m^\pm = \nabla_m \pm \frac{1}{8} H_{mnp} f^{np}$$

SUSY & BI \Rightarrow EOMs always for CY
 $\Leftrightarrow \begin{cases} R(\nabla) \wedge \psi = 0 \\ R(\nabla) \wedge \omega = 0 \end{cases}$
 for non-Kahler

Need to vary these eq's too

\rightarrow extra "moduli" for ∇ (can remove by field redef [de Ossa-Svanes: 14])

• extra extension bundle ~~for~~

\Leftrightarrow extra Atiyah map on cpt str mod space

$$0 \rightarrow \text{End } V \otimes \text{End } (TX) \rightarrow \widehat{\mathbb{Q}}^* \rightarrow \pi_* X \rightarrow 0$$

$$\Delta^a \in \ker (R + F)$$

6) Last eq: BI for $H = J(d\omega)$

Can construct an extension bundle

$$0 \rightarrow T^*X \rightarrow \hat{\mathbb{Q}} \rightarrow \hat{\mathbb{Q}} \rightarrow 0$$

with ^{diff} operator $\bar{D} = \begin{bmatrix} \bar{\partial} & H \\ 0 & \bar{\partial}_{\bar{Q}} \end{bmatrix}$

$$\bar{D}^2 = 0 \Leftrightarrow dH(J(d\omega)) = \frac{\alpha'}{4} (\text{tr}(F_1 F) - \text{tr}(R_1 R))$$

The (analogue of the) Atiyah map takes

$$x = \begin{pmatrix} k \\ \alpha \\ \Delta \end{pmatrix} \xrightarrow{\text{cpl spt}} bdl \bar{\partial}_t A$$

to $T^{(1,0)}X$ -valued form

$$H: \Omega^{(0,q)}(X, \hat{\mathbb{Q}}) \rightarrow \Omega^{(q, q+1)}(X, T^{*(1,0)}X)$$

$$H(x) = i (-1)^q \Delta^p \wedge (d\omega)_{p,mn} dx^m \wedge dx^n - \frac{\alpha'}{4} (\text{tr}(\alpha_1 F) - \text{tr}(R_1 R))$$

Result

heterotic structure \leftrightarrow holomorphic structure
on (X, V, H) $\bar{\partial}$ on \hat{Q} .

infinitesimal
moduli of het
structure \leftrightarrow infinitesimal deformations
of hol. structure
on \hat{Q}

References for deformation:

Candolras - de la Ossa (gen CY geometry)

Becker - Tseng - Yau '06 (Hull-Strominger geometry)

Anderson - Gray - Lukas - Orant '10, 11

\rightarrow Anderson - Gray - Sharpe '14, de la Ossa - Swanes '14

{ Garcia - Fernandez - Rubro - Trifunovic '15

Boragna - Heekmati '13

generalized geometry approach