

Bibliography

- F-theory reviews

- Denef 0803.1194 (also a bit of algebraic geometry)

- Weigand 1009.3497

- Maharana and Palti 1212.0555

- Algebraic geometry

- Tristan Hubsch: "Calabi-Yau manifolds: A bestiary for physicists"

IIB supergravity

Introduce RR-axion

$$\tau = C_0 + \frac{i}{g_s} \rightarrow \text{string coupling constant } (\equiv e^\phi)$$

axio-dilaton

$$G_3 = F_3 - \tau H_3$$

(Locally

$$F_3 = dC_2$$

$$H_3 = dB_2$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$$

In the Einstein frame

$$(g_{MN}^E = e^{-\phi/2} g_{MN}^S)$$

strings frame metric

$$S_{IIB} = \frac{2\pi}{\ell_s^8} \left[\int d^{10}x \sqrt{-g} R - \frac{1}{2} \int \frac{1}{(\text{Im}\tau)^2} dz \wedge * d\bar{z} + \frac{1}{\text{Im}\tau} G_3 \wedge * \bar{G}_3 \right]$$

$$\ell_s \equiv 2\pi\alpha' + \frac{1}{2} \tilde{F}_5 \wedge * \tilde{F}_5 + C_4 \wedge H_3 \wedge F_3 + (\text{fermionic terms})$$

Plus the constraint

$$*\tilde{F}_5 = \tilde{F}_5.$$

This action is invariant under $SL(2, \mathbb{R})$

Represented by 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$.

$$\begin{pmatrix} F \\ H \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}$$

$$\tilde{F}_5 \rightarrow \tilde{F}_5$$

$$g_{MN}^E \rightarrow g_{MN}^E$$

$$\tau \rightarrow \frac{az + b}{cz + d}$$

This is for the tree-level action. Instantons (more precisely, $D(-1)$ -instantons) are known to break $SL(2, \mathbb{R}) \rightarrow SL(2, \mathbb{Z})$, which is believed to be the exact duality group of ~~the~~ IIB string theory on \mathbb{R}^{10} .

$SL(2, \mathbb{Z})$: 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$.

This is also the group of large diffeomorphisms of a two-torus (preserving the complex structure).

$$T^2 = \mathbb{C} / \mathbb{L} \rightarrow \text{a lattice}$$

where

$$\mathbb{L} = \{ ne_1 + me_2 \mid n, m \in \mathbb{Z} \}$$

Here $e_1, e_2 \in \mathbb{C}$ define the lattice generators. We are interested in complex structure only, so we fix $e_1 = 1$ and as convention dictates, call $e_2 = z$.

Clearly, $(1, z)$ and $(1, z+1)$ define the same lattice.

$$\text{Also } (1, z) \sim (z, -1) \sim (1, -1/z)$$

\downarrow rotation, \downarrow rescaling
 keeping orientation

In terms of the lattice, these are represented by

$$T: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (\text{Induces } z \rightarrow z+1)$$

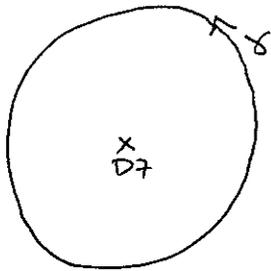
$$S: \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{Induces } z \rightarrow -1/z)$$

T and S generate $SL(2, \mathbb{Z})$.

[Vafa '96]: Suggestion: Understand the behavior of $z = G_0 + i/g_s$ in terms of the complex structure z of a T^2 .

The D7-brane is the magnetic monopole for C_0 (viewed as a 0-form connection):

$$dF_1 = S^2(z) \quad \text{with locally, away from } z=0, F_1 = dC_0.$$



As we go around γ ,

$$\Delta C_0 = \int_{\gamma} dC_0 = \int_{\gamma} F_1 \stackrel{\text{Stokes' theorem}}{=} \int_D dF_1 = \int_D S^2(z) = 1$$

where D is the disk bounded by γ :

$$\partial D = \gamma.$$

So a D7 brane induces a monodromy $C_0 \rightarrow C_0 + 1$ around it, or equivalently $z \rightarrow z + 1$.

In terms of the torus fibration image, by consistency, the T^2 should degenerate over $z=0$.

□ General lesson: degenerations of the $T^2 \leftrightarrow$ 7-brane locations. □

→ How to understand this torus fibration?

- + Just a way of encoding the IIB data.
- + Alternatively, a true geometric background, in M-theory, in a suitable limit. (Discussed later.)
- + No understanding (yet?) of the 12d theory directly. Seems to involve a $(10, 2)$ signature, still mysterious.

The main virtue of the geometrization is that it allows us to use powerful techniques in algebraic geometry to construct and analyze interesting backgrounds.

M-theory limit

The most common definition of F-theory is via the "F-theory limit of M-theory."

Consider M-theory on T^2 , with complex structure τ , and take the volume of T^2 to zero.

$$M / T^2 \xrightarrow{\substack{\text{one } S^1 \\ \text{small}}} IIA / S^1 \xrightarrow{\tau\text{-dualize}} IIB / S^1$$

In the $\text{vol}(T^2) \rightarrow 0$ limit one has $R(\tilde{S}^1) \rightarrow \infty$, so we recover IIB in \mathbb{R}^{10} . Tracking the dualities carefully (see Denef's review, for example), one has that

$$C_0 + i/g_s = \tau(T^2)$$

in the resulting 10d IIB theory.

This can be generalized. Consider a CY_n with fiber T^2 , and base B_{n-1} . Taking $\text{vol}(T^2) \rightarrow 0$:

$$M / CY_n \rightarrow IIB / B_{n-1}$$

with a locally varying $C_0 + i/g_s$. Whenever we have 7-branes in IIB, the ~~elliptic~~ T^2 fiber of CY_n will degenerate.

Observation: Note that B_{n-1} is in general not Calabi-Yau. Supersymmetry is nevertheless preserved in IIB due to the non-constant $C_0 + i/g_s$, which enters the supersymmetry equations for the background:

A bit of algebraic geometry

Much of modern F-theory (and IIB) model building is done using toric geometry, a natural generalization of projective space:

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*}$$

point at the origin.

↑
after just "Pⁿ"

\mathbb{C}^* equivalence of (z_1, \dots, z_{n+1})
and $(\lambda z_1, \dots, \lambda z_{n+1})$ for any $\lambda \in \mathbb{C}^*$.

" "
 $\mathbb{C} - \{0\}$

Natural generalizations:

$$\{0\} \rightarrow \mathbb{Z} \xleftarrow{\text{"Stanley-Reisner ideal"}}$$

some submanifold (subject to some consistency conditions).

$$\downarrow (\mathbb{C}^*) \rightarrow (\mathbb{C}^*)^k \text{ with non-uniform weights.}$$

Summarized in the weight table

	z_1	\dots	z_{n+1}
\mathbb{C}_1^*	q_1^1	\dots	q_1^{n+1}
\vdots	\vdots	\ddots	\vdots
\mathbb{C}_k^*	q_k^1	\dots	q_k^{n+1}

$$q_i^j \in \mathbb{Z}$$

Meaning that we impose the identifications

$$(z_1, \dots, z_{n+1}) \sim (\lambda_1^{q_1^1} z_1, \dots, \lambda_1^{q_1^{n+1}} z_{n+1})$$

$$\sim \dots \sim (\lambda_k^{q_k^1} z_1, \dots, \lambda_k^{q_k^{n+1}} z_{n+1})$$

for all $(\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^*)^k$.
(That is, $\lambda_j \neq 0$ for all j).

Examples

$$\rightarrow \mathbb{P}^1: \mathbb{C}_1^* \left| \begin{array}{cc} z_1 & z_2 \\ 1 & 1 \end{array} \right.$$

with $Z = \{z_1 = z_2 = 0\}$.

(Exercise: convince yourself that $\mathbb{P}^1 = S^2$ topologically. Hint: recall the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$.)

→ Weighted projective space

$$\mathbb{P}^{231} \quad \mathbb{C}_1^* \left| \begin{array}{ccc} x & y & z \\ 2 & 3 & 1 \end{array} \right. \quad \text{with } Z = \{x=y=z=0\}.$$

→ Note that the ^{complex} dimension of the resulting space is $\boxed{n+1-k}$.

A key theorem/problem:

→ □ There is no compact toric CY space. ←

- Compact toric: \mathbb{P}^n .
- Compact CY: T^2 (which is not toric, despite the name)
- Toric CY: Conifold $\mathbb{P}^1 \times \mathbb{P}^1$ with $Z = \{z=w=0\}$.
or $\{x=y=0\}$.
or simply $Z = \emptyset$.

We construct compact CY spaces by taking hypersurfaces, or complete intersections, on toric spaces. That is, if A is toric, we take $X = \{f(z) = 0\} \subset A$, choosing $f(z)$ such that the hypersurface is CY.

• Essential results:

① $c_1(TA)|_X = c_1(TX) + c_1(NX|_A)$ ["Adjunction formula"]

(For CY X we want $c_1(TX) = 0$, so

$$c_1(TA)|_X = c_1(NX|_A).)$$

② For a toric space

$$c(TA) \equiv 1 + c_1(TA) + c_2(TA) + \dots$$

$$= \prod_{i=1}^{n+1} (1 + [D_i])$$

$$\Rightarrow c_1(TA) = \sum [D_i]$$

③ $[f]$ is the cohomology class of ~~the~~ Poincaré dual to the homology class of $\{f=0\}$. f is a homogeneous polynomial under $(\mathbb{C}^*)^k$: $(\mathbb{C}^*)^k(f) = \lambda_1^{a_1} \dots \lambda_k^{a_k} f$.

$[f] = [g]$ in cohomology iff $\frac{f}{g}$ is invariant under $(\mathbb{C}^*)^k$.

④ $c_1(NX|_A) = [f]$, if $X = \{f=0\} \subset A$.

Example: The quintic Calabi-Yau threefold.

$$A = \mathbb{P}^4 = \mathbb{C}^* \left/ \begin{array}{c} z_1 \ z_2 \ z_3 \ z_4 \ z_5 \\ | \ 1 \ 1 \ 1 \ 1 \ 1 \end{array} \right. \quad \text{with } Z = \{z_1 = z_2 = z_3 = z_4 = z_5 = 0\}$$

$$c_1(TA) = \sum [z_i] = 5 [z_1] \quad (\text{Since } [z_i] = [z_j] \ \forall i, j).$$

We take $f = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5$, for instance.

$$c_1(NX|A) = [f] = [z_1^5] = 5 [z_1].$$

Note $[p^P] = p [f]$.

So $c_1(TX) = c_1(TA) - c_1(NX|A) = 0$, and X is CY_3 .

T^2 : Along very similar lines, one can construct a CY_1 by taking a cubic (such as $z_1^3 + z_2^3 + z_3^3$) inside \mathbb{P}^2 .

It is more common in the literature to take instead a sextic in $\mathbb{P}^{2,3,1}$. That is, (Weierstrass form)

$$A = \mathbb{C}^* \left/ \begin{array}{c} x \ y \ z \\ | \ 2 \ 3 \ 1 \end{array} \right. \quad \text{with } Z = \{x = y = z = 0\}$$

We have $[x] = 2 [z]$, $[y] = 3 [z]$, and thus $c_1(TA) = 6 [z]$.

The most general homogeneous sextic is of the form:

$$ay^2 + bxyz + cx^3 + dx^2z^2 + exz^4 + hz^6 + ky^3z^3 = 0$$

We assume $a \neq 0$ and $c \neq 0$. Then one can redefine coordinates such that the above can be rewritten as [Exercise: do this]

$$y^2 = x^3 + fxz^4 + gz^6.$$

Such a CY_1 is necessary a T^2 , by Hirzebruch-Riemann-Roch:

$$2 - 2g = \chi = \int_x e = \int_x c_1(TX) = 0 \Leftrightarrow g = 1.$$

\downarrow genus \downarrow Euler number Euler class = "top Chern class"

So it makes sense to ask for the map between (f, g) and z .

$$\underline{\tau \rightarrow (f, g)}$$

(Useful reference: <http://dlmf.nist.gov/23>)
 ("Digital library of mathematical functions")

$$f = -15 \sum_{\substack{u \in \mathbb{L} \\ u \neq 0}} u^{-4}$$

with \mathbb{L} the lattice generated by $(1, \tau)$.

$$g = -35 \sum_{\substack{u \in \mathbb{L} \\ u \neq 0}} u^{-6}$$

(Known also as "holomorphic Eisenstein series" $\frac{24}{5} E_4$ and E_6)

Note that under $\tau \rightarrow \tau+1$ clearly $(f, g) \rightarrow (f, g)$, while under $\tau \rightarrow -1/\tau$ $(f, g) \rightarrow (\tau^4 f, \tau^6 g)$ due to the lattice rescaling. This can be reabsorbed into the definition of " τ ", so in general (f, g) and $(\lambda^4 f, \lambda^6 g)$ define T^2 s with the same complex structure (i.e. τ up to $SL(2, \mathbb{Z})$).

$$y^2 = x^3 + f x z^4 + g z^6$$

?? \leftarrow same c.s.

$$y^2 = x^3 + f (\lambda z)^4 + g (\lambda z)^6$$

$$\underline{(f, g) \rightarrow \tau}$$

The right object to consider turns out to be "Klein's J -invariant":

$$j(\tau) = 12^3 \frac{4f^3}{4f^3 + 27g^2}$$

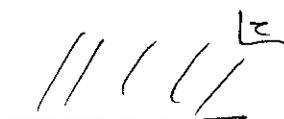
(Using "physics" conventions, where $j(i) = 12^3$.)

This is $SL(2, \mathbb{Z})$ invariant:

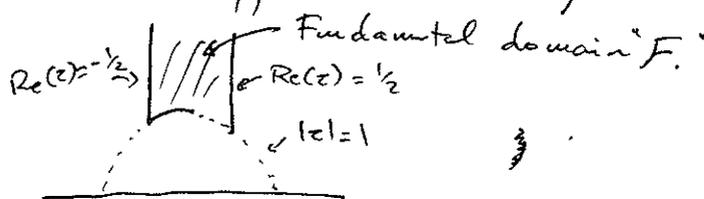
$$j(\tau) = j(\tau+1) = j(-1/\tau)$$

[Exercise: prove this from the $SL(2, \mathbb{Z})$ action on (f, g) above]

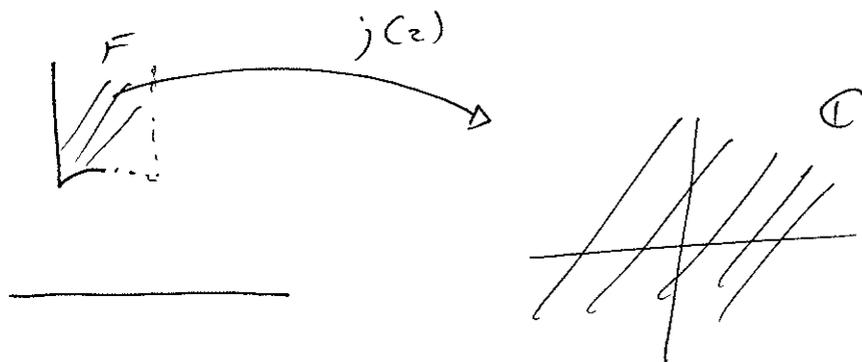
τ naturally lives on the upper half-plane
 ($\text{Im}(\tau) \geq 0$)



Under $SL(2, \mathbb{Z})$ any point in this upper half plane can be mapped to the "fundamental domain":



$j(z)$ gives a one-to-one map between F and \mathbb{C} :



$j^{-1}(z): \mathbb{C} \rightarrow F$ is known too in terms of hypergeometrics, see for instance 1107.2388.

Discriminant

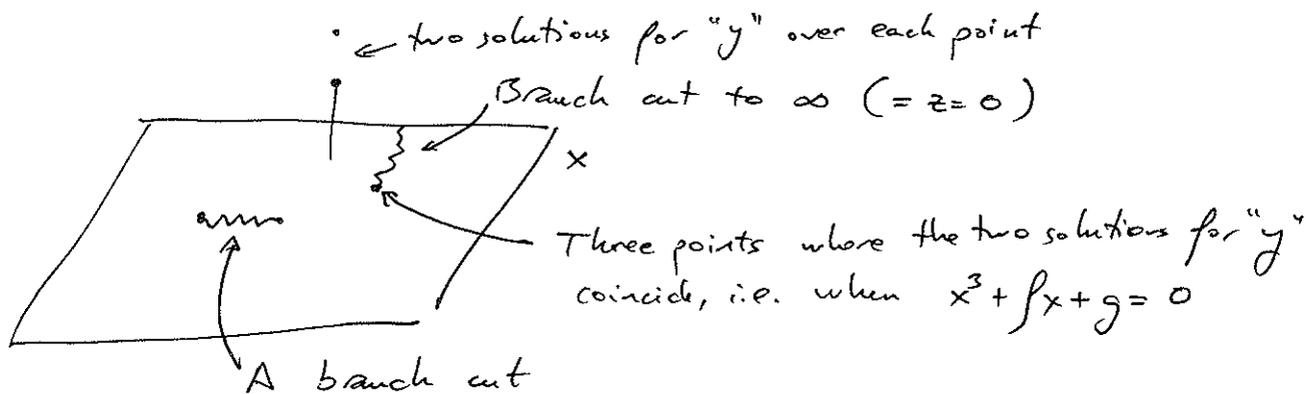
The denominator $\Delta = 4f^3 + 27g^2$ is known as the discriminant of the T^2 . To see what this means, view

$$y^2 = x^3 + fx + g$$

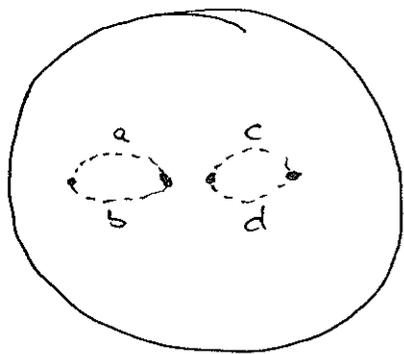
as a double cover ($y = (x^3 + fx + g)^{1/2}$) of the "x" plane. We can momentarily "forget" about the z coordinate, which gives a compactification of the "x" plane at infinity. [Exercise: this is due to the \mathbb{C}^* identification $(x, y, z) = (\lambda^2 x, \lambda^3 y, \lambda z)$, which allows us to set $z=1$ when $z \neq 0$. Convince yourself of this.]

[Ex. Prove that $z=0$ corresponds to a single point in T^2]

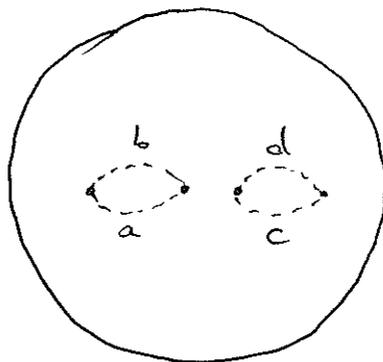
Schematically:



Bringing in the branch point from ∞ , we have a branched double cover of P^1 . Splitting the two branches, and resolving the branch cuts for clarity:

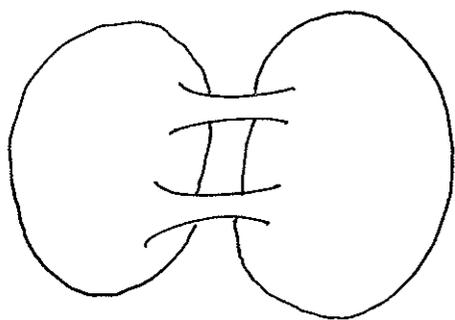


Branch 1

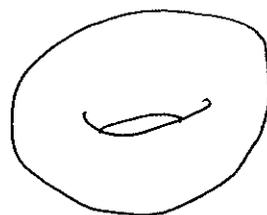


Branch 2

One should glue both branches along the dashed lines, as indicated by the letters. It is easy to see that the result of the gluing is



Topologically
 \sim



The resulting T^2 will be singular whenever two roots of $x^3 + fx + g = (x - e_1)(x - e_2)(x - e_3) = 0$ coincide, i.e. when

$$\tilde{\Delta} = (e_1 - e_2)(e_1 - e_3)(e_2 - e_3) = 0,$$

and one can check that

$$\Delta = 4f^3 + 27g^2 = -2\tilde{\Delta}.$$

So the T^2 is singular iff $\Delta = 0$. [Exercise: show that the same conclusion follows from $d\hat{f} = \hat{f} = 0$, with $\hat{f} = x^3 + fx + g$]

(at $z=0$, for instance)

Close to a single \circ of Δ one can go to a duality frame for z where

$$j(z) \approx q^{-1} + 744 + 196884q + \mathcal{O}(q^2)$$

with $q = e^{2\pi i z}$ and $\text{Im}(z) \gg 1$ (i.e. weak coupling).

$$\text{So } (e^{2\pi i z})^{-1} \sim \frac{1}{z} \Rightarrow z \approx \frac{1}{2\pi i} \log(j(z)).$$

So the monodromy around an isolated \circ of Δ is $\tau \rightarrow \tau + 1$, as we argued before was the case for a D7.

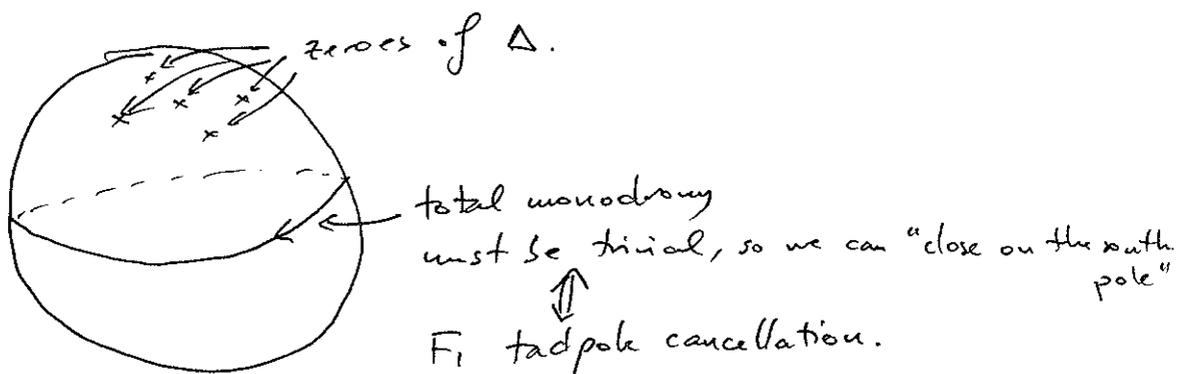
Lesson: $\Delta = 0$ at the location of D7 branes.

Notice that this was for a particular choice of duality frame. In a general duality frame, the monodromy around a single \circ of Δ is

$$M_{z=0} = g \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} g^{-1}$$

with $g \in SL(2, \mathbb{Z})$, by conjugation. This allows for constructing global (compact) solutions as long as the total monodromy is trivial:

$$\prod_{z_i \in \{z \mid \Delta(z)=0\}} M_{z=z_i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



This is complicated to solve in general, but here is where the algebraic methods pay off. They make constructing global solutions simple, since geometric consistency is ensured by the construction.

Example: $K3$

We build it as a T^2 fibration over \mathbb{P}^1 , in Weierstrass form for the fiber:

$$(*) \quad y^2 = x^3 + f(s, t) x z^4 + g(s, t) z^6$$

in an ambient toric space given by

	\mathbb{P}^1 base		$\mathbb{P}^{2,3,1}$ fiber		
	s	t	x	y	z
\mathbb{C}_1^*	1	1	a	b	0
\mathbb{C}_2^*	0	0	2	3	1

$$Z = \{s=t=0\} \cup \{x=y=z=0\}$$

(Note that we can always set the charge of "z" under \mathbb{C}_1^* to 0, by redefining the \mathbb{C}^* generators adequately without losing generality, we have done so.)

Denote by φ the degree of f under \mathbb{C}_1^* , and γ the degree of g under \mathbb{C}_1^* . (For consistency f and g must be homogeneous polynomials in (s, t) .) Homogeneity of $(*)$ then requires:

$$2b = 3a = a + \varphi = \gamma$$

On the other hand, we have

$$c_1(NX|_A) = [c_2] = 2b[s] + 6[z]$$

while

$$c_1(TA) = (2+a+b)[s] + 6[z]$$

So the Calabi-Yau condition implies

$$0 = c_1(TX) = c_1(TA) - c_1(NX|A) = (2+a-b) [s]$$

The solution to these equations is

$$a=4$$

$$b=6$$

$$\varphi=8$$

$$\gamma=12$$

[Exercise: check]

So the ambient space is

$$\begin{array}{c|ccccc} & s & t & x & y & z \\ \hline \mathbb{C}_1^* & 1 & 1 & 4 & 6 & 0 \\ \mathbb{C}_2^* & 0 & 0 & 2 & 3 & 1 \end{array}$$

$$Z = \{s=t=0\} \cup \{x=y=z=0\}$$

Notice that the degree of Δ'' is $4p^3 + 27g^2$, so in this case we have 24 points in \mathbb{P}^1 where $\Delta=0$, that is 24 7-branes at points in the \mathbb{P}^1 base.

Brane Collisions

We have seen that single zeroes of Δ correspond (in some duality frame) to D7 branes.

One can get non-abelian theories by considering higher order zeroes of Δ . (Generalizing the familiar observation that N D7s on top of each other are described at low energies by a $U(N)$ $N=1$ 8d SYM theory.)

The ways that this can happen have been classified by Kodaira. If we are studying the physics at $z=0$, we look to the orders of vanishing of f, g, Δ . I.e., if

$$f \sim z^p f_0 \quad \text{with } f_0(z=0) \neq 0, \text{ we say that}$$

$$\text{ord}(f) = p.$$

The possibilities at complex codimension one (i.e. in cases like K3) are given by

$\text{ord}(f)$	$\text{ord}(g)$	$\text{ord}(\Delta)$	fiber type	Gauge group in 8d
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	$SU(n)$
≥ 1	1	2	II	none
1	≥ 2	3	III	$SU(2)$
≥ 2	2	4	IV	$SU(3)$
2	≥ 3	$n+6$	I_n^*	$SO(8+2n)$
≥ 2	3	$n+6$	I_n^*	$SO(8+2n)$
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8
≥ 4	≥ 6	≥ 12	non-minimal	unknown

→ Comments:

- + $USp(2n)$ groups are also possible in 8d, by adding flux to I_n^* cases. [hep-th/9712028]
- + The last row corresponds to "non-minimal" singularities, which cannot be resolved in a CY way. Whether these make sense in string theory is an open problem. (See for instance hep-th/9812028)
- + I have only listed the non-abelian algebras. The U(1) factors are trickier, and in the K3 case can be best understood from duality with the heterotic on $\tilde{\tau}^2$, for instance. [hep-th/9602022]