Inflation

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Further Readings

《宇宙学基本原理》

龚云贵编著

http://www.ph.utexas.edu/~ygong/correct.pdf
http://www.ph.utexas.edu/~ygong/cosmorev.pdf

1001.5259

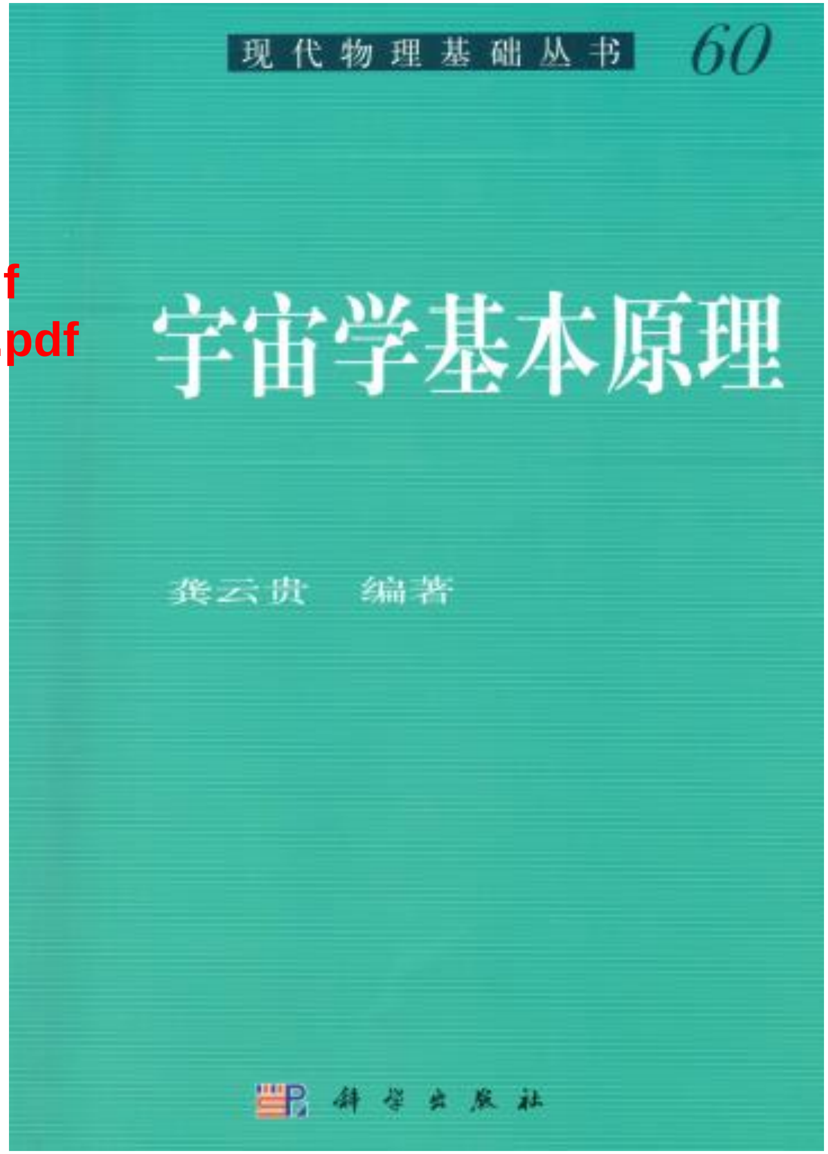
0907.5424

0902.1529

1302.4640

0903.5158

0802.3688

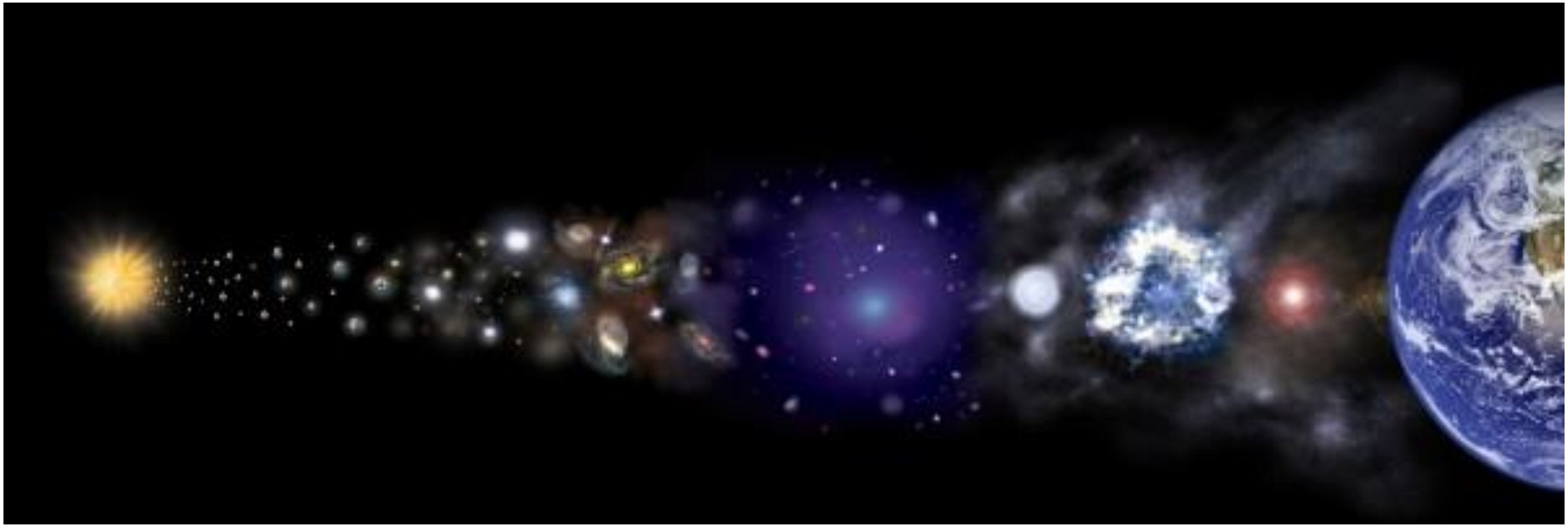


Outline

- Standard Cosmology
- Cosmological problems
- Inflation
- Slow-roll parameters
- Quantum Fluctuations and Power Spectrum
- Inflationary models

The Universe

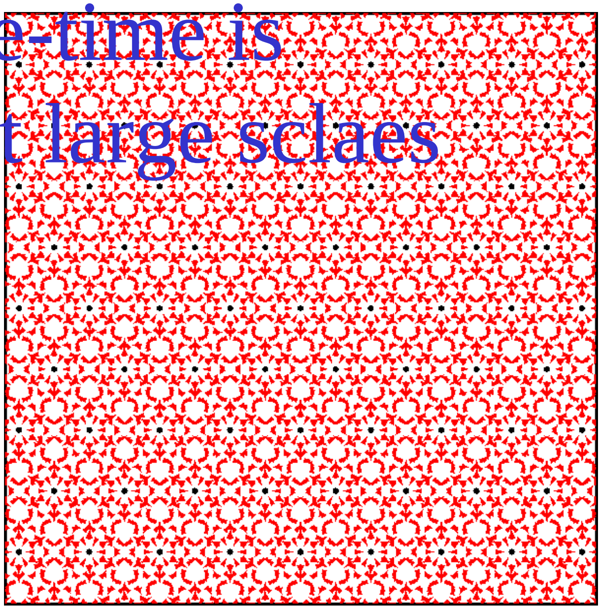
- Universe (宇宙): Two words in Chinese
- The first word means all spaces around us
- The second word means the whole time
- Universe：All spaces and the whole time



Cosmology

- Cosmology: study the whole universe
- Fundamental forces: only gravity and electromagnetic forces are long range forces
- Gravity: Einstein's general relativity
- Cosmological principle: Space-time is Isotropic and Homogeneous at large sclaes

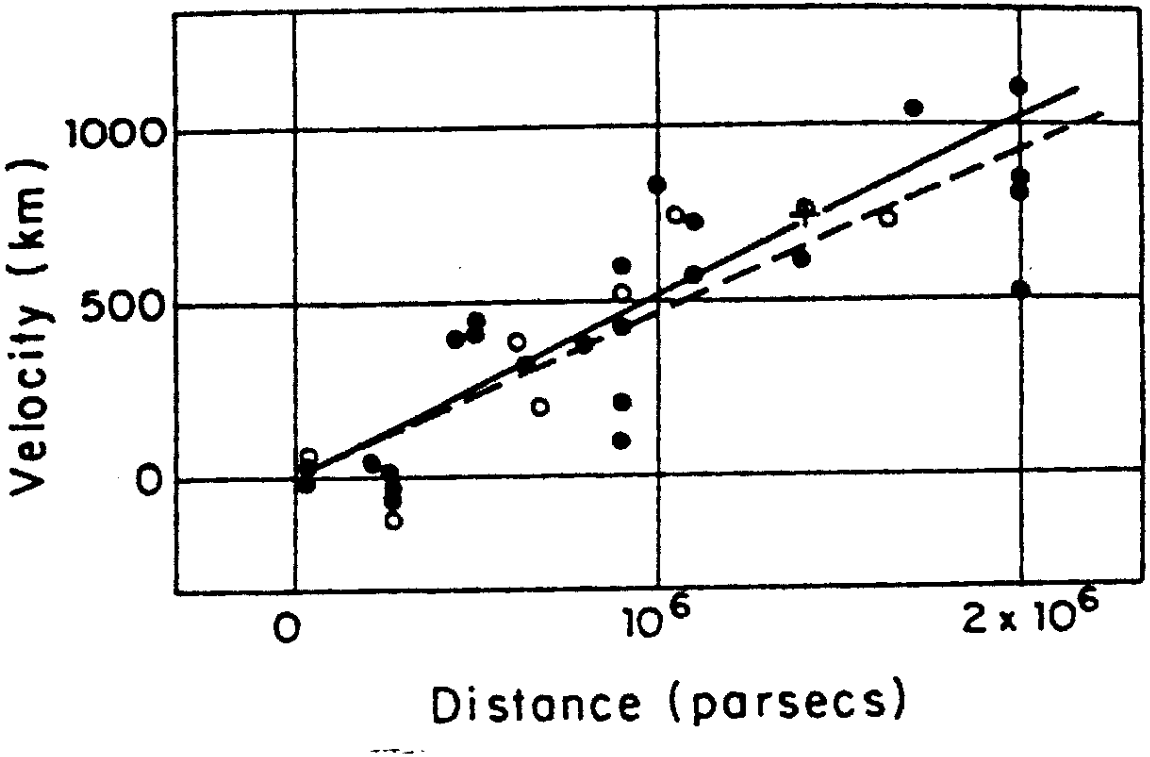
14000Mpc~10^{26} m
Galaxy size: a few Mpc



12.1 Gyr, photons from all galaxies shown

Hubble's Law

- Expanding Universe

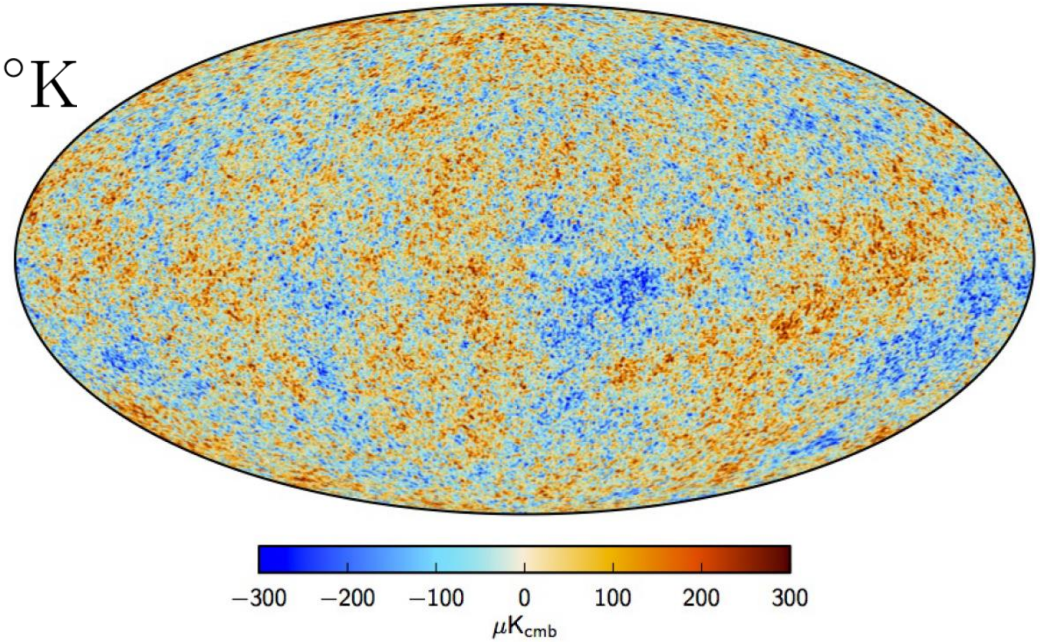


1929

Hubble

Cosmological Principle

$$T = 2.72548 \pm 0.00057\ ^\circ\mathrm{K}$$

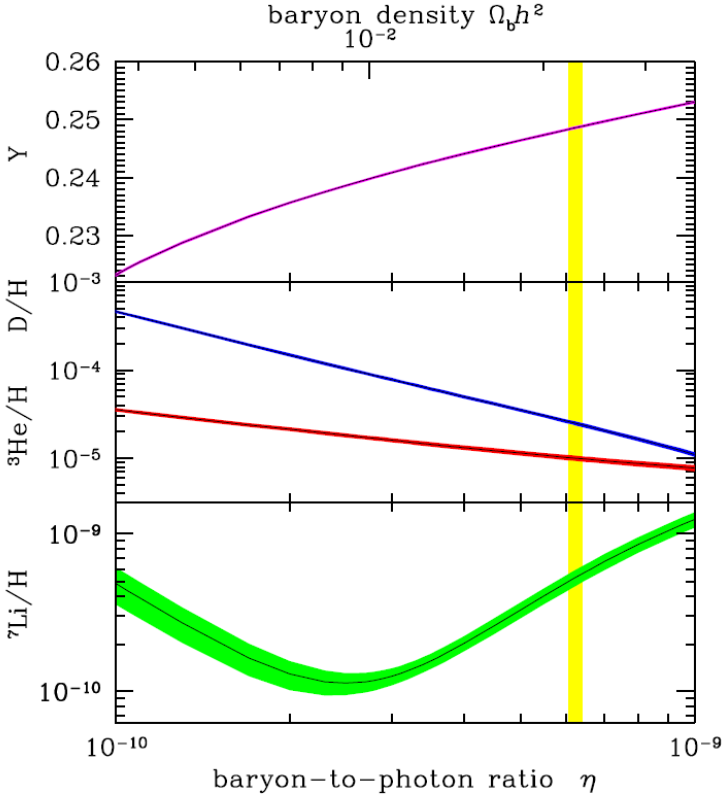


Universe 380,000 years old,13.7 billion years ago

arXiv: 1502.01582

Big Bang Cosmology

- Big Bang (Hoyle 1949)
- Prediction of CMB (Gamov): confirmed in 1965
- Explanation of the primordial abundances of elements
- Thermal history
- …..



Standard Cosmology

- Robertson-Walker

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

Scale factor   Spatial curvature

- Energy-momentum tensor

$$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p)U_\mu U_\nu$$

$$T^{\mu\nu}_{;\nu} = 0 \longrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \qquad p = f(\rho)$$

Equation of state

- Dust

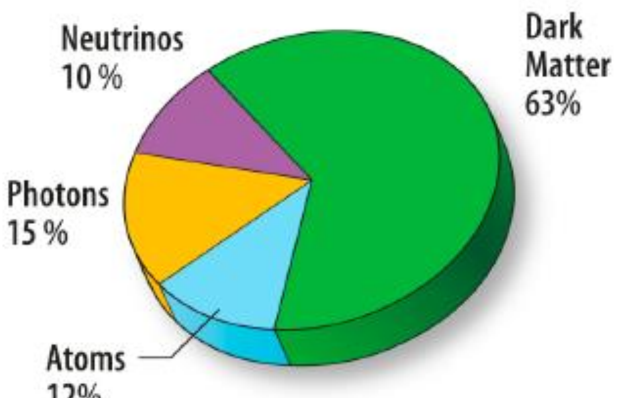
$$w = \frac{p}{\rho}$$

$$w = 0, \quad \rho \propto a^{-3}$$

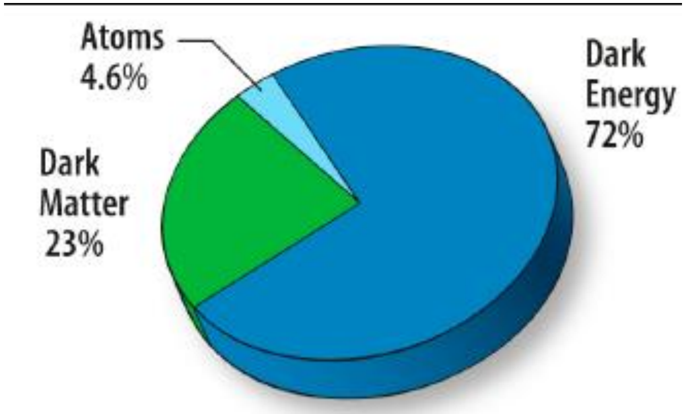
- Radiation

$$w = 1/3, \quad \rho \propto a^{-4} \propto T_{CMB}^4$$

More general $\rho \propto a^{-3(1+w)}$

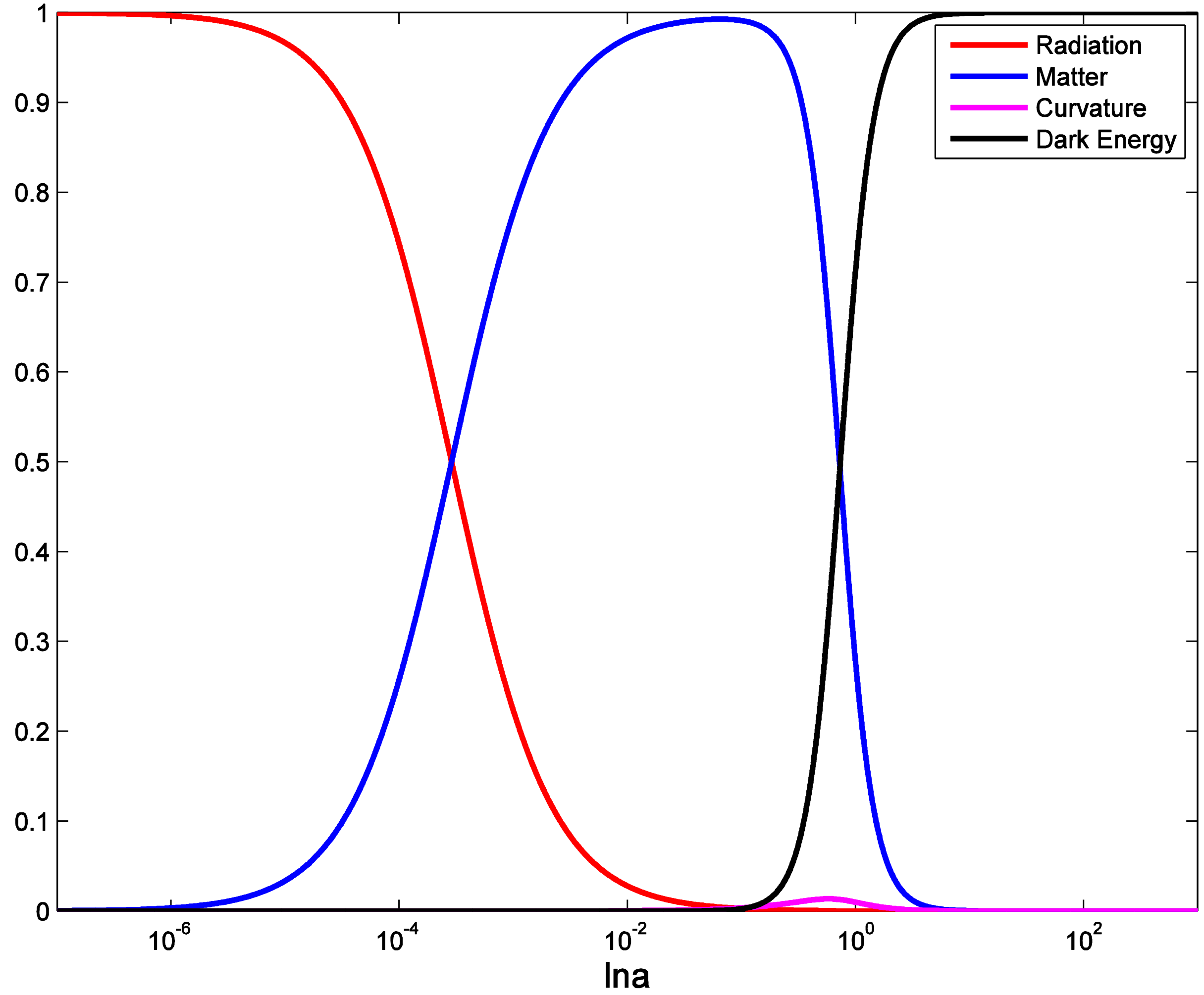


13.7 BILLION YEARS AGO
(Universe 380,000 years old)



TODAY

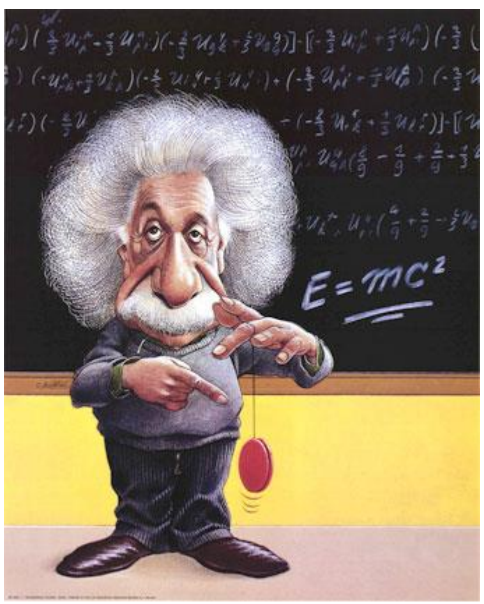
The evolution of different matter

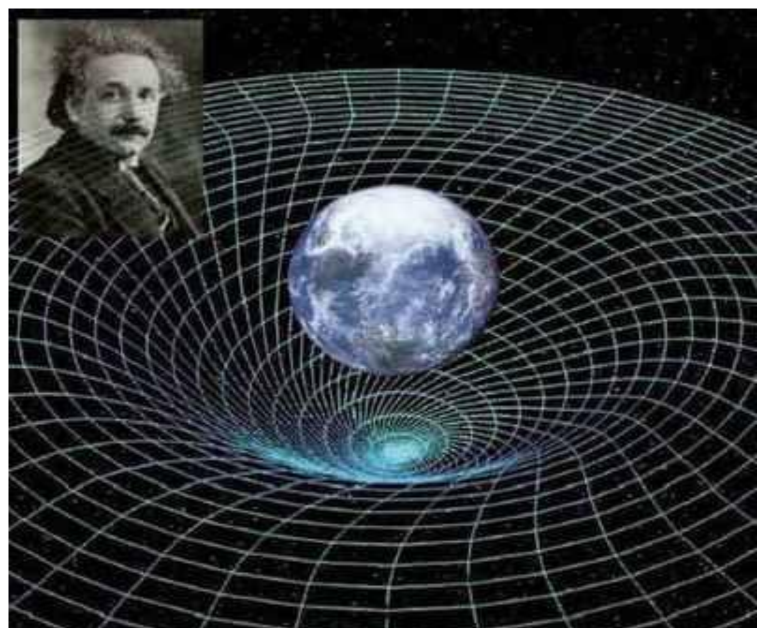


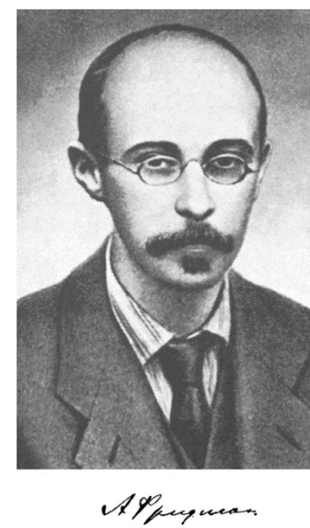
Standard cosmology

- Einstein's general relativity $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
- Friedmann equation

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \qquad H(t) = \frac{\dot{a}}{a}$$







Standard cosmology

- Friedmann Eq. $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

$$H(t) = \frac{\dot{a}}{a}$$

- Acceleration $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Energy conservation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \qquad p = f(\rho)$$

- Deceleration parameter

$$q(t) \equiv -\frac{\ddot{a}}{aH^2} = -\frac{1}{aH^2}\frac{d^2a}{dt^2}$$

Matter domination

- Friedmann equation

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left[1 - 2q_0 + 2q_0\left(\frac{a_0}{a}\right)\right]$$

$$\Omega_{k0} = 1 - \Omega_{m0} = 1 - 2q_0$$

- Einstein-de Sitter universe

$$K = 0, \quad \Omega_k = 0 \qquad q_0 = 1/2, \quad \Omega_{m0} = 1$$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left(\frac{a_0}{a}\right) \qquad a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t} \qquad \rho_m = \rho_{m0}\left(\frac{t}{t_0}\right)^{-2} = \frac{1}{6\pi G t^2}$$

Radiation dominated

- radiation $w = p/\rho = 1/3$ $\qquad \rho_r = \rho_{r0}\left(\frac{a_0}{a}\right)^4$

$$q_0 = \frac{8\pi G\rho_{r0}}{3H_0^2} = \Omega_{r0} \qquad \rho_r + 3p_r = 2\rho_r$$

$$H_0^2 + \frac{K}{a_0^2} = \frac{8\pi G\rho_{r0}}{3}$$

$$\Omega_{k0} = 1 - \Omega_{r0} = 1 - q_0$$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\left[1 - q_0 + q_0\left(\frac{a_0}{a}\right)^2\right]$$

$$a(t) = a_0(2H_0 q_0^{1/2} t)^{1/2}\left(1 + \frac{1-q_0}{2q_0^{1/2}}H_0 t\right)^{1/2}$$

Summary

- Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$
- FRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

- MD $\rho(t) \propto a^{-3}, \quad a(t) \propto t^{2/3}$
- RD $\rho(t) \propto a^{-4}, \quad a(t) \propto t^{1/2}$
- CMB $T = 2.72548°\mathrm{K}$

Horizons

- Particle horizon: The boundary between observable universe and the regions that light has not reached (unobservable)

$$d_{PH}(t) = \int_0^{r_H} \sqrt{g_{rr}}dr = a(t)\int_0^t \frac{dt'}{a(t')}$$

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

$$d_{PH}(t) = \begin{cases} 2H_0^{-1}(a/a_0)^{3/2} = 2H_0^{-1}(1+z)^{-3/2}, \ \text{MD}, \\ H_0^{-1}(a/a_0)^2 = H_0^{-1}(1+z)^{-2}, \ \text{RD}, \end{cases}$$

$K = 0$ $\qquad$ $d_{PH} \propto H^{-1}$

Redshift $\quad 1 + z = \dfrac{a_0}{a}$

Event horizon

- The boundary in space-time beyond which events cannot affect an outside observer

$$d_{EH}(t) = a(t) \int_t^{\infty} \frac{dt'}{a(t')}$$

de-Sitter Universe $\quad a(t) = a_0 \exp(Ht)$

$$d_{EH}(t) = H^{-1}$$

For matter or radiation domination, there is no event horizon

Apparent horizon

- The boundary between light rays that are directed outwards and moving outwards, and those directed outward but moving inward.
- Apparent horizons are observer dependent

$$d_{AH} = \left(H^2 + \frac{K}{a^2} \right)^{-1/2}$$

$$d_{AH} = H^{-1}, \quad K = 0$$

Hubble horizons

- Hubble horizon $d_H \propto H^{-1}$
- Co-moving Hubble horizons d_H/a

$d_H/a \propto a^{1/2}$ **Matter domination**

$d_H/a \propto a$ **Radiation domination**

$d_H/a \propto a^{-1}$ **de-Sitter, inflation**

- Co-moving particle horizon

$d_{pH}/a \propto a^{1/2}$ **Matter domination**

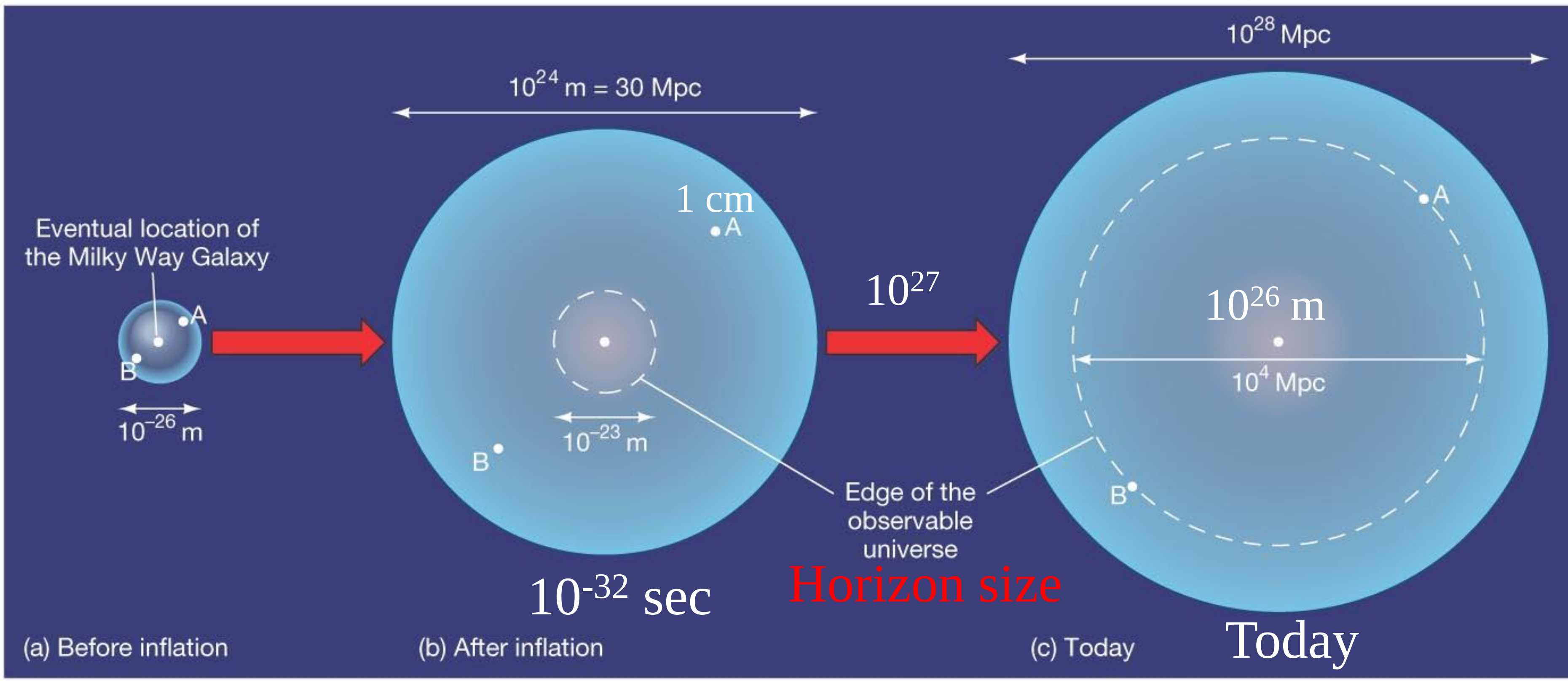
$d_{pH}/a \propto a$ **Radiation domination**

Horizon Problems

- The expansion speed is less than light speed

$$t = 10^{-32}\text{s},\ 2ct = 10^{-23}\text{m}$$
$$t_0 = 10^{18}\text{s},\ 2ct_0 = 10^{27}\text{m} = 10^4\text{Mpc}$$

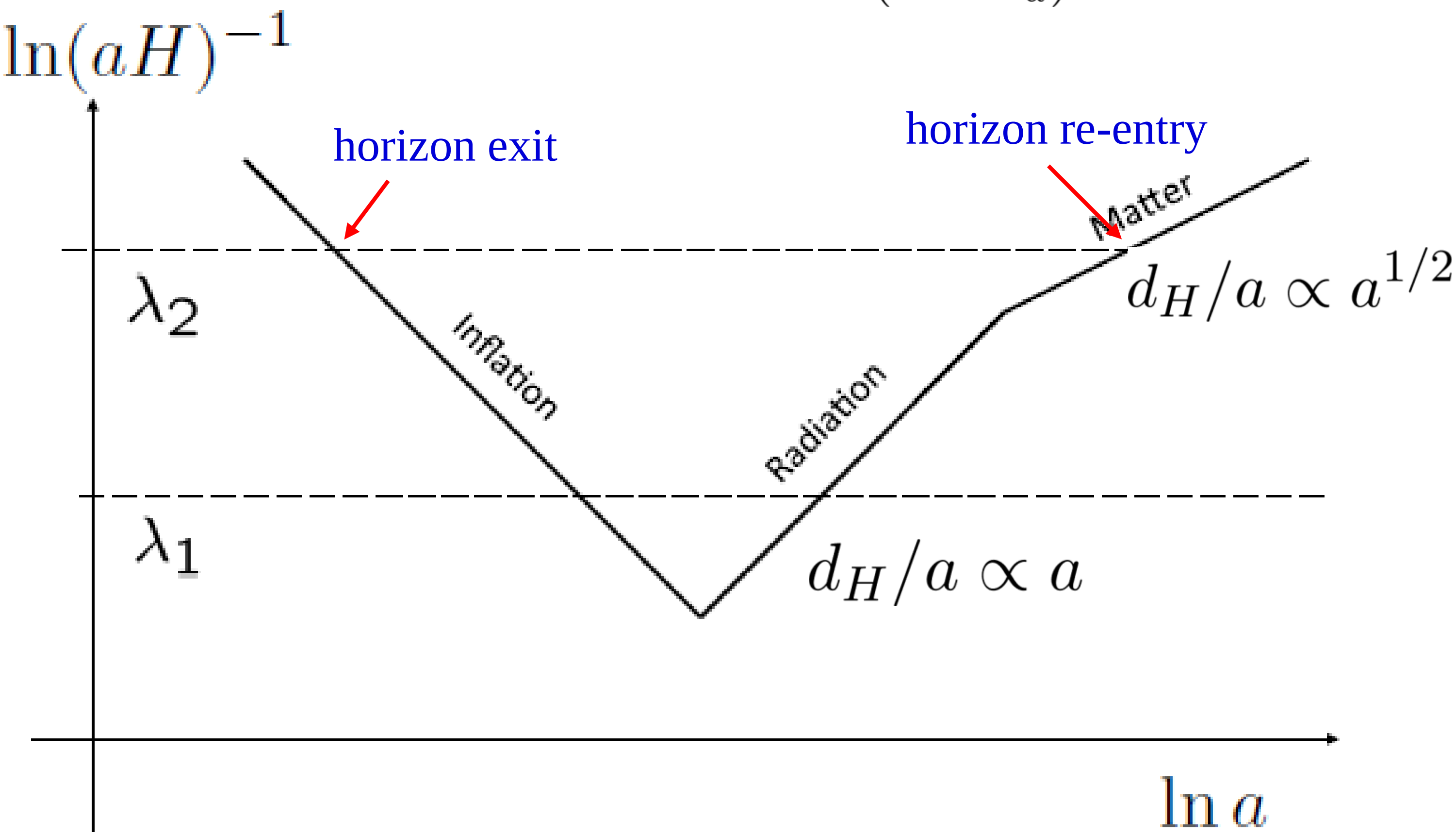
$$\frac{a_0}{a} = (1+z_{eq})^{1/4}\left(\frac{t_0}{t}\right)^{1/2} = 10^{27}$$



Horizon problem

- $z_d = 1100$ the angle subtended by the horizon

$$(1+z_d)^{-1/2} \approx 1.6^\circ$$



Flatness problem

- Why $\Omega_K \approx 0$ at the beginning

Curvature density

$$\Omega_K(z) = -K/(a^2H^2) = \Omega_{K0}(1+z)^2/E^2(z)$$

MD

$$E^2(z) \simeq (1+z)^3, \quad \Omega_K(z) \simeq \Omega_{K0}/(1+z)$$

Matter-Radiation Equality $\Omega_K \sim 10^{-4}\Omega_{K0}$

RD

$$E^2(z) \sim (1+z)^4, \quad \Omega_K(z) \sim \Omega_{K0}/(1+z)^2$$

Problems

- Flatness Problem: Why does the universe appear so flat? not clearly open or closed
- Relics Problem: Why do we see no monopoles?
- Horizon Problem

The Universe looks the same everywhere in the sky that we look? The entire universe must have been at uniform temperature near beginning, There has not been enough time since the big bang for light to travel between two parts on opposite horizons

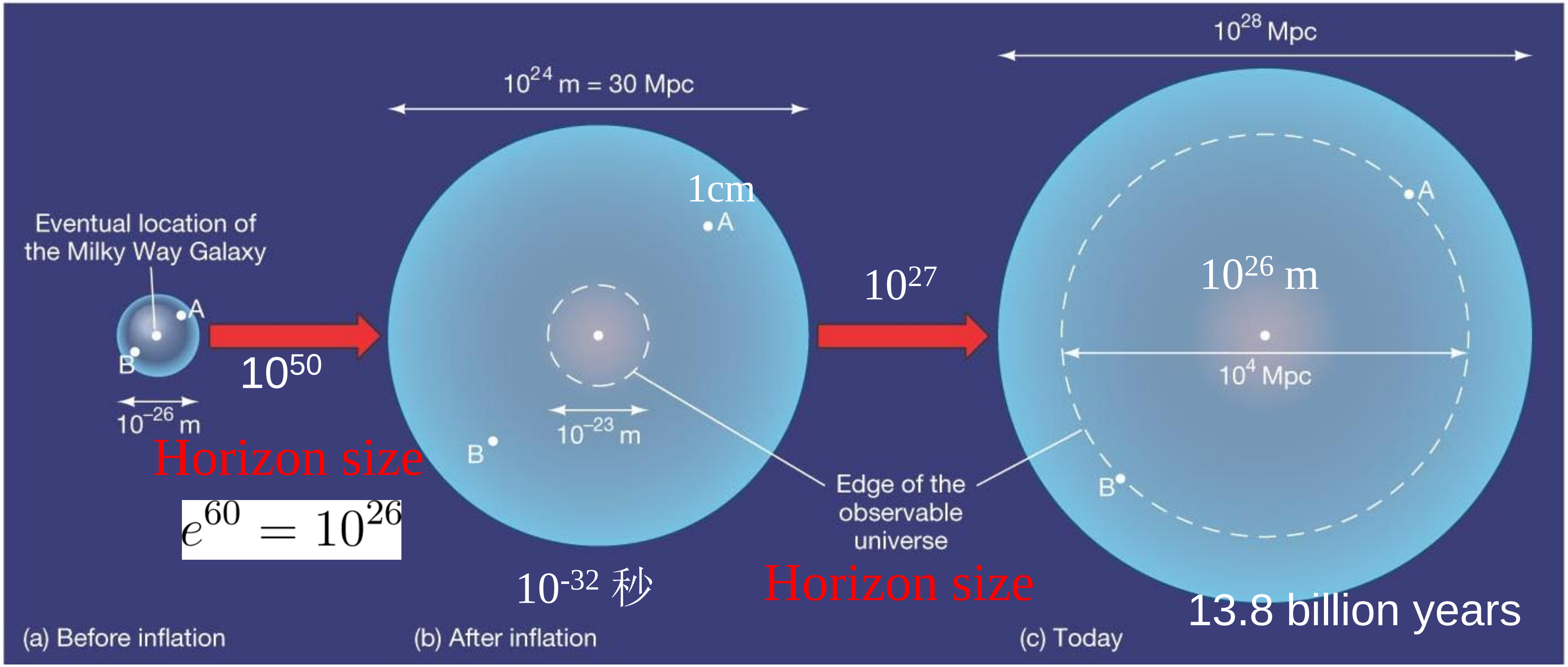
- Dark energy: repulsive force
- Dark matter

Inflation Theory

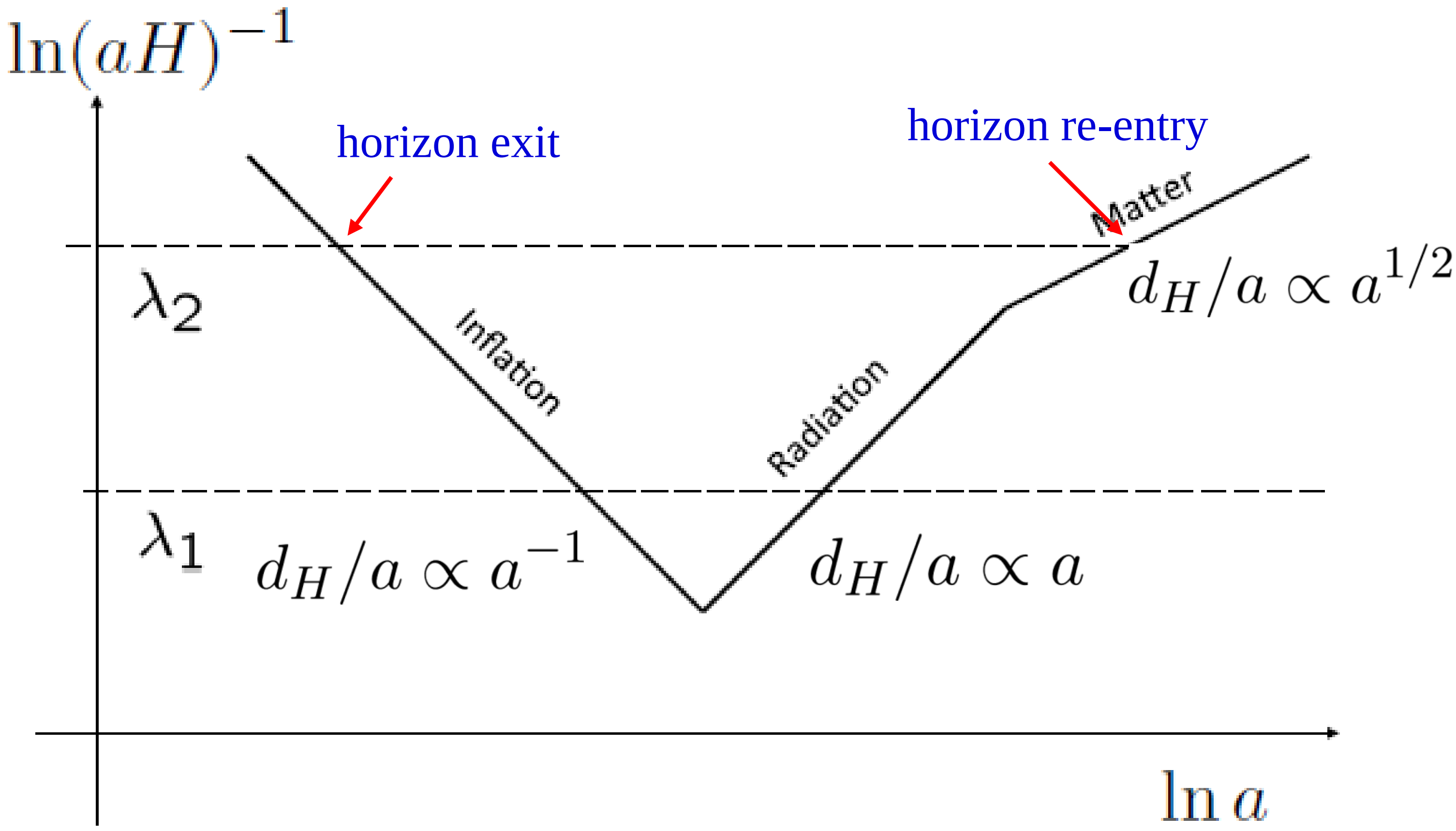
- Guth (1980s): The Universe expanded exponentially fast at very early time

Inflationary Solutions

- Horizon Problem: Observable universe was extremely small before inflation, all regions could be in causal contact



Inflationary solution



Inflationary Solution

- Flatness: Inflation pushes the Universe towards flatness (stretch away any unevenness)

  Longer inflation → Flatter Universe

- Relics: Inflation greatly dilutes any relics

  → We should not observe them today

Inflation

- Conditions

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\rho + 3p < 0$$

$$\ddot{a} > 0 \Longleftrightarrow \frac{d}{dt}\left(\frac{1}{aH}\right) < 0$$

- Inflation is equivalent to the decrease in co-moving Hubble horizon, and it can solve the problems in the standard cosmology

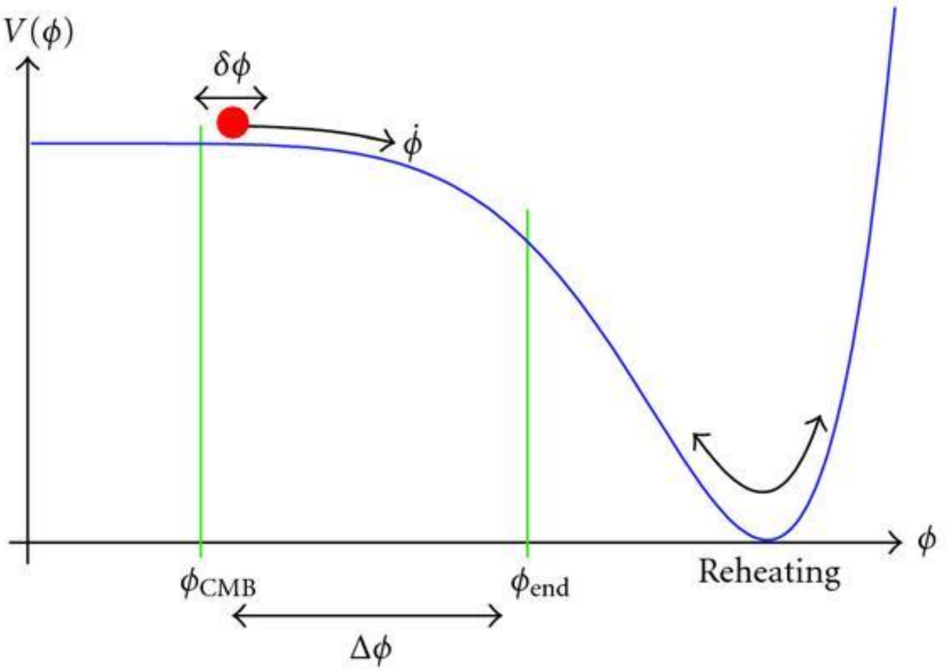
Inflationary Models

- Inflation: accelerated expansion, repulsive force
- Scalar field: if potential energy is bigger than kinetic energy, drives accelerated expansion
- Flat potential:

to get enough inflation

slow-roll inflation

slow-roll parameters



Scalar field

- Lagrangian $\mathscr{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$

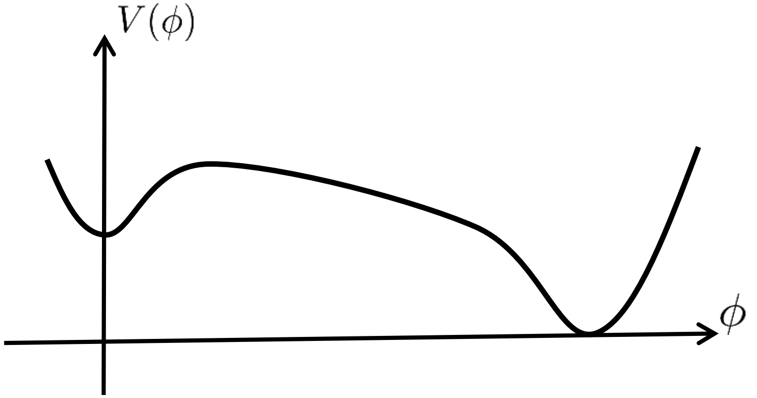
$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}\mathscr{L}_\phi) = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathscr{L}_\phi.$$

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)U_\mu U_\nu$$

$$\rho = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi), \quad p = \mathscr{L}_\phi$$

$$U_\mu = \partial_\mu\phi/\sqrt{-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}$$

$$\boxed{\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}}$$



Models with scalar fields

- Cosmological equations

$$H^2 = \frac{8\pi G}{3}\left(\cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi)\right)$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \Longleftrightarrow \quad \cancel{\ddot{\phi}} + 3H\dot{\phi} + V'(\phi) = 0$$

$$V'(\phi) = dV(\phi)/d\phi$$

- Slow-roll approximation

$$\dot{\phi}^2 \ll 2V(\phi) \qquad |\ddot{\phi}| \ll 3H|\dot{\phi}|$$

$$H^2 \simeq \frac{8\pi G}{3}V(\phi) \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \approx -V(\phi)$$

$$3H\dot{\phi} \simeq -V'(\phi) \qquad \rho + 3p \approx -2V(\phi) < 0$$

Slow-roll

- Slow-roll conditions $\dot\phi^2 \ll 2V(\phi)$ $|\ddot\phi| \ll 3H|\dot\phi|$

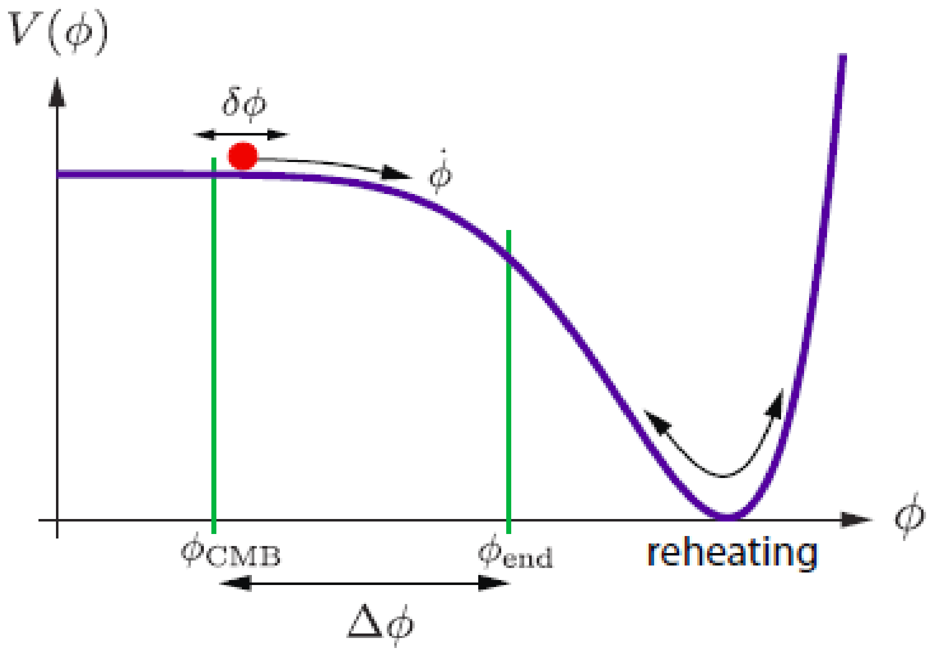
$$H^2 \simeq \frac{8\pi G}{3} V(\phi)$$

$$\bar\epsilon = \frac{\dot\phi^2}{2V(\phi)} = \frac{1}{48\pi G}\left(\frac{V'}{V}\right)^2 \ll 1$$

$$3H\dot\phi \simeq -V'(\phi)$$

$$|\bar\eta| = \left|\frac{\ddot\phi}{3H\dot\phi}\right|$$

$$= \frac{1}{24\pi G}\left|\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right| \ll$$



Slow-roll parameters

- Slow-roll $\dot{\phi}^2 \ll 2V(\phi) \quad |\ddot{\phi}| \ll 3H|\dot{\phi}|$

$$\bar{\epsilon} = \frac{\dot{\phi}^2}{2V(\phi)} = \frac{1}{48\pi G}\left(\frac{V'}{V}\right)^2 \ll 1 \qquad 3H\dot{\phi} \simeq -V'(\phi)$$

$$|\bar{\eta}| = \left|\frac{\ddot{\phi}}{3H\dot{\phi}}\right| = \frac{1}{24\pi G}\left|\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right| \ll 1$$

$$\epsilon = 3\bar{\epsilon} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \ll 1$$

$$\epsilon,\ \eta \sim 1$$

$$\eta = \frac{1}{8\pi G}\frac{V''}{V} \ll 1$$

End of inlation

$$\xi = \frac{1}{(8\pi G)^2}\frac{V'V'''}{V^2} \ll 1$$

Hubble flow parameters

- Hubble flow slow-roll parameters

$$\epsilon_H = \frac{1}{4\pi G}\left(\frac{H'}{H}\right)^2 = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V} = -\frac{\dot{H}}{H^2} \approx \epsilon \qquad H' = dH/d\phi$$

$$\eta_H = \frac{1}{4\pi G}\frac{H''}{H} = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\ddot{H}}{2H\dot{H}} \approx 3\bar{\eta} = \eta - \epsilon \qquad H'' = d^2H/d\phi^2$$

$$\xi_H = \frac{1}{(4\pi G)^2}\frac{H'H'''}{H^2} = \frac{\dddot{\phi}}{H^2\dot{\phi}} - \left(\frac{\ddot{\phi}}{H\dot{\phi}}\right)^2 = \frac{\dddot{H}}{2H^2\dot{H}} - 2\eta_H^2$$

$$^{(n)}\beta_H = \frac{1}{4\pi G}\left(\frac{(H')^{n-1}d^{n+1}H/d\phi^{n+1}}{H^n}\right)^{1/n}$$

PRD 50 (94) 7222

$$\epsilon = \left(\frac{1-\eta_H/3}{1-\epsilon_H/3}\right)^2 \epsilon_H \qquad \epsilon_H \approx \epsilon - \frac{4}{3}\epsilon^2 + \frac{2}{3}\epsilon\eta$$

$$\eta = \frac{\epsilon_H + \eta_H - \eta_H^2 - \xi_H}{1-\epsilon_H/3} \qquad \eta_H \approx \eta - \epsilon + \frac{8}{3}\epsilon^2 - \frac{8}{3}\epsilon\eta - \frac{1}{3}\eta^2 - \frac{1}{3}\xi$$

EOM of scalar field

- EOM

$$V(\phi) = 3\left(1 - \frac{1}{3}\epsilon_H\right) M_{pl}^2 H^2 \qquad M_{pl}^2 = 1/(8\pi G)$$

$$V'(\phi) = -3\left(1 - \frac{1}{3}\eta_H\right) H\dot{\phi}$$

$$\frac{\ddot{a}}{a} = H^2(\phi)[1 - \epsilon_H(\phi)]$$

- End of inflation

$$\ddot{a} > 0 \Longrightarrow \epsilon_H < 1$$

$$\epsilon_H = 1 \qquad \epsilon \sim 1, \quad |\bar{\eta}| \sim 1$$

Number of e-foldings

- Number of e-foldings before the end of inflation

$$N(t) = \int_t^{t_f} d\ln a(t) \qquad\qquad dN = -H dt$$

$$\frac{d\ln H}{dN} = \epsilon_H \approx \epsilon, \qquad \frac{d\ln \epsilon_H}{dN} = 2(\eta_H - \epsilon_H) \approx 2(\eta - \epsilon)$$

- parameters $\dot{\epsilon}_H = 2H\epsilon_H(\epsilon_H - \eta_H)$

$$\dot{\eta}_H = H(\epsilon_H \eta_H - \xi_H)$$

- The total number of e-foldings

$$N(\phi_e, \phi_i) = \ln[a(t_e)/a(t_i)] = \int_{t_i}^{t_e} H dt \approx -8\pi G \int_{\phi_i}^{\phi_e} \frac{V(\phi)}{V'(\phi)} d\phi$$

Number of e-foldings

- The number of e-foldings from horizon exit to the end of inflation $k = aH$

$$
\begin{aligned}
\frac{k}{a_0 H_0} &= \frac{a_* H_*}{a_0 H_0} = \frac{a_*}{a_e}\frac{a_e}{a_{reh}}\frac{a_{reh}}{a_0}\frac{H_*}{H_0} \\
&= e^{-N_*}\left(\frac{\rho_e}{\rho_{reh}}\right)^{-1/3}\left(\frac{\rho_{r0}}{\rho_{reh}}\right)^{1/4}\left(\frac{\rho_*}{\rho_{c0}}\right)^{1/2} \\
&= e^{-N_*}\left(\frac{\rho_{reh}^{1/4}}{\rho_e^{1/4}}\right)^{1/3}\left(\frac{\rho_*^{1/4}}{\rho_e^{1/4}}\right)\left(\frac{\rho_*^{1/4}}{10^{16}\mathrm{Gev}}\right)\left(\frac{10^{16}\mathrm{Gev}}{\rho_{c0}^{1/4}}\right)\left(\frac{\rho_{r0}^{1/4}}{\rho_{c0}^{1/4}}\right),
\end{aligned}
$$

$$
N_* = 60.86 - \ln h - \ln\frac{k}{a_0 H_0} - \frac{1}{3}\ln\frac{V_e^{1/4}}{\rho_{reh}^{1/4}} + \ln\frac{V_*^{1/4}}{V_e^{1/4}} - \ln\left(\frac{10^{16}\mathrm{Gev}}{V_*^{1/4}}\right)
$$

$$
\boxed{N_* \geq 50 - 60}
$$

Lyth Bound

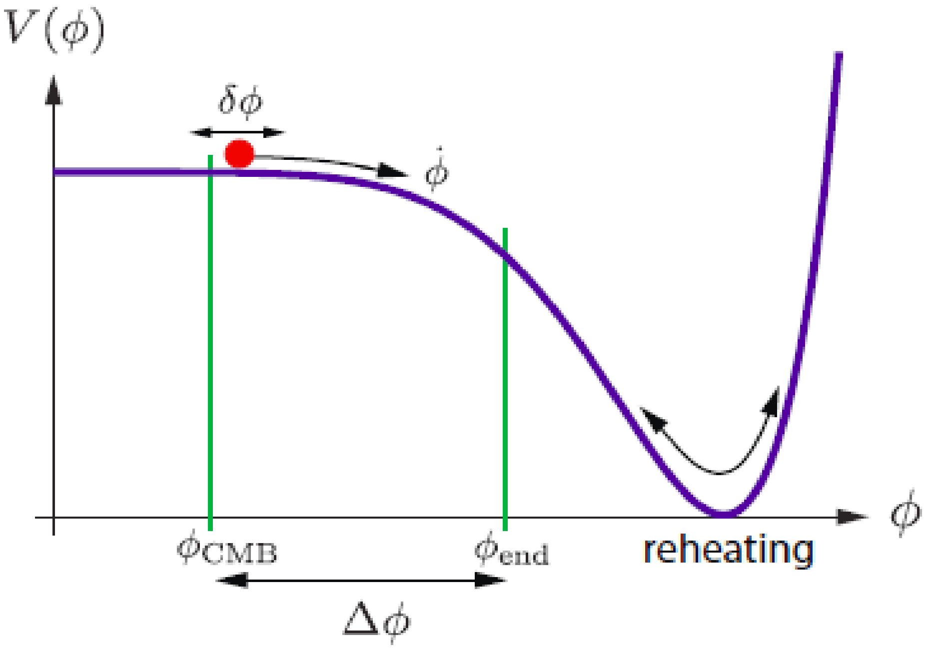
- Number of e-folds

$$N(t) = \int_t^{t_e} d\ln a(t) = \int_t^{t_e} H(t)dt = \int_\phi^{\phi_e} \frac{H}{\dot{\phi}} d\phi$$

$$N(\phi) = -\frac{1}{2M_{pl}^2}\int_\phi^{\phi_e} \frac{H}{H'} d\phi = \frac{1}{M_{pl}}\int_{\phi_e}^{\phi} \frac{1}{\sqrt{2\epsilon_H}} d\phi$$

$$\Delta\phi > N(\phi_*)\sqrt{2\epsilon(\phi_*)}M_{pl}$$

$$\boxed{\frac{\Delta\phi}{M_{pl}} > N(\phi_*)\sqrt{2\epsilon(\phi_*)}}$$



D.H. Lyth, PRL 78 (97) 1861

Attractors

- Hamilton-Jacobi Formulation

$$[H'(\phi)]^2 - \frac{3}{2M_{pl}^2}H^2(\phi) = -\frac{1}{2M_{pl}^4}V(\phi) \qquad \delta\phi = 0$$

$$H(\phi) = H_0(\phi) + \delta H(\phi)$$

$$H_0'\delta H' \approx \frac{3}{2M_{pl}^2}H_0\delta H$$

$$\delta H(\phi) = \delta H(\phi_i)\exp[-3N(\phi)],\ \ N(\phi) = \frac{1}{2M_{pl}^2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)}{H_0'(\phi)}d\phi$$

Salopek & Bond, PRD 42 (90) 3936
Liddle, Parsons & Barrow, PRD 50 (94) 7222

- Attractor: Whatever the initial conditions are, the scalar field will enter the slow-roll trajectories if the scalar field satisfies the slow-roll conditions.

Reheating

- After the end of inflation, all the energy of the universe is stored in the inflaton, and the temperature is extremely low.
- A process of energy transfer is needed to keep thermal equilibrium, and recovers the standard thermal history.

$$\dot{\rho} + (3H + \Gamma)\rho = 0$$ **Particle decay rate** Γ

$$T_{reh} \sim \sqrt{M_{pl}\Gamma}$$

$$\frac{T_{reh}}{1\ \mathrm{GeV}} \sim \left(\frac{m}{10^6\ \mathrm{GeV}}\right)^{3/2} \qquad m \gtrsim 10^4\ \mathrm{GeV}$$

Mode decomposition

- Helmholtz theorem $\vec{A}=\vec{B}+\vec{C}, \quad \vec{\nabla}\cdot\vec{C}=0$

$$\vec{B}=-\vec{\nabla}\phi, \quad \nabla\times\vec{B}=0$$

$$v_i = v_i^S + v_i^V \qquad \nabla\times v^S = 0 \qquad \nabla\cdot v^V = 0$$

$$v_i^S = -\frac{ik_i}{k}V \qquad k_i v_i^V = 0$$

$$\Pi_{ij} = \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T$$

$$\Pi_{ij}^S = \left(-\frac{k_i k_j}{k^2}+\frac{1}{3}\delta_{ij}\right)\Pi \qquad \Pi_i^i$$

$$\Pi_{ij}^V = -\frac{i}{2k}(k_i\Pi_j + k_j\Pi_i) \qquad k_i\Pi_i = 0$$

$$k_i\Pi_{ij}^T = 0$$

The metric tensor

- Vector mode: It decays when the universe expands
- Scalar mode

$$
\begin{aligned}
ds^2 =& a^2(\tau)\{-(1+2A)d\tau^2 + 2\nabla_i B d\tau dx^i \\
&+ [(1-2D)\gamma_{ij} + 2(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2)E]dx^i dx^j\},
\end{aligned}
$$

$$
{}^{(3)}R = \frac{4}{a^2}\nabla^2\left(D + \frac{1}{3}\nabla^2 E\right)
$$

$$
{}^{(3)}R = \frac{6K}{a^2}
$$

Background

Curvature perturbation

Degrees of freedom (4): $A,\ B,\ D,\ E$

Independent variable

- Variables: scalar mode

Degrees of freedom (5): $A, \ B, \ D, \ E, \ \delta\phi$

Coordinate transformation (2): $x^\mu \to \xi^\mu$

- Fix a gauge: 5-2=3

Three independent DOF

- Bianchi identity: 2

1 independent DOF left

Quantum fluctuations

- Scalar perturbations

$$ds^2 = a^2(\tau)\{-(1+2A)d\tau^2 + 2\nabla_i B d\tau dx^i \qquad d\tau = dt/a$$

$$+ [(1-2D)\gamma_{ij} + 2(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2)E]dx^i dx^j\},$$

- Action

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right]$$

$$\delta_2 S = \frac{1}{2}\int\left(v'^2 - \gamma^{ij}v_{,i}v_{,j} + \frac{z''}{z}v^2\right)d^3x d\tau, \quad v' = dv/d\tau$$

$$v = a[\delta\phi + (\phi_0'/\mathscr{H})D] = a[\delta\phi^{gi} + (\phi_0'/\mathscr{H})\Phi] = -z\mathscr{R}$$

$$\delta\phi^{gi} = \delta\phi + \phi_0'(B - E'), \quad \mathscr{R} = \Phi + \mathscr{H}\delta\phi^{gi}/\phi_0', \quad z = \frac{a\phi_0'}{\mathscr{H}}$$

ADM Foramlism

- ADM decomposition

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

$$g^{00} = -\frac{1}{N^2},\ \ g^{0i} = \frac{N^i}{N^2},\ \ g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2}$$

$$\gamma^{ik}\gamma_{kj} = \delta^i_j \quad N^i = \gamma^{ij} N_j$$

Lapse function N

Shift function N^i

Extrinsic curvature $K_{ij} = \frac{1}{2N}\left(\frac{\partial \gamma_{ij}}{\partial t} - \nabla_i N_j - \nabla_j N_i\right) \equiv \frac{E_{ij}}{N}$

Gravitational action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R = \frac{1}{16\pi G}\int dt d^3x N\sqrt{\gamma}\,[\,{}^{(3)}R + K_{ij}K^{ij} - (\gamma^{ij}K_{ij})^2],$$

Quantum fluctuations

- The action

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right]$$

$$S = \frac{1}{2}\int dt d^3x\sqrt{\gamma}\,\Big[N\,{}^{(3)}R - 2NV + N^{-1}(E_{ij}E^{ij} - E^2)$$
$$+N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - N\gamma^{ij}\partial_i\phi\partial_j\phi\Big]$$

$$M_{pl}^2 = 1$$

Hamiltonian constraint

$${}^{(3)}R - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - \gamma^{ij}\partial_i\phi\partial_j\phi = 0.$$

Momentum constraint

$$\nabla_i\left[N^{-1}(E^i_j - \delta^i_j E)\right] = N^{-1}(\dot{\phi} - N^i\phi_{,i})\phi_{,j} \qquad \phi_{,i} = \partial_i\phi$$

Background

- FRW metric $N=1, \quad N_i=0, \quad \gamma_{ij}=a^2\delta_{ij}$

$$E_{ij}=H\gamma_{ij}, \quad E^{ij}=H\gamma^{ij}, \quad E^{ij}E_{ij}=3H^2, \quad E=3H,$$

$$E_{ij}-\gamma_{ij}E=-2H\gamma_{ij}, \quad E^{ij}E_{ij}-E^2=-6H^2, \quad {}^{(3)}R=0.$$

- The action

$$S_0=\frac{1}{2}\int dtd^3x\, a^3\left(\dot{\phi}^2-2V-6H^2\right)$$

- EOM

$$\dot{H}=-\frac{1}{2}\dot{\phi}^2,$$

$$\ddot{\phi}+3H\dot{\phi}+\frac{dV}{d\phi}=0$$

- Hamiltonian constraint

$$6H^2=\dot{\phi}^2+2V$$

Quantum fluctuation

- Variables: scalar mode

  Degrees of freedom (5): $A,\ B,\ D,\ E,\ \delta\phi$

  Coordinate transformation (2): $x^\mu \to \xi^\mu$

- Fix a gauge: $E=0,\ \delta\phi=0$
- Gauge: uniform field gauge

$$N=1+N_1,\quad N^i=\psi_{,i}+N_T^i,\quad \gamma_{ij}=a^2(1+2\zeta)\delta_{ij},$$
$$\gamma^{ij}=a^{-2}(1-2\zeta)\delta_{ij},\quad N_i=a^2(\psi_{,i}+N_T^i),\quad \partial_i N_T^i=0$$

Three independent DOF $N_1,\ \psi,\ \zeta$

$A,\ B,\ D$

Quantum fluctuation

- Perturbations

To the 1st order $^{(3)}R = -\frac{4}{a^2}\nabla^2\zeta, \quad E_{ij} = H\gamma_{ij} + a^2\dot{\zeta}\delta_{ij} - a^2\psi_{,ij},$

$$E = 3H + 3\dot{\zeta} - \nabla^2\psi, \quad E^{ij} = H\gamma^{ij} + a^{-2}\dot{\zeta}\delta_{ij} - a^2\psi_{,ij},$$

$$E^{ij}E_{ij} - E^2 = -6H^2 - 12H\dot{\zeta} + 4H\nabla^2\psi,$$

$$E_{ij} - \gamma_{ij}E = -2Ha^2\delta_{ij} - 4a^2H\zeta\delta_{ij} - 2a^2\dot{\zeta}\delta_{ij} - a^{-2}(\psi_{,ij} - \delta_{ij}\nabla^2\psi).$$

Momentum constraint

$$\nabla_i\left[N^{-1}(E^i_j - \delta^i_j E)\right] = N^{-1}(\dot{\phi} - N^i\phi_{,i})\phi_{,j} \qquad \phi_{,i} = \partial_i\phi$$

$$\phi_{,i} = 0$$

First order approximation

- Constraints

Momentum constraint $H\partial_j N_1 = \partial_j \dot{\zeta}$ $\longrightarrow$ $N_1 = \dot{\zeta}/H$

Hamiltonian constraint $\nabla^2\psi + \frac{1}{a^2}\nabla^2\left(\frac{\zeta}{H}\right) - \frac{\dot{\phi}^2}{2H^2}\dot{\zeta} = 0$

$$\psi = -\frac{\zeta}{a^2 H} + \chi, \quad \nabla^2\chi = \frac{\dot{\phi}^2}{2H^2}\dot{\zeta}$$

$$\delta_1 S = \frac{1}{2}\int dt d^3x\, a^3 \left[3\zeta(\dot{\phi}^2 - 6H^2 - 2V) - 12H\dot{\zeta}\right].$$

Second order of approximation

- To the second order

$$\gamma_{ij} = a^2 e^{2\zeta}\delta_{ij} = a^2(1+2\zeta+2\zeta^2)\delta_{ij}, \quad \gamma^{ij} = a^{-2}e^{-2\zeta}\delta_{ij},$$

$$\sqrt{\gamma} = a^3 e^{3\zeta} = a^3\left(1+3\zeta+\frac{9}{2}\zeta^2\right), \quad {}^{(3)}R = \frac{1}{a^2}e^{-2\zeta}\left[-4\nabla^2\zeta - 2(\zeta_{,i})^2\right],$$

$$E_{ij} = a^2\left[(H+\dot{\zeta}+2H\zeta+2H\zeta^2+2\zeta\dot{\zeta}-\psi_{,k}\zeta_{,k})\delta_{ij} - (1+2\zeta)\psi_{,ij}\right],$$

$$E = 3(H+\dot{\zeta}) - \nabla^2\psi - 3\psi_{,k}\zeta_{,k},$$

$$E^{ij} = a^{-2}\left[(H+\dot{\zeta}-2H\zeta+2H\zeta^2-2\zeta\dot{\zeta}-\psi_{,k}\zeta_{,k})\delta_{ij} - (1-2\zeta)\psi_{,ij}\right],$$

$$E^{ij}E_{ij} - E^2 = -6H^2 - 12H\dot{\zeta} - 6\dot{\zeta}^2 + 4H\nabla^2\psi + 4(\dot{\zeta}-3H\zeta)\nabla^2\psi.$$

Second order approximation

- The action (to the second order)

$$\begin{aligned}\delta_2 S &= \frac{1}{2}\int dt d^3x \frac{\dot\phi^2}{H^2}\left[a^3\dot\zeta^2 - a(\zeta_{,i})^2\right] \\ &= \frac{1}{2}\int d\tau d^3x \frac{a^2\phi'^2}{\mathscr{H}^2}\left[\zeta'^2 - (\zeta_{,i})^2\right] \\ &= \frac{1}{2}\int d\tau d^3x \left[v'^2 - (v_{,i})^2 + \frac{z''}{z}v^2\right],\end{aligned}$$

Simple harmonic oscillators

$$v = a\phi'\zeta/\mathscr{H},\ \phi' = d\phi/d\tau,\ \mathscr{H} = d\ln a/d\tau$$

$$z = \frac{a\phi_0'}{\mathscr{H}}$$

$$\delta_2 S = \frac{1}{2}\int\left(v'^2 - \gamma^{ij}v_{,i}v_{,j} + \frac{z''}{z}v^2\right)d^3x d\tau, \quad v' = dv/d\tau$$

$$v = -z\mathscr{R}$$

Quantization

- Canonical quantization

Conjugate momentum $\pi(\tau,\vec{x}) = \delta L/\delta v' = v'(\tau,\vec{x})$

Hamiltonian

$$H = \int (v'\pi - L)\sqrt{\gamma}d^3x = \frac{1}{2}\int \left(\pi^2 + \gamma^{ij}v_{,i}v_{,j} - \frac{z''}{z}v^2\right)\sqrt{\gamma}d^3x$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{v}(\tau,\vec{x}')] = [\hat{\pi}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = 0,$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'),$$

$$\int \delta^{(3)}(x-x')\sqrt{\gamma}d^3x = 1$$

$$i\hat{v}' = [\hat{v},\ \hat{H}], \quad i\hat{\pi}' = [\hat{\pi},\ \hat{H}]$$

Quantization

- quantization

$$\hat{v}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}[v_k(\tau)a_k e^{i\vec{k}\cdot\vec{x}} + v_k^*(\tau)a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}]$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{v}(\tau,\vec{x}')] = [\hat{\pi}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = 0,$$

$$[\hat{v}(\tau,\vec{x}),\ \hat{\pi}(\tau,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'),$$

$$[a_k,\ a_{k'}] = [a_k^\dagger,\ a_{k'}^\dagger] = 0, \quad [a_k,\ a_{k'}^\dagger] = \delta^{(3)}(\vec{k}-\vec{k}')$$

- Bunch-Davies vaccuum $a_k|0\rangle = 0$

$$v_k^*\frac{dv_k}{d\tau} - v_k\frac{dv_k^*}{d\tau} = -i$$

Mukhanov-Sassaki Eq.

- Mode function $\boxed{v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0}$

Asymptotic solution

- EOM

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

- Well inside the horizon

$$v_k(\tau) \to \frac{1}{2k} e^{-ik\tau}, \quad k \to \infty \qquad\qquad v_k'' + k^2 v_k \approx 0$$

$$v_k^* \frac{dv_k}{d\tau} - v_k \frac{dv_k^*}{d\tau} = -i$$

- Superhorizon

$$v_k(\tau) \propto z, \quad k \to 0 \qquad\qquad v_k'' - \frac{z''}{z} v_k = 0$$

Comoving curvature perturbation $\mathscr{R} = \dfrac{v}{z}$ is a constant

The parameters

- Slow-roll expansion

$$z = \frac{a\phi_0'}{\mathscr{H}}$$

$$\frac{z'}{z} = aH\left(1 - \frac{\dot{H}}{H^2} + \frac{\ddot{H}}{2H\dot{H}}\right) = aH(1 + \epsilon_H - \eta_H)$$

$$\frac{z''}{z} = 2a^2H^2\left(1 + \epsilon_H - \frac{3}{2}\eta_H + \epsilon_H^2 + \frac{1}{2}\eta_H^2 - 2\epsilon_H\eta_H + \frac{1}{2}\xi_H\right)$$

$$a''/a = 2a^2H^2 - a^2H^2\epsilon_H$$

$$\mathscr{H} = \frac{a'}{a} = \frac{da/d\tau}{a} = aH$$

Quantum fluctuation

- First order $$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

$$\frac{z''}{z} = 2a^2H^2\left(1 + \epsilon_H - \frac{3}{2}\eta_H + \epsilon_H^2 + \frac{1}{2}\eta_H^2 - 2\epsilon_H\eta_H + \frac{1}{2}\xi_H\right)$$

$$\frac{d}{d\tau}\left(\frac{1}{aH}\right) = -1 + \epsilon_H \qquad aH = -1/[(1-\epsilon_H)\tau]$$

$$v_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right) v_k = 0 \qquad \nu = 3/2 + 2\epsilon_H - \eta_H \approx \text{常数}$$

$$v_k(\tau) = \sqrt{-\tau}[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)]$$

$$c_2(k) = 0$$

Boundary condition $$v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2}\sqrt{-\tau}H_\nu^{(1)}(-k\tau)$$

Quantum fluctuations

- Super-horizon

$$H_\nu^{(1)}(x \ll 1) \sim \sqrt{\frac{2}{\pi}} e^{-i\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} x^{-\nu}$$

$$v_k(\tau) = e^{i(\nu-1/2)\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{1/2-\nu} \propto z$$

- Co-moving curvature perturbation on super-horizon

$$|\mathscr{R}_k| = \left|\frac{v_k}{z}\right| = \left|\frac{H}{\dot{\phi}_0} \frac{v_k}{a}\right| = \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{H}{\dot{\phi}_0} \frac{H}{\sqrt{2k^3}} \left(\frac{k}{2aH}\right)^{3/2-\nu}, \quad k < aH.$$

$$aH = -1/[(1-\epsilon_H)\tau]$$

Quantum fluctuation

- Power spectrum

$$\hat{v}_k = v_k a_k + v_k^* a_k^\dagger$$

$$\boxed{\langle \hat{v}_{k_1} \hat{v}_{k_2}^* \rangle = v_{k_1} v_{k_2}^* \langle 0 | a_{k_1} a_{k_2}^\dagger | 0 \rangle} = v_{k_1} v_{k_2}^* \langle 0 | [a_{k_1}, \; a_{k_2}^\dagger] | 0 \rangle$$
$$= |v_{k_1}|^2 \delta^{(3)}(\vec{k}_1 - \vec{k}_2),$$

Why need quantization?

$$\langle \mathscr{R}_{k_1} \mathscr{R}_{k_2}^* \rangle = \langle \hat{v}_{k_1} \hat{v}_{k_2}^* \rangle / z^2 = \left| \frac{v_{k_1}}{z} \right|^2 \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$$
$$= (2\pi^2/k^3) \delta^3(\vec{k}_1 - \vec{k}_2) \mathscr{P}_{\mathscr{R}}(k_1)$$

$$|\mathscr{R}_k| = \left| \frac{v_k}{z} \right|$$

Parameterization

$$\mathscr{P}_{\mathscr{R}} = \frac{k^3}{2\pi^2} |\mathscr{R}_k|^2 = A_{\mathscr{R}}(k_*) \left(\frac{k}{k_*} \right)^{n_s - 1 + \frac{1}{2} n_s' \ln(k/k_*) + \cdots}$$

Red tilt $n_s - 1 < 0$

Blue tilt $n_s - 1 > 0$

Pivotal scale

Primordial power spectrum

- Power spectrum

$$|\mathscr{R}_k| = \left|\frac{v_k}{z}\right| = \left|\frac{H}{\dot{\phi}_0}\frac{v_k}{a}\right| = \frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{H}{\dot{\phi}_0}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{2aH}\right)^{3/2-\nu}, \quad k < aH.$$

$$\mathscr{P}_{\mathscr{R}} = \frac{k^3}{2\pi^2}|\mathscr{R}_k|^2 = 2^{2\nu-3}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2\left(\frac{H}{\dot{\phi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}\Bigg|_{k=aH}.$$

$$\nu = 3/2 + 2\epsilon_H - \eta_H$$

Horizon exit

$$\mathscr{P}_{\mathscr{R}} \approx [1 + 2(2 - \ln 2 - \gamma)(2\epsilon_H - \eta_H) - 2\epsilon_H]\left(\frac{H}{\dot{\phi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2$$

$$\mathscr{P}_{\mathscr{R}} = A_{\mathscr{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1+\frac{1}{2}n_s'\ln(k/k_*)+\cdots}$$

Spectral tilt

- The power spectrum

$$\mathscr{P}_{\mathscr{R}} = 2^{2\nu-3}\left(\frac{\Gamma(\nu)}{\Gamma(3/2)}\right)^2\left(\frac{H}{\dot{\phi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}\Bigg|_{k=aH}.$$

$$\mathscr{P}_{\mathscr{R}} \approx [1+2(2-\ln 2-\gamma)(2\epsilon_H-\eta_H)-2\epsilon_H]\left(\frac{H}{\dot{\phi}_0}\right)^2\left(\frac{H}{2\pi}\right)^2$$

- Spectral index

$$\nu = 3/2 + 2\epsilon_H - \eta_H$$

$$n_s - 1 = \frac{d\ln \mathscr{P}_{\mathscr{R}}}{d\ln k}\Bigg|_{k=aH} = 2\eta_H - 4\epsilon_H \approx 2\eta - 6\epsilon$$

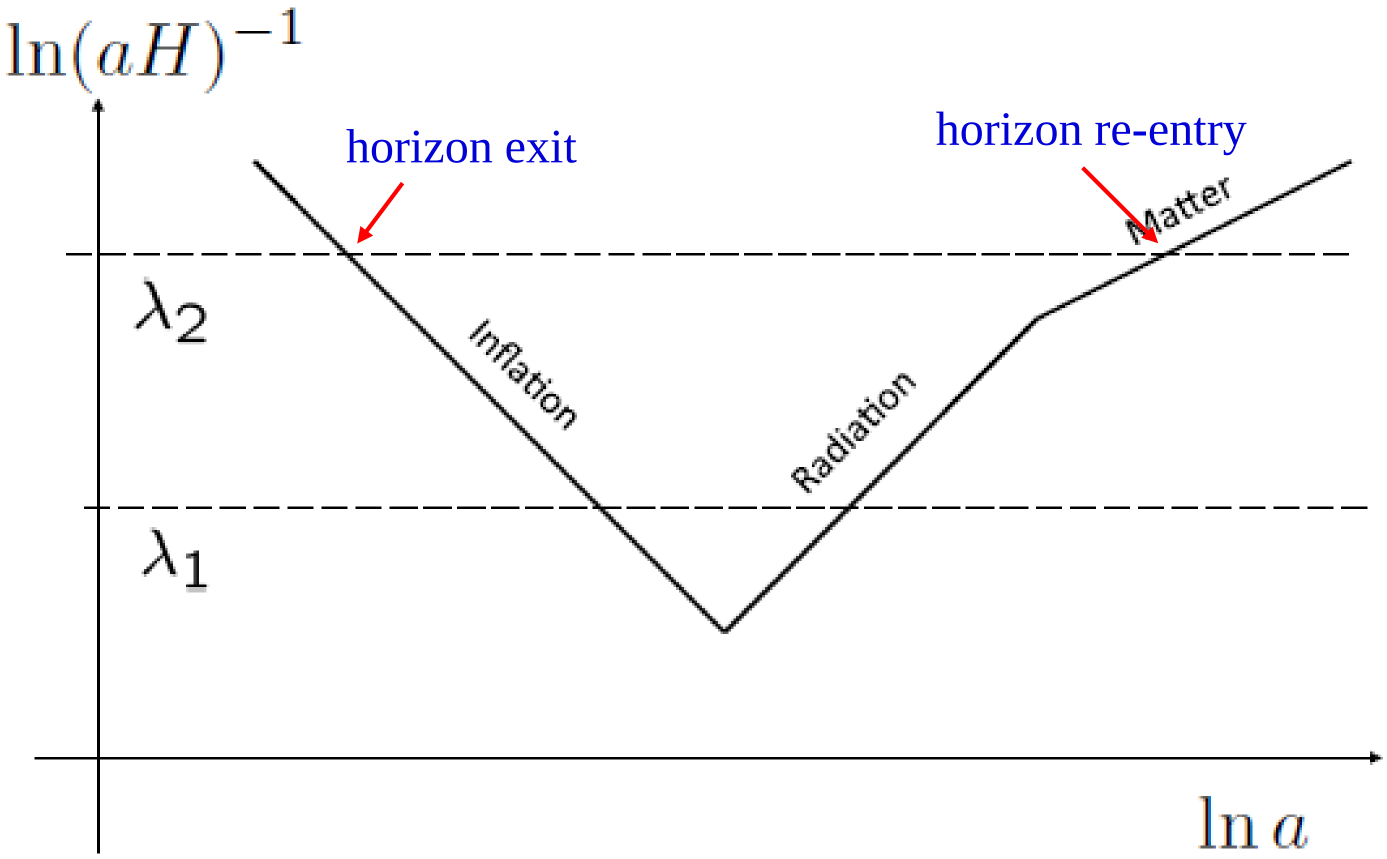
Note: all the values are evaluated at the horizon exit

$$n_s' = \frac{dn_s}{d\ln k}\Bigg|_{k=aH} = 10\epsilon_H\eta_H - 8\epsilon_H^2 - 2\xi_H$$

$$d\ln k = (1-\epsilon_H)H\,dt$$

$$\dot{\epsilon}_H = 2H\epsilon_H(\epsilon_H - \eta_H) \qquad \dot{\eta}_H = H(\epsilon_H\eta_H - \xi_H)$$

Amplification of Quantum Fluctuation



Tensor perturbations

- GWs $$ds^2 = a^2[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j],$$

$$h_{ii} = 0,\ \partial_i h_{ij} = 0$$

$$N = a,\ \gamma_{ij} = a^2(\delta_{ij} + h_{ij}),\ N_i = 0,\ E_{ij} = \gamma'_{ij}/2$$

- To the second order

$$\sqrt{\gamma} = a^3\left(1 - \frac{1}{4}h_{ij}h_{ij}\right),$$

$$E_{ij}E^{ij} - (\gamma^{ij}E_{ij})^2 = \frac{1}{4}(h'_{ij})^2 + \frac{a'}{a}(h_{ij}h_{ij})' - 6\left(\frac{a'}{a}\right)^2,$$

$${}^{(3)}R = a^{-2}\left[-\frac{1}{4}(\partial_k h_{ij})^2 + \partial_k(h_{ij}\partial_k h_{ij}) + \frac{1}{2}\partial_j(h_{ik}\partial_i h_{kj}) - \partial_j(h_{ik}\partial_k h_{ij})\right]$$

Quantum fluctuation of GWs

- The action to the second order

$$\delta_2 S = \frac{1}{64\pi G}\int d\tau d^3x[(h'_{ij})^2 - (\partial_l h_{ij})^2]a^2$$

$$\hat{h}_{ij}(x,\tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{s=+,\times} [\epsilon^s_{ij}(k)h^s_k(\tau)a_k e^{i\vec{k}\cdot\vec{x}} + (\epsilon^s_{ij}(k)h^s_k(\tau))^* a^\dagger_k e^{-i\vec{k}\cdot\vec{x}}]$$

$$\epsilon_{ii} = k^i\epsilon_{ij} = 0 \qquad \epsilon^s_{ij}\epsilon^{s'}_{ij} = 2\delta_{ss'}$$

$$u^s_k(\tau) = \frac{a}{\sqrt{16\pi G}} h^s_k(\tau)$$

$$\delta_2 S = \sum_s \frac{1}{2}\int d\tau d^3k \left[\left(\frac{du^s_k}{d\tau}\right)^2 - \left(k^2 - \frac{a''}{a}\right)(u^s_k)^2\right]$$

Quantum fluctuation of GWs

- Mode function

$$\frac{d^2u_k^s}{d\tau^2}+\left(k^2-\frac{a''}{a}\right)u_k^s=\frac{d^2u_k^s}{d\tau^2}+\left(k^2-\frac{\mu^2-1/4}{\tau^2}\right)u_k^s=0$$

$$\mu=3/2+\epsilon_H \qquad a''/a=2a^2H^2-a^2H^2\epsilon_H$$

- Asymptotic condition

$$u_k^s(\tau)=\frac{\sqrt{\pi}}{2}e^{i(\mu+1/2)\pi/2}\sqrt{-\tau}H_\mu^{(1)}(-k\tau)$$

- Perturbations on super-horizon

$$u_k^s(\tau)=e^{i(\mu-1/2)\pi/2}2^{\mu-3/2}\frac{\Gamma(\mu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{1/2-\mu}$$

Quantum fluctuation of GWs

- The power spectrum

$$\mathscr{P}_T = \frac{k^3}{\pi^2}\sum_{s=+,\times}\left|\frac{2\sqrt{8\pi G}\,u_k^s}{a}\right|^2 = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T+\frac{1}{2}n_T'\ln(k/k_*)+\cdots}$$

$$= (64\pi G)2^{2\mu-3}\left(\frac{\Gamma(\mu)}{\Gamma(3/2)}\right)^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\mu}.$$

$$\mu = 3/2 + \epsilon_H$$

$$\mathscr{P}_T \approx 64\pi G[1+(1-\ln 2-\gamma)\epsilon_H]\left(\frac{H}{2\pi}\right)^2$$

$$\mathscr{P}_T = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_t+\frac{1}{2}n_t'\ln(k/k_*)+\cdots}$$

The tensor spectral tilt

- The spectral index of tensor mode

$$n_T = \frac{d\ln \mathscr{P}_T}{d\ln k} = 3 - 2\mu = -2\epsilon_H$$

- The tensor to scalar ratio

$$\mathscr{P}_{\mathscr{R}} \approx [1 + 2(2 - \ln 2 - \gamma)(2\epsilon_H - \eta_H) - 2\epsilon_H] \left(\frac{H}{\dot\phi_0}\right)^2 \left(\frac{H}{2\pi}\right)^2$$

$$\mathscr{P}_T \approx 64\pi G[1 + (1 - \ln 2 - \gamma)\epsilon_H] \left(\frac{H}{2\pi}\right)^2$$

$$r = \frac{\mathscr{P}_T}{\mathscr{P}_{\mathscr{R}}} = 16\epsilon_H = 16\epsilon = -8n_T \qquad \epsilon_H = -\frac{\dot H}{H^2} = 4\pi G \left(\frac{\dot\phi_0}{H}\right)^2$$

The tensor perturbations

- Tensor mode $ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij}^T)dx^i dx^j]$

$$h_{ij}^{T\prime\prime} + 2\mathscr{H} h_{ij}^{T\prime} + k^2 h_{ij}^T = 16\pi G a^2 P \Pi_{ij}^T$$

- GWs

$$h_{ij} = h_k^+ \epsilon_{ij}^+ + h_k^\times \epsilon_{ij}^\times$$

$$\frac{\partial^2 h_{ij}}{\partial t^2} + 3H\frac{\partial h_{ij}}{\partial t} + \left(\frac{k}{a}\right)^2 h_{ij} = 0 \qquad \Pi_{ij}^T \approx 0$$

Damped oscillations

- RD

$$h_k^s = j_0(k\tau) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}(k\tau)}{(k\tau)^{1/2}} \qquad j_0(x) = \sin(x)/x$$

- MD

$$h_k^s = \frac{3j_1(k\tau)}{k\tau} = 3\sqrt{\frac{\pi}{2}}\frac{J_{3/2}(k\tau)}{(k\tau)^{3/2}} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Fitting formula

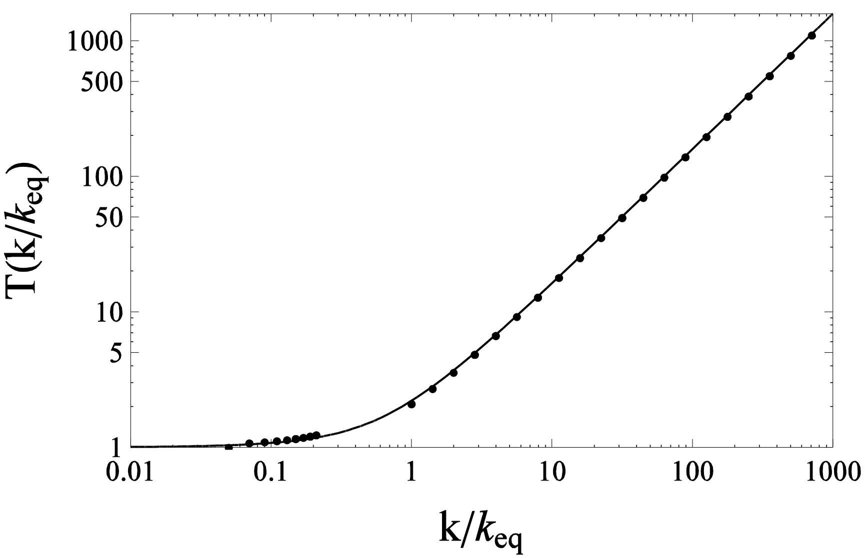
- GWs at present

astro-ph/9306029

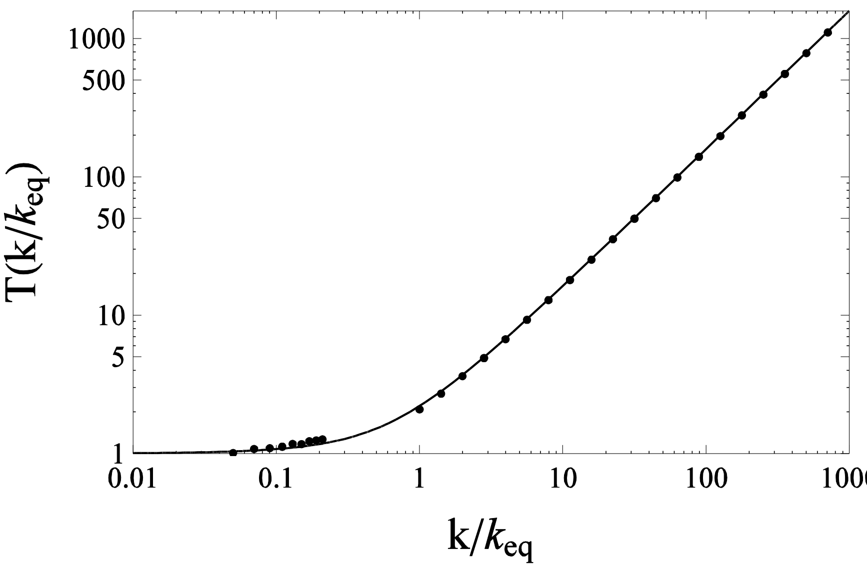
$$h_k^s = h_k^s(0)T(k/k_{eq})\frac{3j_1(k\tau)}{k\tau}$$

$$h_k^s = j_0(k\tau) \quad \textbf{RD}$$

$$T(y) = \sqrt{1+\frac{4}{3}y+\frac{5}{3}y^2}, \quad y = k/k_{eq}$$



$h = 0.4$



$h = 0.6727$

Gravitational waves

- GWs

$$\rho_{GW} = \frac{1}{32\pi G}\langle \nabla_t h_{ij}^{(1)} \nabla_t h^{ij}_{(1)} \rangle = \frac{1}{32\pi G}\langle (\dot{h}_{ij})^2 \rangle$$
$$= \frac{M_{pl}^2}{4a^2}\langle (h'_{ij})^2 \rangle$$
$$= \frac{M_{pl}^2}{2}\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{a^2}\sum_{s=+,\times} |h_k^s|^2$$
$$= \frac{M_{pl}^2}{4\pi^2}\int dk \frac{k^4}{a^2}\sum_{s=+,\times} |h_k^s|^2,$$

$$M_{pl}^2 = \frac{1}{8\pi G}$$

$$h_{ij}^{(1)} = a^2 h_{ij}$$

$$h^{ij}_{(1)} = a^{-2} h_{ij}$$

The energy of GWs

- The energy density

$$\frac{d\rho_{GW}}{d\ln k} = \frac{M_{pl}^2}{4\pi^2 a^2} k^5 \sum_{s=+,\times} |h_k^s|^2$$
$$= \frac{M_{pl}^2}{4}\left(\frac{k}{a}\right)^2 \mathscr{P}_T$$

$$\Omega_{GW} = \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln k} = \frac{1}{3M_{pl}^2 H^2}\frac{M_{pl}^2}{4}\left(\frac{k}{a}\right)^2 \mathscr{P}_T$$
$$= \frac{1}{12}\left(\frac{k}{aH}\right)^2 \mathscr{P}_T$$

$$\mathscr{P}_T = A_T k^{n_T} |T(k)|^2 \left(\frac{3j_1(k\tau)}{k\tau}\right)^2$$

The energy of primordial GWs

- Energy density $\mathscr{P}_T = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T}|T(k)|^2\left(\frac{3j_1(k\tau)}{k\tau}\right)^2$

$$\langle\cos^2(k\tau)\rangle = 1/2 \qquad = \frac{9}{2}A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T}|T(k)|^2\frac{1}{k^4\tau^4}$$

$$\Omega_{GW} = \frac{3}{8}A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T}|T(k)|^2\frac{1}{a^2(H\tau)^2(k\tau)^2}$$

- At present $\tau = \tau_0,\ a_0 = 1,\ H_0\tau_0 = 2$

$$\Omega_{GW} = \frac{3}{32}A_T(k_*)|T(k/k_{eq})|^2\left(\frac{k}{k_*}\right)^{n_T}(k\tau_0)^{-2}$$

$$= \frac{1}{16\pi^2}\frac{V_*}{M_{pl}^4}|T(k/k_{eq})|^2\left(\frac{k}{k_*}\right)^{n_T}(k\tau_0)^{-2}$$

M.S. Turner, M. White, J.E. Lidsy, PRD 48 (93) 4613

The energy of primordial GWs

- spectrum $$T(y) = \sqrt{1 + \frac{4}{3}y + \frac{5}{3}y^2}, \quad y = k/k_{eq}$$

Low frequency $y \ll 1, \quad T(y) \approx 1$

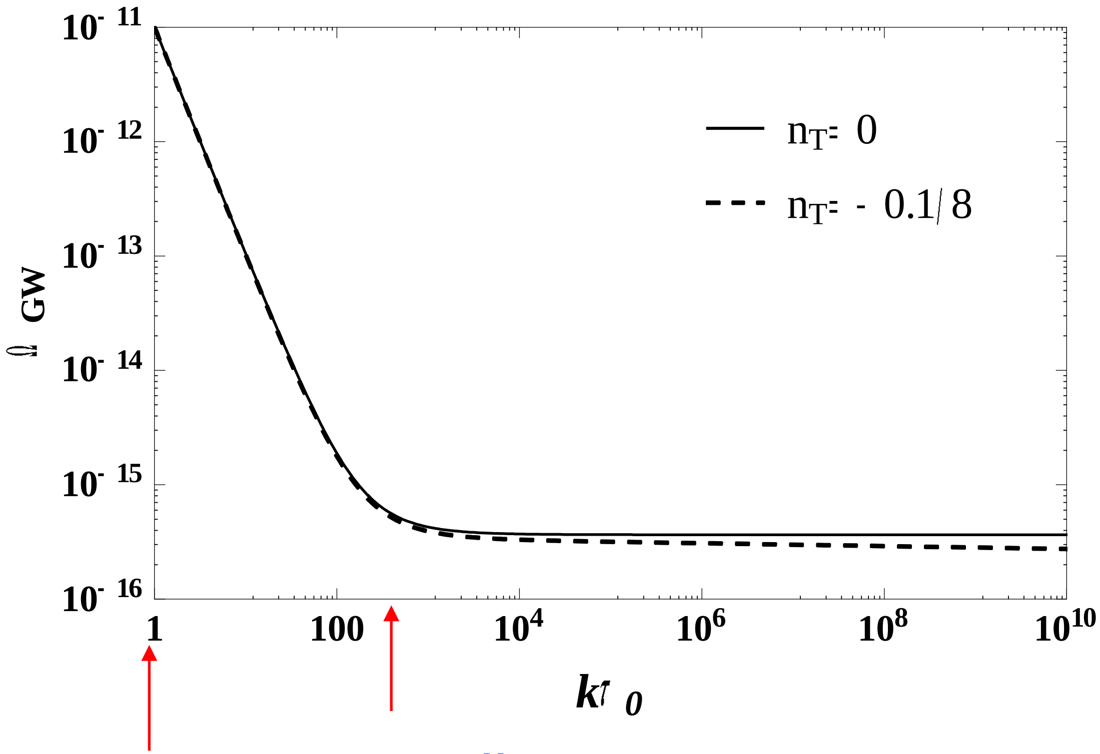
$$\Omega_{GW} \propto k^{n_T - 2} \propto f^{n_T - 2}$$

High frequency $y \gg 1, \quad T(y) \propto y$

$$\Omega_{GW} \propto k^{n_T} \propto f^{n_T}$$

$$\Omega_{GW} = \frac{3}{32} A_T(k_*) |T(k/k_{eq})|^2 \left(\frac{k}{k_*}\right)^{n_T} (k\tau_0)^{-2}$$
$$= \frac{1}{16\pi^2} \frac{V_*}{M_{pl}^4} |T(k/k_{eq})|^2 \left(\frac{k}{k_*}\right)^{n_T} (k\tau_0)^{-2}$$

The spectrum



arXiv: 1502.02114
1502.01589

$$\Omega_{m0} = 0.3156$$

$$H_0 = 67.27 \text{ km/s/Mpc}$$

$$T_{\gamma 0} = 2.725 \text{ K}$$

$$r = 0.1$$

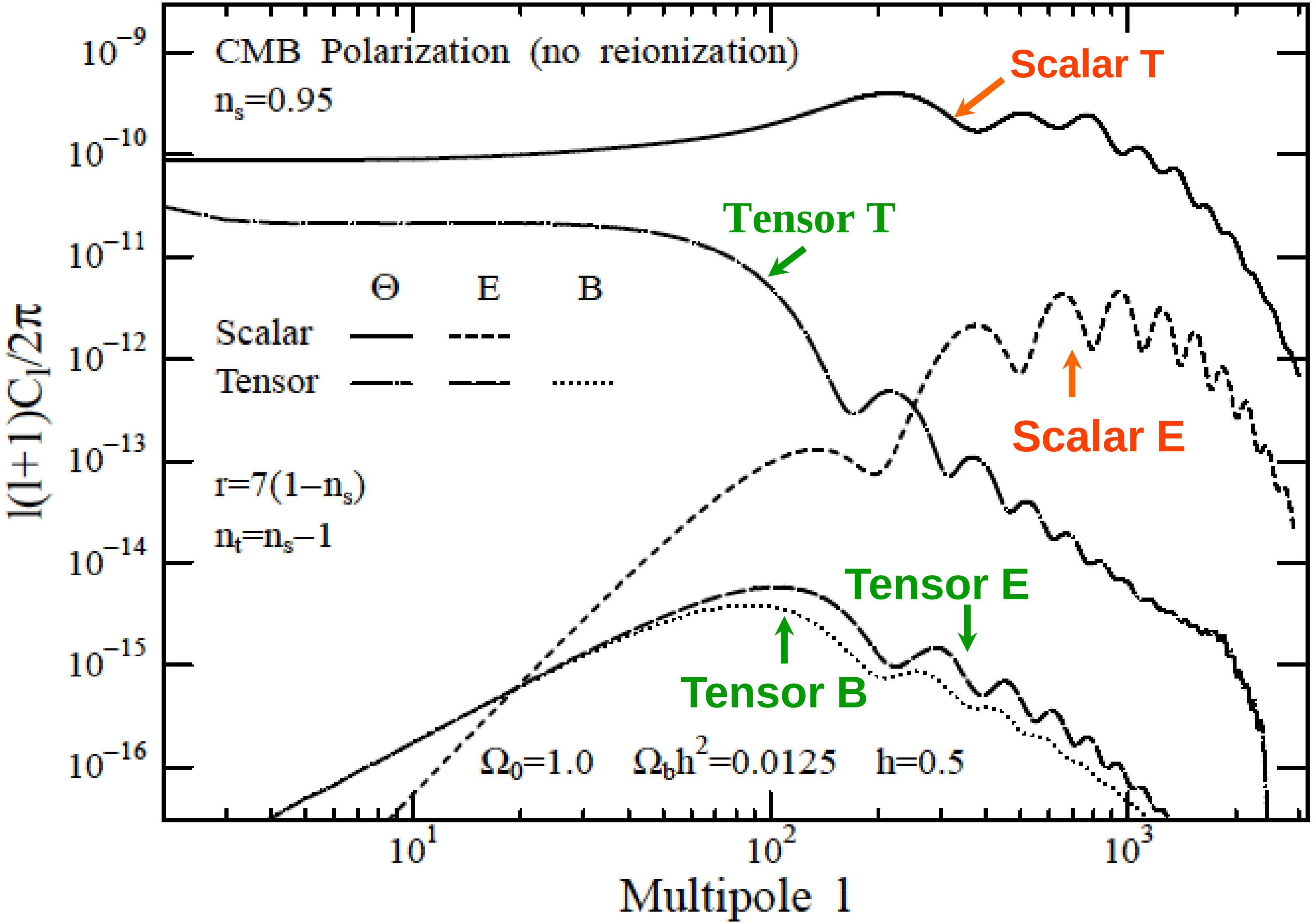
$$A_s = 2.2 \times 10^{-9}$$

Size of universe
14161.5Mpc
$2.2*10^{-18}$ Hz

Equality
112.1Mpc
$1.0*10^{-16}$ Hz

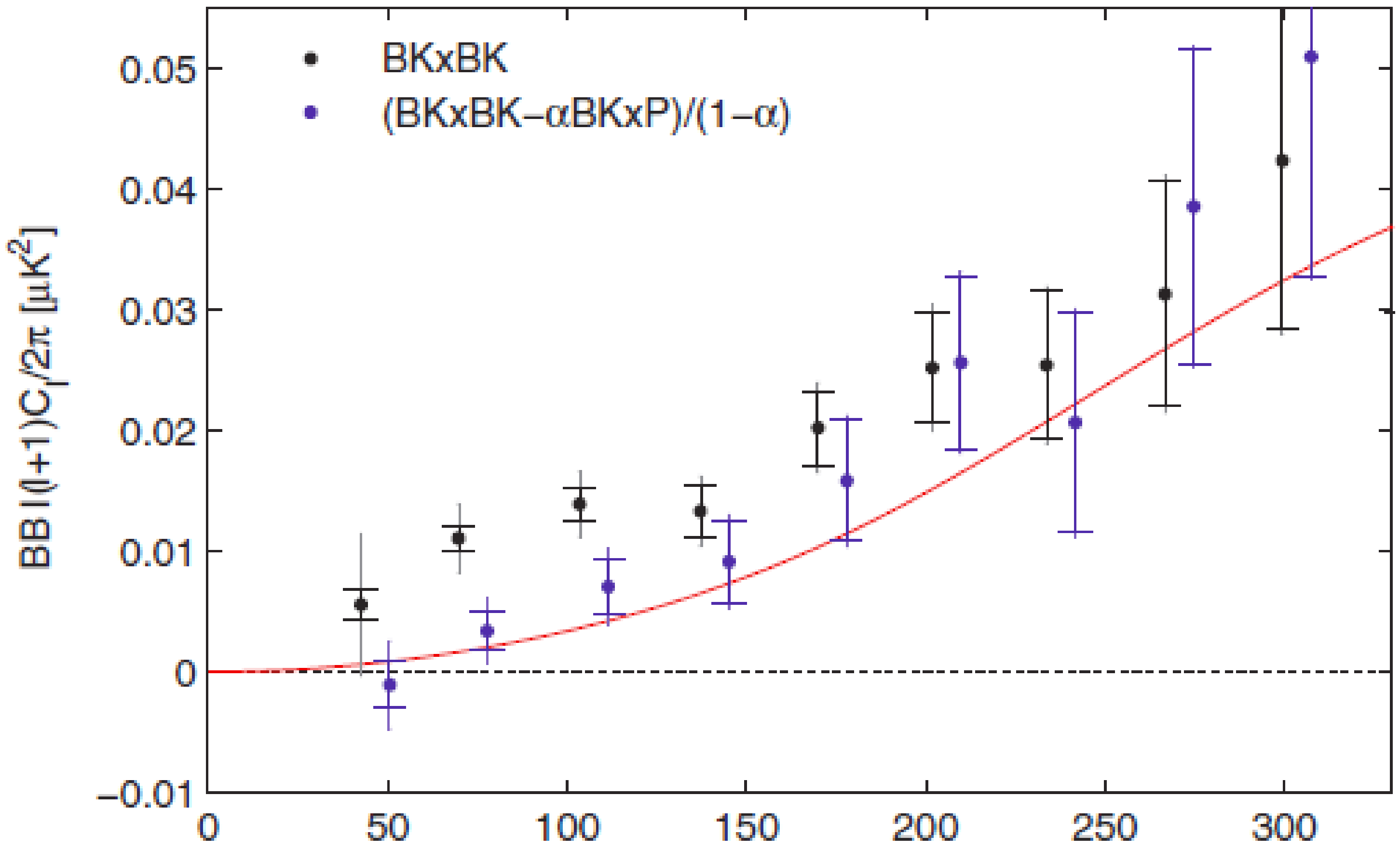
End of inflation~10^{15}GeV
$8.9*10^{7}$Hz

Power Spectrum



Detection of B-mode polarization

- Planck 2015+ BICEP2



The parameterization

- SR parameters

$$\epsilon_H = \frac{1}{4\pi G}\left(\frac{H'}{H}\right)^2 = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V} = -\frac{\dot{H}}{H^2} \approx \epsilon$$

$$\eta_H = \frac{1}{4\pi G}\frac{H''}{H} = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\ddot{H}}{2H\dot{H}} \approx 3\bar{\eta} = \eta - \epsilon$$

$$\xi_H = \frac{1}{(4\pi G)^2}\frac{H'H'''}{H^2} = \frac{\dddot{\phi}}{H^2\dot{\phi}} - \left(\frac{\ddot{\phi}}{H\dot{\phi}}\right)^2 = \frac{\dddot{H}}{2H^2\dot{H}} - 2\eta_H^2$$

$$n_s - 1 = \frac{d\ln \mathscr{P}_{\mathscr{R}}}{d\ln k}\Big|_{k=aH} = 3 - 2\nu = 2\eta_H - 4\epsilon_H \approx 2\eta - 6\epsilon$$

$$n_s' = \frac{dn_s}{d\ln k}\Big|_{k=aH} = 10\epsilon_H\eta_H - 8\epsilon_H^2 - 2\xi_H \approx 16\epsilon\eta - 24\epsilon^2 - 2\xi$$

$$n_T = \frac{d\ln \mathscr{P}_T}{d\ln k} = 3 - 2\mu = -2\epsilon_H \quad r = \frac{\mathscr{P}_T}{\mathscr{P}_{\mathscr{R}}} = 16\epsilon_H = 16\epsilon = -8n_T$$

Spectral index

- To second order

Stewart & Lyth, PLB 302 (93) 171;
Stewart & Gong,PLB 510 (01) 1;
Lidsey etal, Rev. Mod. Phys. 69 (97) 373
S. Habib etal, PRL 89 (02) 281301

$$n_s - 1 = 2\eta_H - 4\epsilon_H - 8(1+C)\epsilon_H^2 + 2(3+5C)\epsilon_H\eta_H - 2C\xi_H$$
$$= -6\epsilon + 2\eta - 2(12C + 5/3)\epsilon^2 + 2\eta^2/3 + 2(8C-1)\epsilon\eta$$
$$+ 2(1/3 - C)\xi \qquad C = -2 + \ln 2 + \gamma$$

$$n_s' = 10\epsilon_H\eta_H - 8\epsilon_H^2 - 2\xi_H \approx 16\epsilon\eta - 24\epsilon^2 - 2\xi$$
$$n_T = -2\epsilon_H[1 + (3+2C)\epsilon_H - 2(1+C)\eta_H]$$
$$= -2\epsilon[1 + (4C + 11/3)\epsilon - 2(C + 2/3)\eta]$$

$$r = 16\epsilon_H[1 + 2C(\epsilon_H - \eta_H)] = 16\epsilon[1 + 2(C - 1/3)(2\epsilon - \eta)]$$

Planck2015 Results

- Planck

$$n_s = 0.968 \pm 0.006, \quad n_s' = -0.0085 \pm 0.0076$$

$$r_{0.002} < 0.11, \quad 95\%$$

arXiv: 1502.02114

- Planck+BICEP2

$$\ln(10^{10} A_s) = 3.062 \pm 0.029$$

$$r_{0.002} < 0.10, \quad 95\%$$

PRL114 (15) 101301

$$n_s - 1 = 2\eta_H - 4\epsilon_H \approx 2\eta - 6\epsilon$$

$$n_s' = 10\epsilon_H \eta_H - 8\epsilon_H^2 - 2\xi_H \approx 16\epsilon\eta - 24\epsilon^2 - 2\xi$$

$$r = 16\epsilon = -8n_T$$

$$\frac{\Delta\phi}{M_{pl}} > N(\phi_*)\sqrt{2\epsilon(\phi_*)}$$

$$\boxed{N_* \geq 50 - 60}$$

Summary

- Given potential, calculate the slow-roll parameters
- Calculate ϕ_e by requiring $\mathrm{Max}(\epsilon,\ \eta) \sim 1$
- Calculate ϕ_*

$$N_* = \int_{\phi_e}^{\phi_*} \frac{1}{\sqrt{2\epsilon(\phi)}} d\phi = 50 - 60$$

- Use the value of ϕ_* to calculate the slow-roll parameters at the horizon exit
- Use the slow-roll parameters at the horizon exit to calculate the observables

Power-law inflation

- Exponential potential (exact solution)

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right)$$

$$a(t) = a_0 t^p \qquad \epsilon = 1/3p, \;\; \eta = 2/p$$

$$\phi = \sqrt{2p}\, M_{pl} \ln\left(\sqrt{\frac{V_0}{p(3p-1)}}\frac{t}{M_{pl}}\right)$$

$$n_s = 1 - 2/p \qquad r = 16/p = 8(1-n_s)$$

- de-Sitter universe

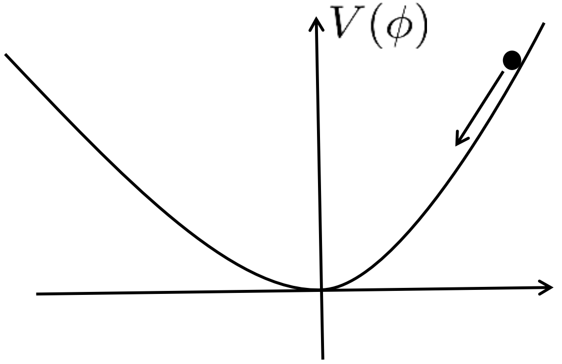
$$\epsilon_H = \eta_H = 0 \qquad \nu = 3/2$$

Scale invariant spectrum $n_s = 1$

The chaotic inflation

Power-law potential

$$V(\phi) = \frac{\lambda}{p} m_{pl}^4 \left(\frac{\phi}{m_{pl}} \right)^p$$



$$\epsilon = \frac{p^2}{16\pi} \left(\frac{m_{pl}}{\phi} \right)^2, \quad \eta = \frac{p(p-1)}{8\pi} \left(\frac{m_{pl}}{\phi} \right)^2.$$

$$\phi \gg m_{pl}, \ \epsilon \ll 1, \ \eta \ll 1$$

$$m_{pl}^2 = \frac{1}{G} = 8\pi M_{pl}^2$$

The end of inflation $\phi_e \sim m_{pl}$

$$\frac{\phi_e}{m_{pl}} = \begin{cases} p/\sqrt{16\pi}, & 0 < p < 2, \\ \sqrt{p(p-1)/8\pi}, & p \geq 2. \end{cases}$$

End of inflation

- chaotic potential $$V(\phi) = \frac{1}{2}m^2\phi^2$$

$$\epsilon = 2\eta = 2\left(\frac{M_{pl}}{\phi}\right)^2$$

$$H^2 = \frac{1}{6}\left(\frac{m}{M_{pl}}\right)^2\phi^2$$

$$3H\dot{\phi} = -m^2\phi$$

$$\dot{\phi} = -\sqrt{2/3}\, mM_{pl}$$

Slow-roll approximation $$\phi_e = \sqrt{2}M_{pl}$$

$$\epsilon_H = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V} = 1 \Rightarrow \dot{\phi}^2 = V(\phi)$$

$$\phi_e = \sqrt{\frac{4}{3}}M_{pl}$$

$$\phi_e = 1.155M_{pl}$$

- Numerical result: $$\phi_e = 1.006M_{pl}$$

Chaotic inflation

- Number of e-foldings

$$N_* = -\frac{8\pi}{pm_{pl}^2}\int_{\phi_*}^{\phi_e}\phi d\phi = \frac{4\pi}{p}\left[\frac{\phi_*^2}{m_{pl}^2} - \frac{\phi_e^2}{m_{pl}^2}\right] \approx \frac{4\pi}{p}\frac{\phi_*^2}{m_{pl}^2} - \tilde{n},$$

$$0 < p < 2,\ \tilde{n} = p/4 \quad p \geq 2,\ \tilde{n} = (p-1)/2.$$

$$\epsilon(\phi_*) = \frac{p}{4(N_* + \tilde{n})}, \quad \eta(\phi_*) = \frac{p-1}{2(N_* + \tilde{n})}$$

$$n_s = 1 - \frac{p+2}{2(N_* + \tilde{n})}, \quad r = \frac{4p}{N_* + \tilde{n}} = \frac{8p(1-n_s)}{p+2}$$

$$\xi = \frac{(p-1)(p-2)}{4N_* + \tilde{n})^2} \quad n_s' = -\frac{2+p}{2(N_* + \tilde{n})^2} = -\frac{2(1-n_s)^2}{p+2}$$

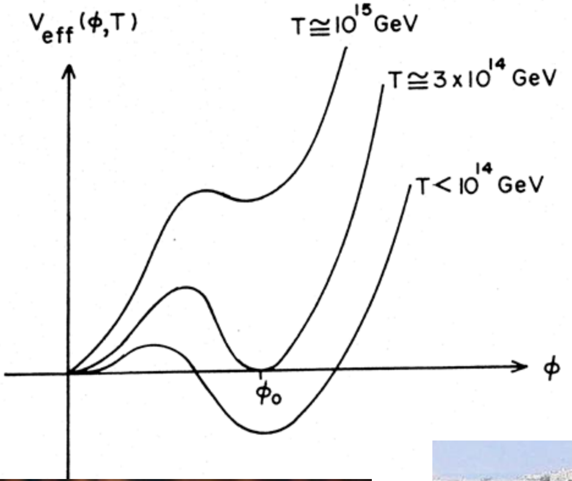
Higgs field

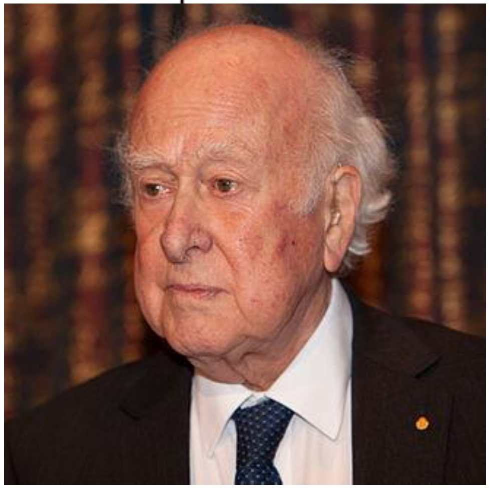
$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2 + bT^2\phi^2$$

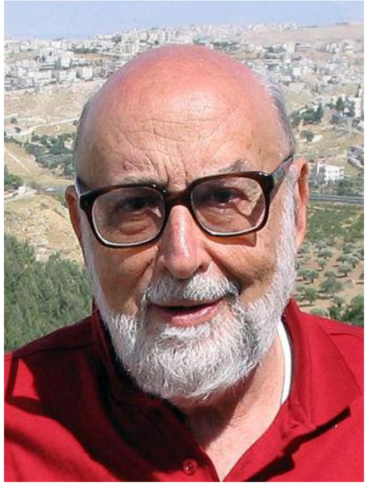
- Higgs particle

$m = 125 \text{ GeV}$ 2012

- 2013 Nobel prize





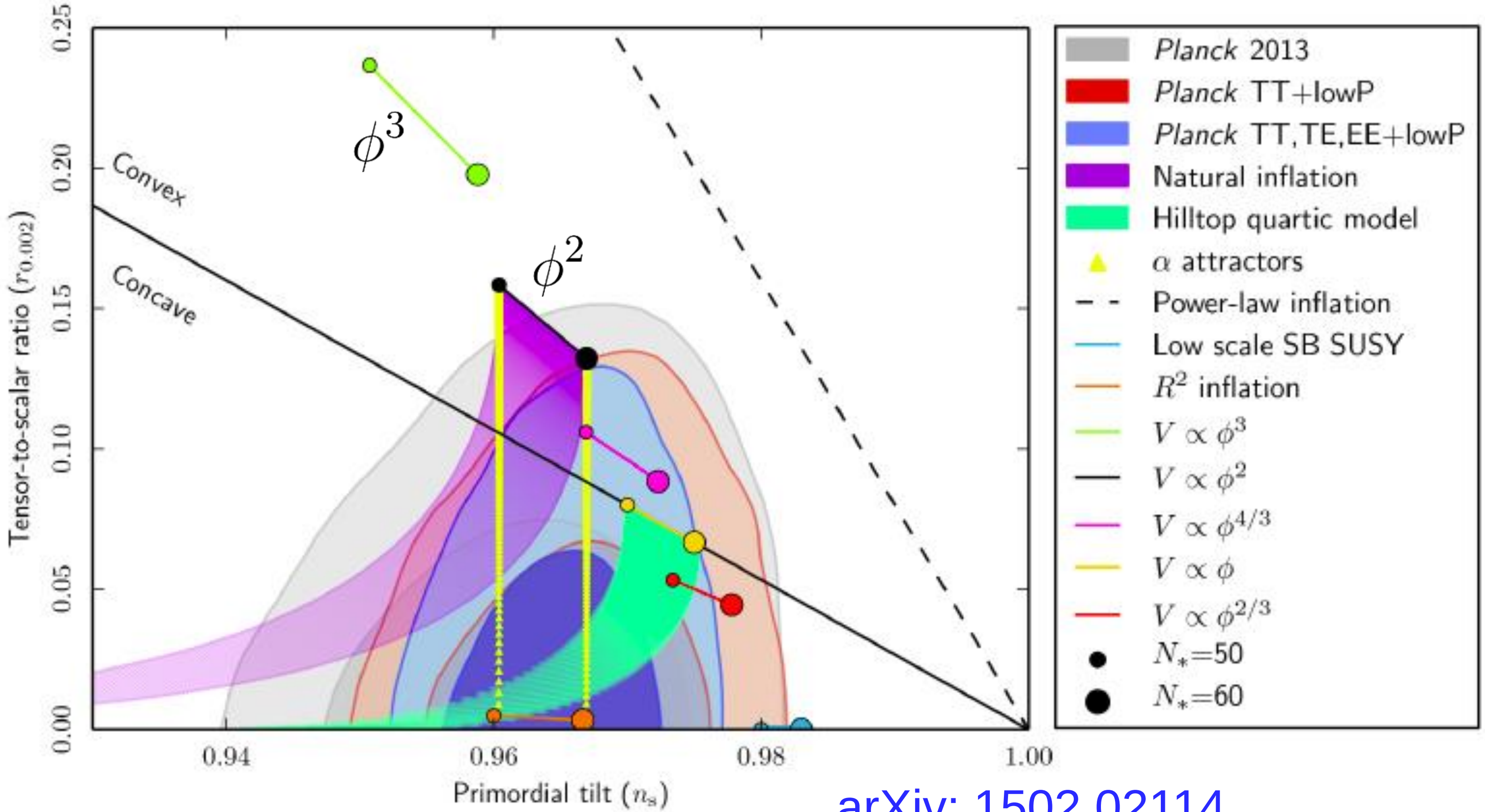


Englert

Higgs field

- potential $V(\phi) = \frac{\lambda}{4}\phi^4, \quad T \gg T_{crt}$

$$n_s = 1 - \frac{3}{N_*}, \quad r = \frac{16}{N_*} \sim 0.27$$



arXiv: 1502.02114

Higgs inflation

- Non-minimal coupling $\xi\phi^2 R$ PLB 659 (08) 703

$$S=\int d^4x\sqrt{-g}\left[\frac{M_{pl}^2+\xi\phi^2}{2}R-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-V(\phi)\right]$$

- Conformal transformation $V(\phi)=\frac{\lambda}{4}\phi^4$

$$\hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu},\quad \Omega^2=1+\frac{\xi\phi^2}{M_{pl}^2},$$

Einstein frame

$$d\psi^2=\left[6M_{pl}^2\left(\frac{d\ln\Omega(\phi)}{d\phi}\right)^2+\frac{1}{\Omega^2(\phi)}\right]d\phi^2$$

$$S=\int d^4x\sqrt{-\hat{g}}\left[\frac{M_{pl}^2}{2}\hat{R}-\frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi-U(\psi)\right]\qquad \kappa=M_{pl}^{-1}$$

$$U(\psi)=\frac{\lambda}{4\kappa^4\xi^2}\left[1+\exp\left(-\frac{2\kappa\psi}{\sqrt{6}}\right)\right]^{-2}\qquad \xi\gg 1,\ \sqrt{\xi}\kappa\phi=\exp(k\psi/\sqrt{6})$$

Higgs inflation

- Spectral index $\xi \gg 1$

Kaiser, PRD 52 (95) 4295

$$n_s = 1 - \frac{8(4N+9)}{(4N+3)^2}, \quad r = \frac{192}{(4N+3)^2}$$

$$n_s \approx 1 - \frac{2}{N}, \quad r \approx \frac{12}{N^2}$$

$$N \sim 60$$

T. Chiba & M. Yamguchi, JCAP 0810, 021

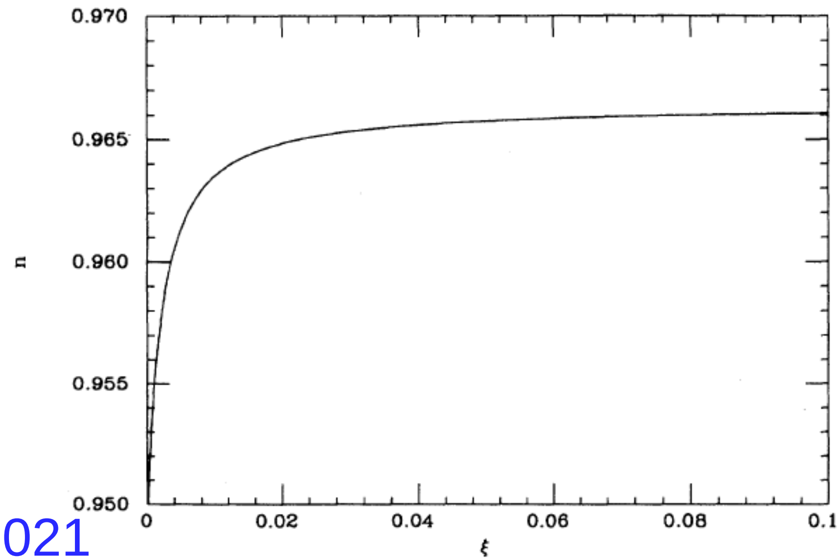
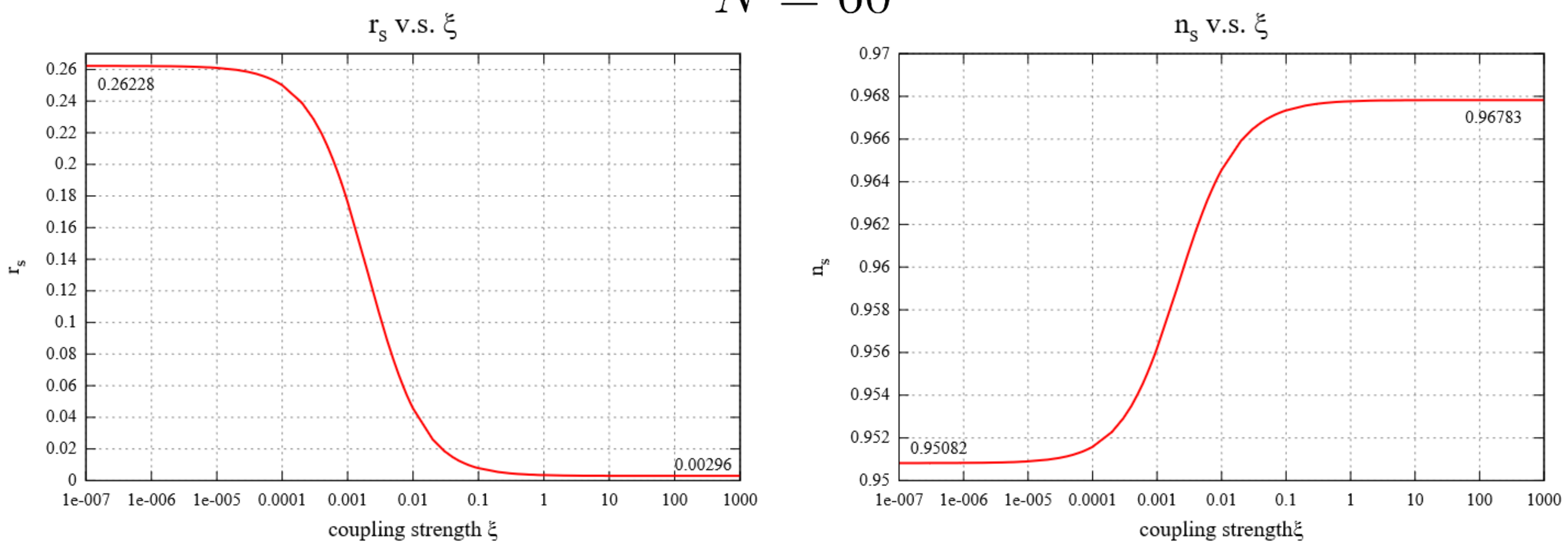


FIG. 2. Second-order results for the spectral index n_s for the model of Sec. IV, based on Eqs. (10), (65), and (67), with $\alpha = 60$. This model only admits chaotic inflation initial conditions.

Higgs inflation

- The effect of coupling constant

$$N = 60$$



$$n_s \approx 1 - \frac{2}{N}, \quad r \approx \frac{12}{N^2}$$

$$n_s = 0.968 \pm 0.006, \quad r_{0.002} < 0.10, \quad 95\%$$

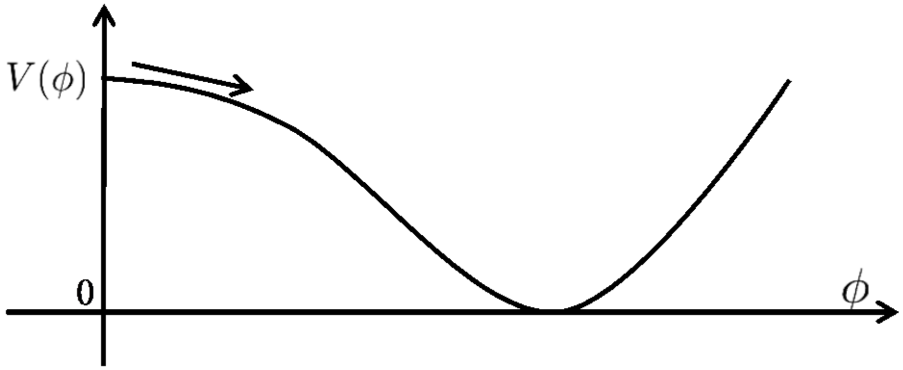
Small-field inflation

- Coleman-Weinberg potential

$$V(\phi) = \frac{B\sigma^4}{2} + B\phi^4[\ln(\phi^2/\sigma^2) - \frac{1}{2}]$$

$\phi \ll \sigma$

$$V(\phi) = \frac{B\sigma^4}{2} - \frac{\lambda\phi^4}{4}$$



Hilltop inflation

- Hill-top model

$$V(\phi) = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^p\right] \quad p > 2$$

$$\epsilon(\phi) = \frac{p^2 M_{pl}^2 (\phi/\mu)^{2p-2}}{2\mu^2\left[1-(\phi/\mu)^p\right]^2}, \quad \eta(\phi) = -\frac{p(p-1)M_{pl}^2(\phi/\mu)^{p-2}}{\mu^2\left[1-(\phi/\mu)^p\right]}$$

- End of inflation

$$\left|\frac{\epsilon}{\eta}\right| = \frac{1}{2}\frac{p}{p-1}\frac{(\phi/\mu)^p}{1-(\phi/\mu)^p} < 1$$

$$\left(\frac{\phi}{\mu}\right)^p < \frac{2p-2}{3p-2} \sim \frac{2}{3}$$

$$\eta(\phi_e) = 1$$

$$\phi_e \approx \left[p(p-1)\right]^{1/(2-p)}\mu^{p/(p-2)}$$

Boubekeur & Lyth, JCAP 0507, 010

Hilltop inflation

- The special case $p = 2$

$$N = \frac{\mu^2}{p}\left[f(\phi_e/\mu) - f(\phi_*/\mu)\right]$$

$$f(x) = \ln x - \frac{1}{2}x^2$$

$$n_s = 0.968 \pm 0.006$$

Planck 2015 constraints

$$r_{0.002} < 0.10, \quad 95\%$$

$$\mu \geq 9M_{pl}, \quad \frac{\phi_*}{\mu} \geq 0.138$$

This case is not small field inflation

Hilltop inflation

- Number of e-foldings $p \neq 2$

$$N = \frac{\mu^2}{p}\left[f(\phi_e/\mu) - f(\phi_*/\mu)\right]$$

$$f(x) = \frac{x^{2-p}}{2-p} - \frac{1}{2}x^2$$

$$\frac{\phi_e}{\mu} \approx \left[\frac{\mu^2}{p(p-1)}\right]^{\frac{1}{p-2}}, \quad \frac{\phi_*}{\mu} \approx \left(\frac{\mu^2}{p[(p-2)N+p-1]}\right)^{\frac{1}{p-2}}$$

$$\frac{\epsilon(\phi_e)}{\eta(\phi_e)} < 1 \rightarrow \mu < \sqrt{\frac{p(p-1)(2p-2)}{3p-2}}\left(\frac{3p-2}{2p-2}\right)^{1/p}$$

Hilltop inflation

- Scalar spectral tilt and tensor to scalar ratio

$$n_s = 1 - \frac{2(p-1)}{(p-2)N + p - 1}$$

$$r \approx \frac{8p^2}{\mu^2}\left[\frac{\mu^2}{p[(p-2)N + p - 1]}\right]^{(2p-2)/(p-2)}$$

Kohri, Lin & Lyth, JCAP 0712, 004

- Large N limit

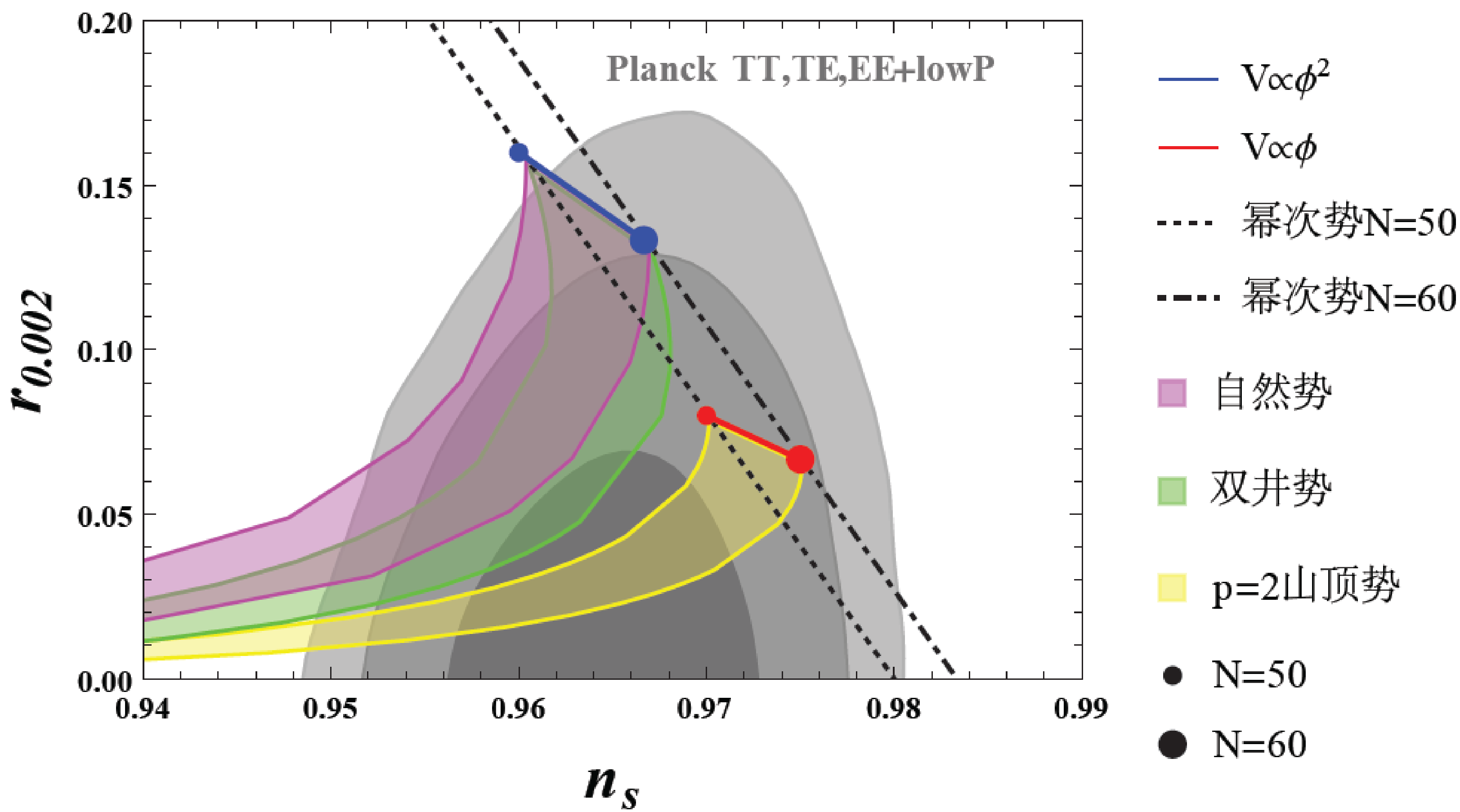
$$n_s \approx 1 - \frac{2(p-1)}{(p-2)N} + \frac{2(p-1)^2}{(p-2)^2N^2}, \quad r \approx \frac{8p^2}{\mu^2}\left[\frac{\mu^2}{p(p-2)N}\right]^{(2p-2)/(p-2)}\left[1 - \frac{2(p-1)^2}{(p-2)^2N}\right]$$

$p = 4$ More exact solution

$$n_s = 1 - \frac{3}{N} + \frac{3\sqrt{36+\mu^4}}{4N^2}, \quad r = \frac{\mu^4}{4N^3} - \frac{3\mu^4\sqrt{36+\mu^4}}{16N^4}$$

Roest, 1309.1285, JCAP 1401, 007

Inflationary models



Natural Inflation

$$V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right)\right]$$

$$\epsilon = \frac{M_{pl}^2}{2f^2}\left[\frac{\sin(\phi/f)}{1+\cos(\phi/f)}\right]^2 \qquad \eta = -\frac{M_{pl}^2}{f^2}\frac{\cos(\phi/f)}{1+\cos(\phi/f)}$$

$$\xi = -\frac{M_{pl}^4}{f^4}\left[\frac{\sin(\phi/f)}{1+\cos(\phi/f)}\right]^2 = -\frac{2M_{pl}^2}{f^2}\epsilon$$

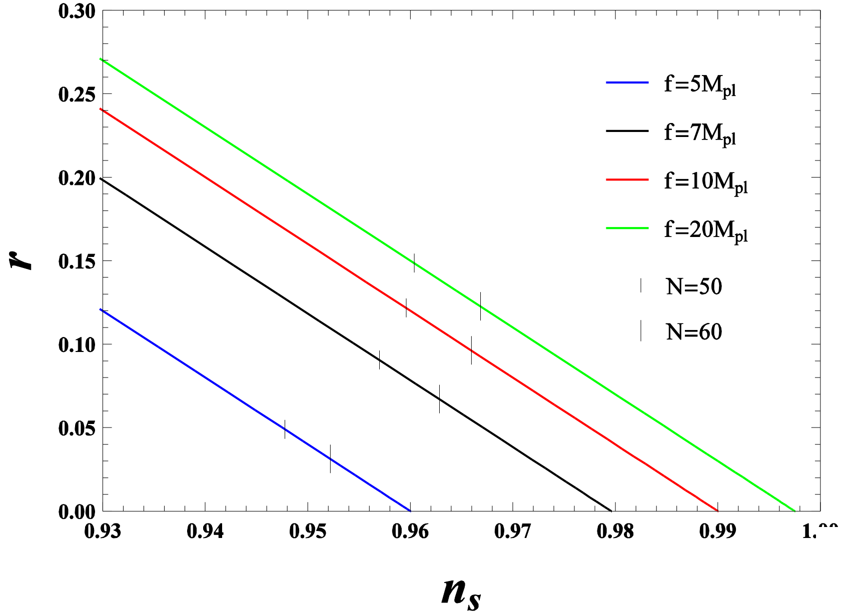
$$\frac{\phi_e}{f} = \arccos\left[\frac{1-2(f/M_{pl})^2}{1+2(f/M_{pl})^2}\right] \qquad N_* = \frac{2f^2}{M_{pl}^2}\ln\left[\frac{\sin(\phi_e/2f)}{\sin(\phi_*/2f)}\right]$$

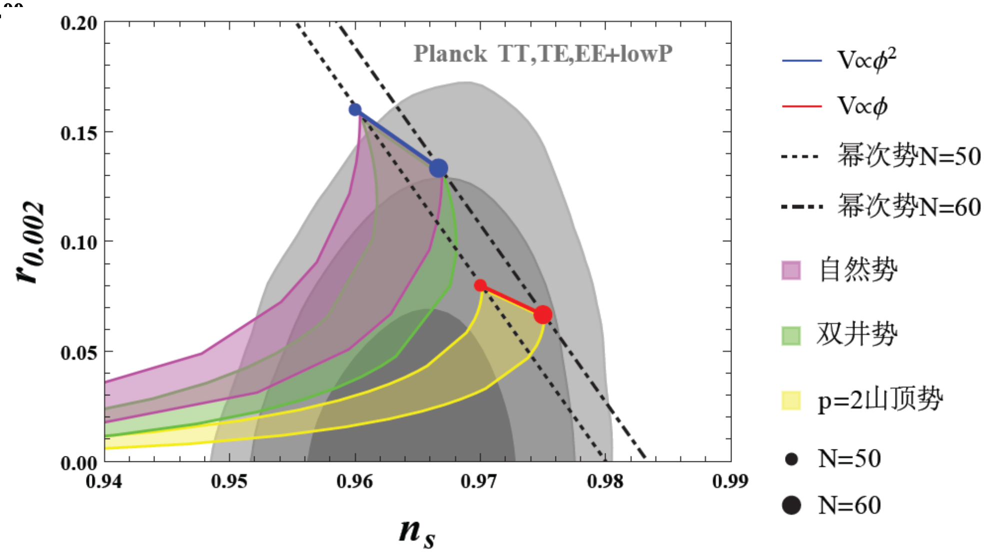
$$n_s \approx 1 - M_{pl}^2/f^2,\ \ f < 1.5M_{pl}$$

Gao and Gong, PLB 734 (14) 41

$$n_s \approx 1 - 2/N,\ \ r \approx 8/N,\ \ f > 1.5M_{pl}$$

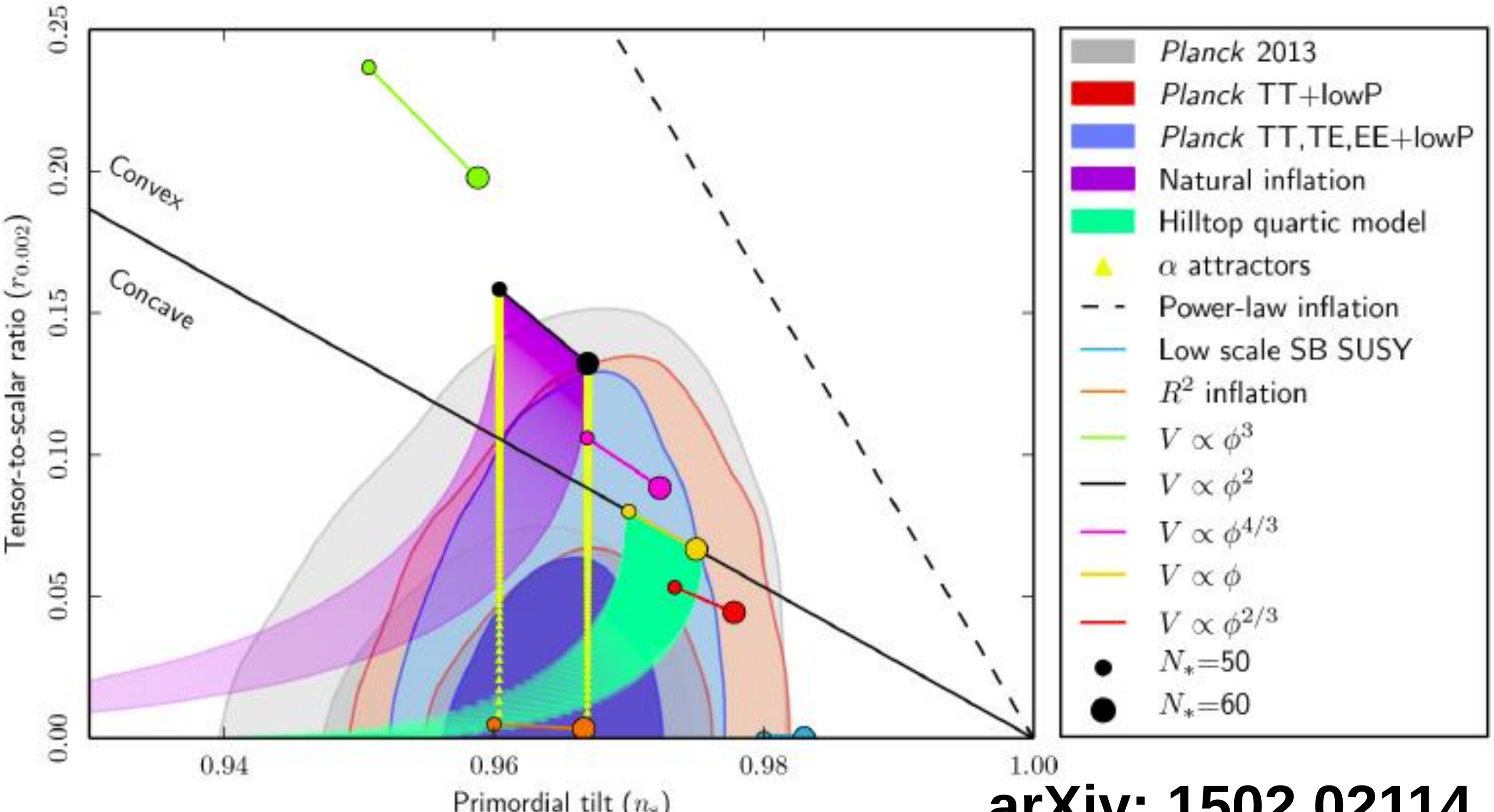
Natural inflation





CMB constraints

- Planck 2015

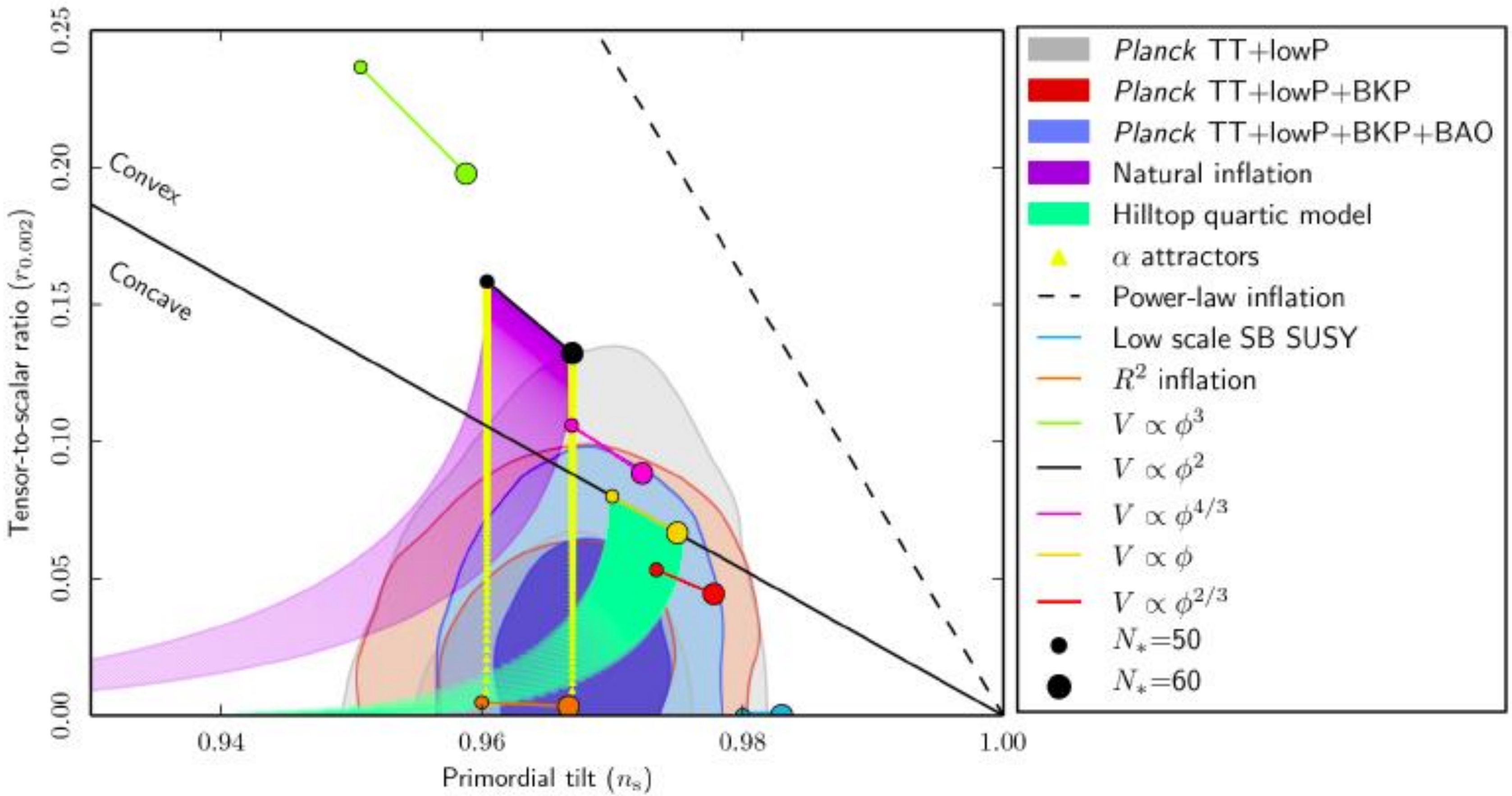


arXiv: 1502.02114

$r_{0.002} < 0.11$ 95% confidence levels

Current Constraints

- Planck 2015 + BICEP2



PRL114 (15) 101301

R+R^2 inflation

- General f(R) theory

$$S=\int d^4x\frac{1}{2}\sqrt{-\tilde{g}}f(\tilde{R})=\frac{1}{2}\int d^4x\sqrt{-\tilde{g}}[h'(\phi)\tilde{R}-\phi h'(\phi)+h(\phi)],$$

$$h'(\phi)=dh(\phi)/d\phi,\ \ h(\phi)=f(\phi)$$

- Conformal transformation

$$g_{\mu\nu}=\Omega(\phi)\tilde{g}_{\mu\nu},\quad \Omega(\phi)=h'(\phi)$$

$$d\psi^2=\left[\frac{3}{2}\frac{(d\Omega/d\phi)^2}{\Omega^2(\phi)}\right]d\phi^2,\quad \Omega(\phi)=h'(\phi)=\exp\left(\sqrt{2/3}\,\psi\right)$$

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}R(g)-\frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi-U(\psi)\right],$$

R+R^2 inflation

- Einstein frame

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}R(g)-\frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi-U(\psi)\right],$$

$$U(\psi)=\frac{1}{2}\exp\left(-2\sqrt{\frac{2}{3}}\psi\right)(\phi(\psi)h'[\phi(\psi)]-h[\phi(\psi)])$$

$$f(R)=R+\alpha R^2$$

$$U(\psi)=\frac{1}{8\alpha}\left[1-e^{-\sqrt{\frac{2}{3}}\psi}\right]^2$$

$$n_s=1-\frac{2}{N_*},\qquad r=\frac{12}{N_*^2}$$

Inflationary models

- Chaotic Inflation (Power-law potential)

$$\epsilon(\phi_*) = \frac{p}{4(N_* + \tilde{n})}, \quad \eta(\phi_*) = \frac{p-1}{2(N_* + \tilde{n})}$$

$$n_s = 1 - \frac{p+2}{2(N_* + \tilde{n})}, \quad r = \frac{4p}{N_* + \tilde{n}}$$

- Hilltop Inflation

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right]$$

Boubekeur & Lyth, JCAP 0507, 010

$$n_s = 1 - \frac{2(p-1)}{(p-2)N_*}, \qquad r \approx \frac{8p^2}{\mu^2}\left[\frac{\mu^2}{p(p-2)N}\right]^{(2p-2)/(p-2)}$$

Universal Attractors

- Natural Inflation $V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right)\right]$

$$n_s \approx 1 - M_{pl}^2/f^2, \;\; f < 1.5 M_{pl}$$

$$\underline{n_s \approx 1 - \frac{2}{N_*}}, \;\; r \approx \frac{8}{N_*}, \;\; f > 1.5 M_{pl}$$

- R^2 Inflation (Starobinsky model)

$$R + R^2$$

$$\underline{n_s = 1 - \frac{2}{N_*}}, \quad r = \frac{12}{N_*^2}$$

Universal Attractors

- non-minimal coupling $\xi\phi^2 R$ with strong coupling $\xi \gg 1$

$$V(\phi) = \frac{\lambda}{4}\phi^4$$

$$n_s = 1 - \frac{2}{N}, \quad r = \frac{12}{N^2}$$

Kaiser, PRD 52 (95) 4295
Bezrukov and Shaposhnikov, PLB 659 (08) 703

- non-minimal coupling with strong coupling

$$\Omega(\phi)R, \ \Omega(\phi) = 1 + \xi f(\phi), \ V_J(\phi) = \lambda^2 f^2(\phi)$$

$$n_s = 1 - \frac{2}{N}, \quad r = \frac{12}{N^2} \qquad \xi \gg 1$$

Kallosh, Linde and Roest, PRL 112 (14) 011303

α Attractors

- The model

$$\mathcal{L}_E = \sqrt{-g}\left[\frac{1}{2}R - \frac{\alpha}{(1-\phi^2/6)^2}\frac{(\partial\phi)^2}{2} - f^2(\phi/\sqrt{6})\right]$$

$$n_s = 1 - \frac{2}{N}, \qquad r = \frac{12\alpha}{N^2}$$

T-model $V = V_0 \tanh^{2n}(\phi/\sqrt{6\alpha})$

E-model $V = V_0\left[1 - \exp(-\sqrt{2/3\alpha}\,\phi)\right]^{2n}$

Galante, Kallosh, Linde, Roest, PRL 114 (15) 141302

$$-\frac{1}{2}K_E(\rho)(\partial\rho)^2 - V_E(\rho)$$

$$K_E = \frac{a_p}{\rho^p} + \cdots \qquad V_E = V_0(1 + c\rho + \cdots)$$

$$n_s = 1 - \frac{p}{p-1}\frac{1}{N}, \qquad r \sim \frac{1}{N^{p/(p-1)}}$$

Potential Reconstruction

- Relations $n_s, \quad r, \quad N$

Lin, Gao & Gong, MNRAS 459 (16) 4029

$$n_s - 1 = -2\epsilon + \frac{d\ln\epsilon}{dN} \qquad r = 16\epsilon$$

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{1}{2}\frac{V'}{V}\frac{d\phi}{dN} = \frac{1}{2}\frac{d\ln V}{dN} > 0$$

$$n_s - 1 \approx -(\ln V)_{,N} + \left(\ln\frac{V_{,N}}{V}\right)_{,N} = \left(\ln\frac{V_{,N}}{V^2}\right)_{,N}$$

$$\phi - \phi_e = \pm\int_0^N \sqrt{2\epsilon(N)}dN$$

$$\epsilon(N), \quad n_s(N), \quad \phi(N) \longrightarrow V(\phi)$$

Model Independent Reconstruction

- The parametrization of the spectral tilt

$$n_s - 1 \approx -\frac{p}{N+\alpha}$$

Creminelli etal. arXiv: 1412.0678
Chiba 1504.07692

$$\epsilon(N) = \frac{p-1}{2(N+\alpha) + C(N+\alpha)^p}$$

Mukahnov EPJC 73 (13) 2486

$$p > 1,\ C \geq 0$$

$$V(N) = \frac{p-1}{A}\left[\frac{1}{(N+\alpha)^{p-1}} + \frac{C}{2}\right]^{-1}$$

- Case 1 $\quad C = 0, \quad V(\phi) = V_0(\phi - \phi_0)^{2(p-1)}$

The parametrization of the spectral tilt

- Case 2 $p = 2$

$$n_s - 1 \approx -\frac{2}{N+\alpha}, \quad r = \frac{16}{C(N+\alpha)^2}$$

$$V(\phi) = V_0 \tanh^2[\gamma(\phi - \phi_0)]$$ T-model

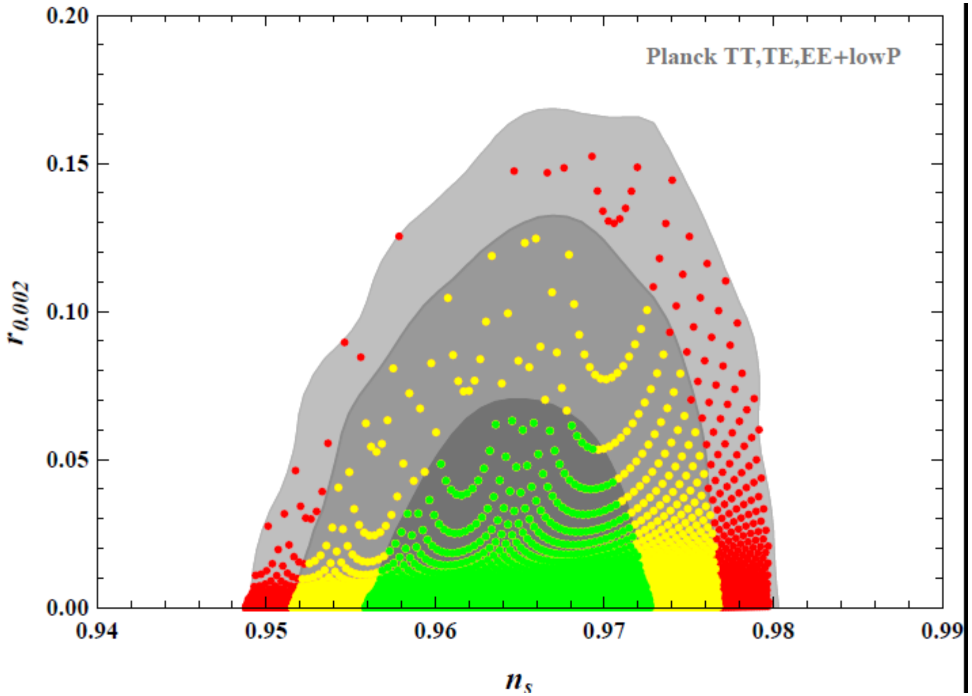
$$C > 1,\ \alpha \ll 1, \quad V(\phi) \approx V_0 \left\{1 - 4\exp\left[-\sqrt{C/2}\,(\phi - \phi_0)\right]\right\}$$

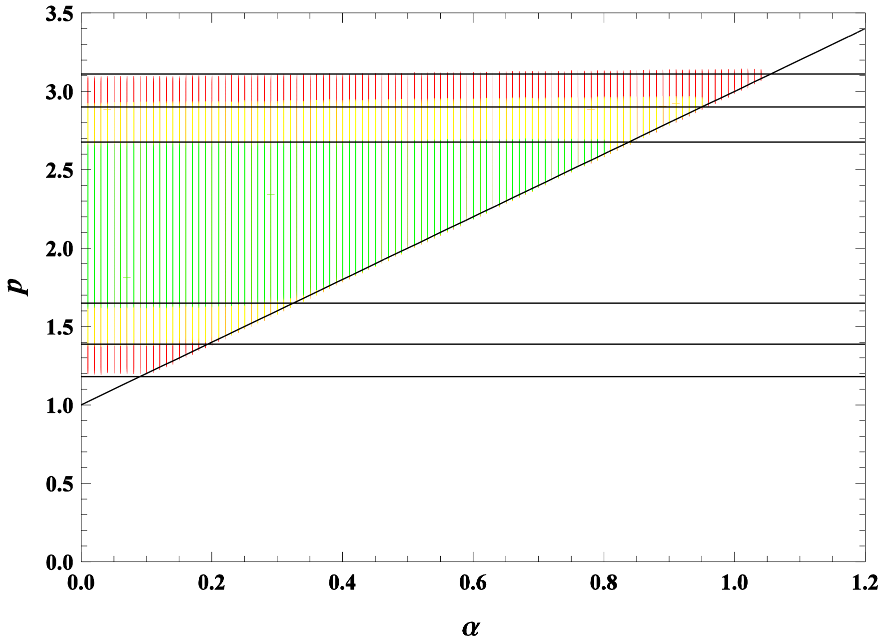
α attractors and Starobinsky model

The parametrization of the spectral tilt

- General case $n_s - 1 \approx -p/(N+\alpha),\ C > 1$

$$V(\phi) = \frac{2(p-1)\alpha^p}{A(p-1-2\alpha)}\left\{1 \mp \left(\frac{2\alpha^p}{p-1-2\alpha}\right)^{1/(2-p)}\left[\frac{2-p}{2\sqrt{p-1}}(\phi-\phi_0)\right]^{-2(p-1)/(2-p)}\right\}^{-1}$$



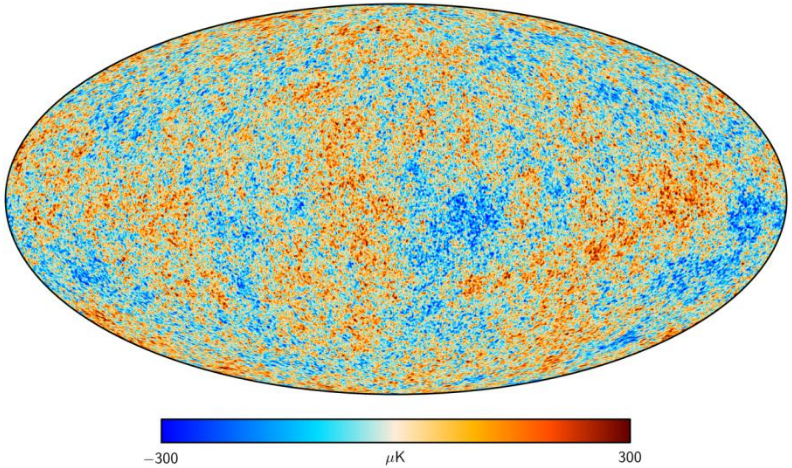


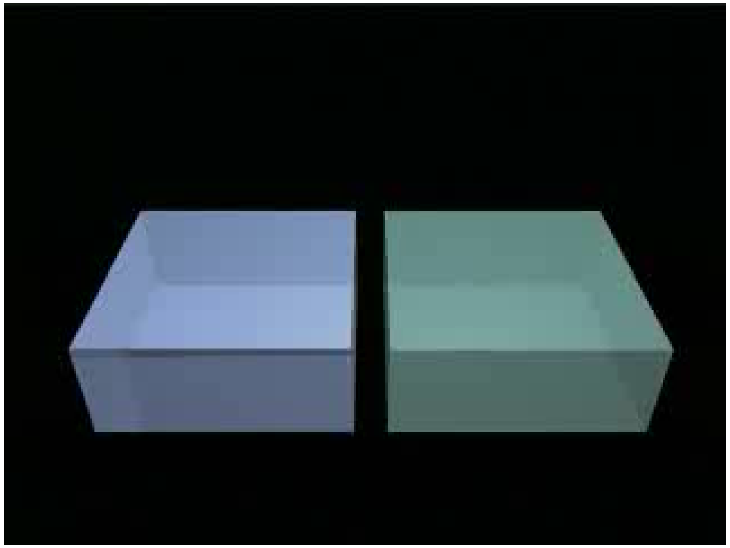
Summary of Inflation

- At $t=10^{-32}$ s, the Universe is about 10^{-24}cm
- In about 10^{-33}s, the Universe expanded exponentially by a factor of 10^{26} $N \sim 60$
- The quantum fluctuation of the inflaton seeds the formation of the large scale structure, and leaves imprints as small anisotropy in CMB (COBE in 1991)
- The power spectrum of the density perturbation is almost Gaussian, adiabatic, and scale invariant

Inflation

- Ripples: The explosive expansion of space during inflation would have created ripples in the fabric of space.
- GW: these gravity waves should have left a signature in the polarization of the last-scattered photons (CMB).





Inflationary models

- The power spectrum is parameterized

$$\mathscr{P}_{\mathscr{R}} = \frac{k^3}{2\pi^2}|\mathscr{R}_k|^2 = A_{\mathscr{R}}(k_*)\left(\frac{k}{k_*}\right)^{n_s-1+\frac{1}{2}n_s'\ln(k/k_*)+\cdots}$$

order of 10^{-9}

$$\mathscr{P}_T = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_t+\frac{1}{2}n_t'\ln(k/k_*)+\cdots} \approx 64\pi G\left(\frac{H}{2\pi}\right)^2$$

$$n_s - 1 = \left.\frac{d\ln\mathscr{P}_{\mathscr{R}}}{d\ln k}\right|_{k=aH} = 3-2\nu = 2\eta_H - 4\epsilon_H \approx 2\eta - 6\epsilon$$

$$r = \frac{A_T}{A_{\mathscr{R}}} = 16\epsilon_H = 16\epsilon = -8n_T \qquad A_T = rA_{\mathscr{R}} \sim H^2 \sim V(\phi)$$

- Energy scale of inflation: measurement of r

Problems

- Too many models: which model of inflation is correct?

  chaotic inflation, Higgs inflation, natural inflation
  Hilltop, Spontaneously broken SUSY, Hybrid
  DBI, D-brane, racetrack, R^2 inflation, α attractors,
  ...

- Why did inflation happen? what is the initial condition?
- What about other parts of inflating universe? We only see a small part, what is the rest?

Conclusion

- Inflation is successful
- Too many inflationary models
- More accurate measurements of CMB needed
- The detection of B-mode polarization is essential for confirming inflation

Thank You