Bound states, virtual states, resonances in the Friedrichs model

Zhiguang Xiao
Collaborators: Zhou Zhi-yong

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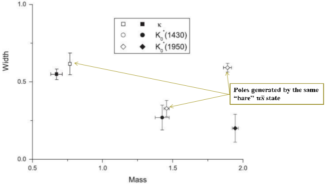
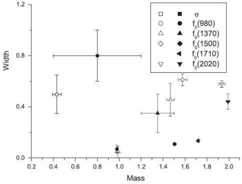
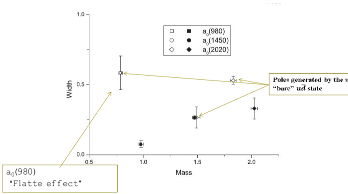
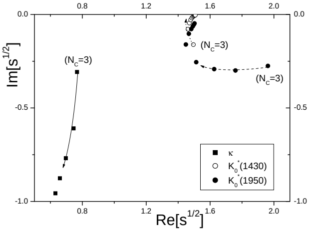
In phenomenology study:

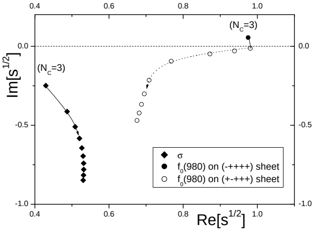
- Couple a bare state with a continuum: dispersive method











- Wave function?
- Probability?

The simplest Friedrichs model

- The system couples one bare state $|1\rangle$ and a continuum state $|\omega\rangle$, which are eigenstates of the free Hamiltonian

$$H_0|1\rangle = \omega_0|1\rangle, \qquad H_0|\omega\rangle = \omega|\omega\rangle.$$

- Orthonormal condition: $\langle 1|1\rangle = 1$, $\langle 1|\omega\rangle = 0$, and $\langle\omega|\omega'\rangle = \delta(\omega - \omega')$

  Completeness: $|1\rangle\langle 1| + \int_0^\infty d\omega|\omega\rangle\langle\omega| = 1$
- The free Hamiltonian can be expressed as:

$$H_0 = \omega_0|1\rangle\langle 1| + \int_0^\infty \omega|\omega\rangle\langle\omega|\mathrm{d}\omega$$

- Interaction: $\langle\omega|V|1\rangle = \lambda f(\omega)$, $\langle\omega'|V|\omega\rangle = \langle 1|V|1\rangle = 0$.

$$V = \lambda\int_0^\infty [f(\omega)|\omega\rangle\langle 1| + f^*(\omega)|1\rangle\langle\omega|]\mathrm{d}\omega$$

Solution of energy eigenfunction

- Eigenvalue equation:

$$H|\Psi(x)\rangle = (H_0 + V)|\Psi\rangle = x|\Psi(x)\rangle.$$

- Solution can be expanded as

$$|\Psi(x)\rangle = \alpha(x)|1\rangle + \int_0^\infty \psi(x,\omega)|\omega\rangle \mathrm{d}\omega.$$

- Using $V|1\rangle = \lambda f(\omega)|\omega\rangle$, $V|\omega\rangle = \lambda f^*(\omega)|1\rangle$, we have

$$\begin{aligned}(\omega_0 - x)\alpha(x) + \lambda \int_0^\infty f^*(\omega)\psi(x,\omega)\mathrm{d}\omega =& 0,\\ (\omega - x)\psi(x,\omega) + \lambda f(\omega)\alpha(x) =& 0.\end{aligned}$$

Continuum state solution

$$\begin{aligned}(\omega_0 - x)\alpha(x) + \lambda \int_0^\infty f^*(\omega)\psi(x,\omega)\mathrm{d}\omega =& 0,\\ (\omega - x)\psi(x,\omega) + \lambda f(\omega)\alpha(x) =& 0.\end{aligned}$$

Eigenvalue $x > 0$, real

$$\psi_\pm(x,\omega) = -\tfrac{\lambda\alpha(x)f(\omega)}{\omega - x \pm i\epsilon} + \gamma_\pm(\omega)\delta(\omega - x)\,,$$

$$(\omega_0 - x)\alpha_\pm(x) + \lambda f^*(x)\gamma_\pm(x) - \alpha_\pm(x)\lambda^2 \textstyle\int_0^\infty \tfrac{f(\omega)f^*(\omega)}{\omega - x \pm i\epsilon}\mathrm{d}\omega = 0.$$

Solution: define $\eta^\pm(x) = x - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega)f^*(\omega)}{x - \omega \pm i\epsilon}\mathrm{d}\omega$

$$\alpha_\pm(x) = \lambda\frac{f^*(x)\gamma_\pm(x)}{\eta^\pm(x)}.$$

Choose normalization, $\langle\Psi(x)|\Psi(x')\rangle = \delta(x - x')$, $\gamma_\pm = 1$,

$$|\Psi_\pm(x)\rangle \quad = \quad |x\rangle + \lambda\frac{f^*(x)}{\eta^\pm(x)}\left[|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{x - \omega \pm i\epsilon}|\omega\rangle\mathrm{d}\omega\right]$$

Discrete state solution

Eigenvalue $x \notin (0, +\infty)$

$$\begin{aligned}(\omega_0 - x)\alpha(x) + \lambda \int_0^\infty f^*(\omega)\psi(x,\omega)\mathrm{d}\omega =& 0,\\ (\omega - x)\psi(x,\omega) + \lambda f(\omega)\alpha(x) =& 0.\end{aligned}$$

$$\psi(x,\omega) = -\tfrac{\lambda\alpha(x)f(\omega)}{\omega - x},$$

$$\alpha(x)\Big((\omega_0 - x) - \lambda^2 \textstyle\int_0^\infty \frac{f(\omega)f^*(\omega)}{\omega - x}\mathrm{d}\omega\Big) = \alpha(x)\eta(x) = 0.$$

For $\alpha(x)$ to be nonzero, $\eta(x)$ has to vanish at x.

- The zero point of $\eta(x)$ corresponds to eigenvalues of the full Hamiltonian — discrete states.

Discrete spectrum

Analytic continuation of $\eta_{\pm}(x)$

$$\eta^{I}(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega) f^*(\omega)}{z-\omega} \mathrm{d}\omega$$
$$\eta^{II}(z) = \eta^{I}(z) - 2i\pi G(z), \quad G(z) \equiv \lambda^2 f(z) f^*(z)$$

- There is a unitarity cut on $(0, \infty)$. η is continued to two a sheeted Riemann surface.
- $\eta(x)$ real-analytic, $\eta^*(x) = \eta(x^*)$, $G(x)$ anti-real-analytic, $G^*(x) = -G(x^*)$.

Discrete state solutions

$$\eta^I(x) = x - \omega_0 - \lambda^2 \int_0^\infty \frac{f(\omega) f^*(\omega)}{x - \omega} \mathrm{d}\omega = 0$$
$$\eta^{II}(x) = \eta^I(z) - 2i\pi G(z), \quad G \equiv \lambda^2 f(x) f^*(x)$$

- Bound states: solutions on the first sheet real axis below the threshold.
  If $\omega_0 < \lambda^2 \int_0^\infty \frac{f(\omega) f^*(\omega)}{\omega} \mathrm{d}\omega$, there could be a bound state.

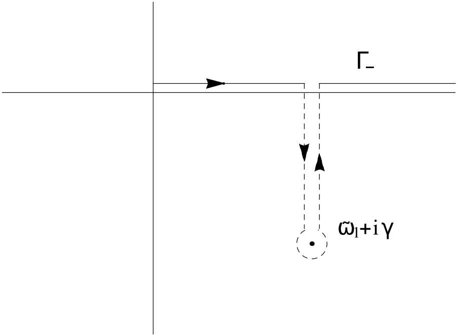
  $$|z_B\rangle = N_B \Big(|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_B - \omega} |\omega\rangle \mathrm{d}\omega \Big)$$

  where $N_B = (\eta'(z_B))^{-1/2} = (1 + \lambda^2 \int d\omega \frac{|f(\omega)|^2}{(z_B - \omega)^2})^{-1/2}$, such that $\langle z_B | z_B \rangle = 1$.

Discrete state solutions

- Resonant states: $\omega_0 >$ threshold, A pair of solutions z_R, z_R^*, on the second sheet complex plane. $\hat{H}|z_R\rangle = z_R|z_R\rangle$

$$|z_R\rangle = N_R\Big(|1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R - \omega]_+}|\omega\rangle\Big),$$
$$|z_R^*\rangle = N_R^*\Big(|1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[z_R^* - \omega]_-}|\omega\rangle\Big),$$



Discrete state solutions

Resonant states:

- Normalization: $\langle z_R|z_R\rangle = 0$, naïve argument, $z_R^* \neq z_R$,

$$\langle z_R|\hat{H}|z_R\rangle = z_R\langle z_R|z_R\rangle = z_R^*\langle z_R|z_R\rangle = 0$$

  $|z_R\rangle$ is not in the Hilbert space — need rigged Hilbert space description.

- Left eigenstates:$\langle \tilde{z}_R|\hat{H} = \langle \tilde{z}_R|z_R$

$$\langle \tilde{z}_R| = N_R\Big(\langle 1| + \lambda\int_0^\infty \mathrm{d}\omega\frac{f(\omega)}{[z_R-\omega]_+}\langle\omega|\Big),$$
$$\langle \tilde{z}_R^*| = N_R^*\Big(\langle 1| + \lambda\int_0^\infty \mathrm{d}\omega\frac{f(\omega)}{[z_R^*-\omega]_-}\langle\omega|\Big).$$

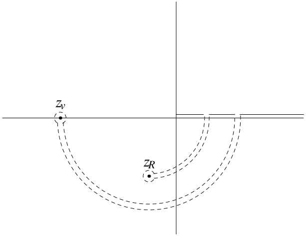
  N_R is a complex normalization parameter, $N_R = (\eta'^{+}(z_R))^{-1/2} = (1+\lambda^2\int d\omega\frac{|f(\omega)|^2}{[(z_R-\omega)_+]^2})^{-1/2}$ such that $\langle \tilde{z}_R|z_R\rangle = 1$

Discrete state solutions

- Virtual states: Solutions on the second sheet real axis below the threshold.

$$|z_v^{\pm}\rangle = N_v^{\pm}\Big(|1\rangle + \lambda\int_0^{\infty}\frac{f(\omega)}{[z_v-\omega]_{\pm}}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle\tilde{z}_v^{\pm}| = \langle z_v^{\mp}|,$$

where
$N_v^- = N_v^{+*} = (\eta'^{+}(z_v))^{-1/2} = (1+\lambda^2\int d\omega \frac{|f(\omega)|^2}{[(z_v-\omega)_+]^2})^{-1/2}$,
such that $\langle\tilde{z}_v^{\pm}|z_v^{\pm}\rangle = 1$.



An example form factor

Choose an example form factor: $|f(\omega)|^2 = \frac{\sqrt{\omega}}{\omega+\rho^2}$, $\rho > 0$

$$\eta(\omega) = \omega - \omega_0 + \frac{\pi\lambda^2}{\sqrt{-\omega}+\rho} = \omega - \omega_0 + \frac{\pi\lambda^2}{-i\sqrt{\omega}+\rho}\,.$$

$$\eta^{II}(\omega) = \omega - \omega_0 + \frac{\pi\lambda^2}{-\sqrt{-\omega}+\rho}\,,$$

Case 1: $\omega_0 > \frac{\pi\lambda^2}{\rho}$, turn on λ slowly

- Three solutions

$$E_{1,2} = \omega_0 - \frac{\pi\lambda^2}{\rho \mp i\omega_0^{1/2}} + O(\lambda^4)\,,$$

$$E_3 = -\rho^2 + 4\gamma\rho + O(\lambda^4) = -\rho^2 + \frac{2\rho\pi\lambda^2}{\omega_0+\rho^2} + O(\lambda^4)$$

- $E_{1,2}$: resonance poles. $\lambda \to 0$, they move the discrete bare state.
- E_3 virtual state: when $\lambda \to 0$, it approaches ρ^2, the pole of the form factor. At $\lambda = 0$, it disappears.

- Resonance poles:

$$|E_1\rangle = N_R\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[E_1-\omega]_+}|\omega\rangle\Big),$$
$$|E_2\rangle = N_R^*\Big(|1\rangle + \lambda\int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{[E_2-\omega]_-}|\omega\rangle\Big),$$

In the $\lambda \to 0$ limit , $|E_{1,2}\rangle \to |1\rangle$.

- Virtual state:

$$|E_3^\pm\rangle = N_v\Big(|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{[E_3-\omega]_\pm}|\omega\rangle \mathrm{d}\omega\Big),$$

At $\lambda = 0$, it dissappears $\eta(z) = z - \omega_0$, no such solution. For $\lambda \neq 0$, it appears,near the pole of the form factor. In the limit of $\lambda \to 0$, $|E_3\rangle \not\to |1\rangle$.

$$\lambda\int_0^\infty \frac{\omega^{1/4}\phi(\omega)}{(\omega+\rho^2)^{1/2}[z_v-\omega]_+}\mathrm{d}\omega = 2\pi i\lambda\frac{z_v^{1/4}\phi(z_v)}{(z_v+\rho^2)^{1/2}} + O(\lambda)$$
$$\sim 2\pi i\frac{e^{-i\pi/4}\rho^{1/2}\phi(e^{-i\pi}\rho^2)}{(-a)^{1/2}} + O(\lambda) \sim \mathcal{O}(\lambda^0)$$

Case 2. $0 < \omega_0 < \frac{\pi\lambda^2}{\rho}$.

1. $\omega_0 = \frac{1}{3}\rho^2$,

There is a triple pole for $\omega_0 = \frac{3}{4}(\pi^3\lambda^4)^{1/3}$

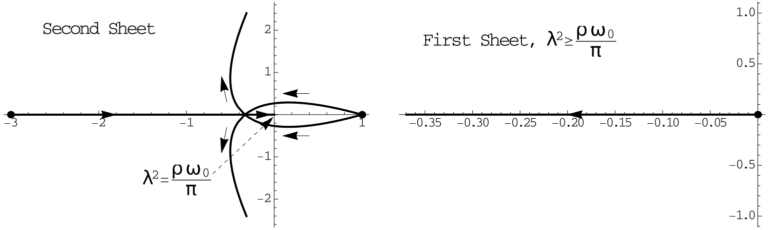
$$|z_v^\pm\rangle = N_v^\pm\Big(|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{[z_v-\omega]_\pm}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle\tilde{z}_v^\pm| = \langle z_v^\mp|\,,$$

$$|z_{v2}^\pm\rangle = -N_{v2}\lambda\int_0^\infty \frac{f(\omega)}{([z_v-\omega]_\pm)^2}|\omega\rangle \mathrm{d}\omega, \quad \langle\tilde{z}_{v2}^\pm| = \langle z_{v2}^\mp|\,,$$

$$|z_{v3}^\pm\rangle = N_{v3}\lambda\int_0^\infty \frac{f(\omega)}{([z_v-\omega]_\pm)^3}|\omega\rangle \mathrm{d}\omega\,, \quad \langle\tilde{z}_{v3}^\pm| = \langle z_{v3}^\mp|\,.$$

Normalization: $\langle\tilde{z}_{v3}^\pm|z_v^\pm\rangle = 1$ and $\langle\tilde{z}_{v2}^\pm|z_{v2}^\pm\rangle = 1$.

Then $N_v = N_{v_2} = N_{v3} = (6/\eta''')^{1/2}$.



2. $\omega_0 < \frac{1}{3}\rho^2$,

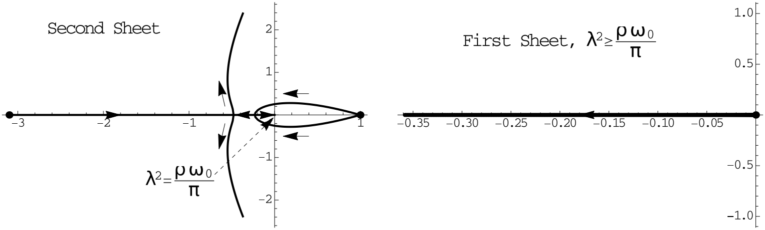
There could be double poles.

$$|z_v^{\pm}\rangle = N_v^{\pm}\Big(|1\rangle + \lambda\int_0^{\infty}\frac{f(\omega)}{[z_v-\omega]_{\pm}}|\omega\rangle \mathrm{d}\omega\Big), \quad \langle\tilde{z}_v^{\pm}| = \langle z_v^{\mp}|\,,$$

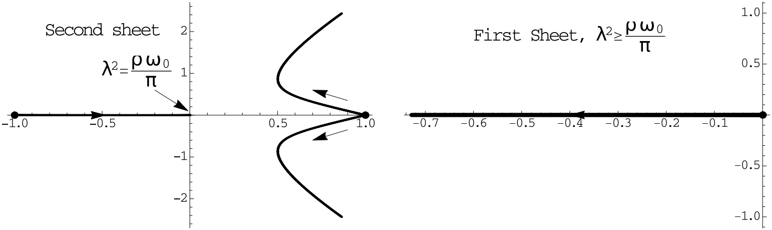
$$|z_{v2}^{\pm}\rangle = -N_{v2}\lambda\int_0^{\infty}\frac{f(\omega)}{([z_v-\omega]_{\pm})^2}|\omega\rangle \mathrm{d}\omega, \quad \langle\tilde{z}_{v2}^{\pm}| = \langle z_{v2}^{\mp}|\,,$$

Normalization: $\langle\tilde{z}_{v2}^{\pm}|z_v^{\pm}\rangle = 1$.

Then $N_v^- = N_{v_2}^- = (2/\eta'')^{1/2}$ and $N_v^+ = N_{v_2}^+ = N_v^{-*}$.

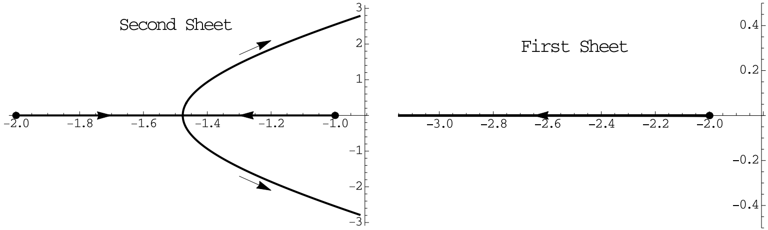


3. $\omega_0 > \frac{1}{3}\rho^2$,



Case 3. $\omega_0 < 0$, always a bound-state pole on the first sheet. A virtual state generated from the formfactor, and a virtual state generated from the discrete bare state.

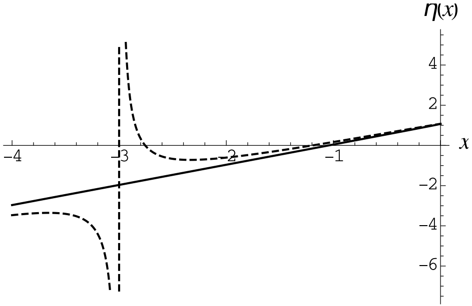
$$z_0 = \omega_0 + \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z_0 - \omega} \mathrm{d}\omega$$



Existence of the virtual states

- Virtual states from the singularity of the form factor, analytically continued $G(\omega) = |f(\omega)|^2$:

$$\eta^I = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega} \mathrm{d}\omega$$
$$\eta^{II}(\omega) = \eta^I(\omega) + 2\pi i \lambda^2 G^{II}(\omega) = \eta^I(\omega) - 2\lambda^2 \pi i \, G(\omega),$$

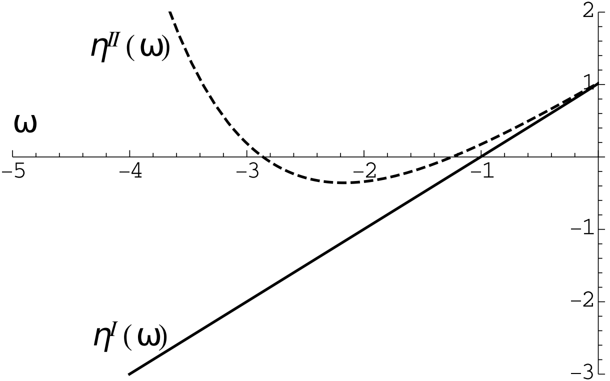


- Virtual state generated from the bare states: $\omega_0 < 0$

Virtual state: another example

Form factor: $G(\omega) = \sqrt{\omega} e^{-\omega}$

$$\begin{aligned}\eta^{II}(\omega) =& \eta^{I}(\omega) - 2\pi i\,\lambda^2 G(\omega)\\ =& \omega - \omega_0 + \lambda^2 \int_0^\infty dx \frac{G(x)}{(x-\omega)} + 2\lambda^2 \pi \sqrt{-\omega} e^{-\omega}.\end{aligned}$$



Completeness relation

- In general, the resonance state and the virtual states do not enter the completeness relation. If there is only continuum eigenstates:

$$\mathbf{1} = \int_0^\infty d\omega |\Psi_+(\omega)\rangle\langle\Psi_+(\omega)|.$$

With one bound state $|E_B\rangle$ eigenstate:

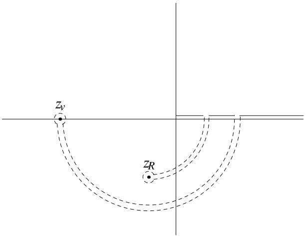
$$\mathbf{1} = |E_B\rangle\langle E_B| + \int_0^\infty d\omega |\Psi_+(\omega)\rangle\langle\Psi_+(\omega)|.$$

Completeness relation

To treat the resonances and virtual states the same as the bound state and Continuum state:

- Petrosky, Prigogine, Tasaki: in solving the large Poincaré problem, propose a definition of continuum state. $|\Psi_+(x)\rangle$ as a distribution, includes the integral contour information

$$
\begin{aligned}
\frac{1}{\eta_d^+(x)} \equiv& \frac{1}{\eta^+(x)} \frac{x-\tilde{\omega}_1+i\gamma}{[x-\tilde{\omega}_1+i\gamma]_+}, \\
|\Psi_\pm(x)\rangle =& |x\rangle + \lambda \frac{f(x)}{\eta_d^\pm(x)} \left[|1\rangle + \lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{x-\omega \pm i\epsilon} |\omega\rangle \right].
\end{aligned}
$$



Completeness relation

- The left state is not modified:

$$\langle\tilde{\Psi}_{\pm}(x)| = \langle x| + \lambda\frac{f(x)}{\eta^{\mp}(x)}\left[\langle 1| + \lambda\int_0^\infty \mathrm{d}\omega\frac{f(\omega)}{x-\omega\mp i\epsilon}\langle\omega|\right].$$

- Using these continuum states, the completeness relation reads,

$$\mathbf{1} = \int_0^\infty d\omega|\Psi_+(\omega)\rangle\langle\tilde{\Psi}_+(\omega)| + |z_R\rangle\langle\tilde{z}_R|.$$

The resonant state enter the completeness relation.

Completeness relation: higher-order pole

When there is an nth-order pole, n degenerate states:

$$
\begin{aligned}
|z^{(1)}\rangle =& N\Big(|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{[z-\omega]_+}|\omega\rangle \mathrm{d}\omega\Big)\,, \\
\langle \tilde{z}^{(1)}| =& N\Big(\langle 1| + \lambda \int_0^\infty \frac{f(\omega)}{[z-\omega]_+}\langle\omega| \mathrm{d}\omega\Big)\,, \\
|z^{(n)}\rangle =& N(-1)^{n-1}\lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{([z-\omega]_+)^n}|\omega\rangle\,, \quad \text{for } n \geq 2\,, \\
\langle \tilde{z}^{(n)}| =& N(-1)^{n-1}\lambda \int_0^\infty \mathrm{d}\omega \frac{f(\omega)}{([z-\omega]_+)^n}\langle\omega|\,, \quad \text{for } n \geq 2\,,
\end{aligned}
$$

$N = (\frac{n!}{\eta^{(n)}(z)})^{1/2} = \Big((-)^{n-1}\frac{\lambda^2}{n!}\int d\omega \frac{|f(\omega)|^2}{([z-\omega]_+)^{n+1}}\Big)^{-1/2}$ is chosen such that $\langle \tilde{z}^{(r)}|z^{(n-r+1)}\rangle = 1$. the completeness relation can also be deduced

$$
\mathbf{1} = \int_0^\infty d\omega |\Psi_+(\omega)\rangle\langle\tilde{\Psi}_+(\omega)| + \sum_{r=1}^{n} |z^{(r)}\rangle\langle\tilde{z}^{(n-r+1)}|.
$$

Coupled channel Friedrichs model

Hamiltonian: $H = H_0 + V$

$$
\begin{aligned}
H &= \omega_0|1\rangle\langle 1| + \int_{a_1}^{\infty} \mathrm{d}\omega\omega|\omega\rangle_{11}\langle\omega| + \int_{a_2}^{\infty} \mathrm{d}\omega\omega|\omega\rangle_{22}\langle\omega| \\
&+ \lambda_1 \int_{a_1}^{\infty} \mathrm{d}\omega[f_1(\omega)|\omega\rangle_1\langle 1| + f_1^*(\omega)|1\rangle\ {}_1\langle\omega|] \\
&+ \lambda_2 \int_{a_2}^{\infty} \mathrm{d}\omega[f_2(\omega)|\omega\rangle_2\langle 1| + f_2^*(\omega)|1\rangle\ {}_2\langle\omega|]
\end{aligned}
$$

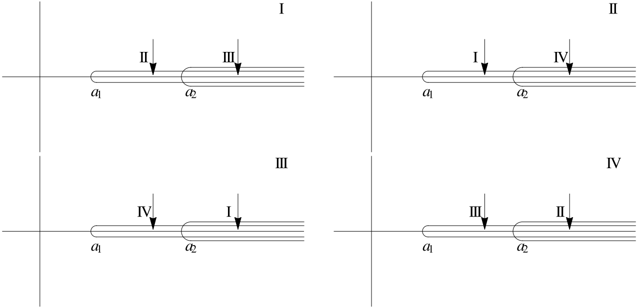
Solution:
Continuous states,

$$
|\Psi_{i\pm}(x)\rangle = |x\rangle_i + \frac{\lambda_i f_i^*(x)}{\eta^{\pm}(x)}\left[|1\rangle + \sum_{j=1,2}\lambda_j \int_{a_j}^{\infty}\mathrm{d}\omega\frac{f_j(\omega)}{x-\omega\pm i\epsilon}|\omega\rangle_j\right].
$$

where $\eta^{\pm}(x) = x - \omega_0 - \lambda_1^2\int_{a_1}^{\infty}\frac{G_1(\omega)}{x-\omega\pm i\epsilon}\mathrm{d}\omega - \lambda_2^2\int_{a_2}^{\infty}\frac{G_2(\omega)}{x-\omega\pm i\epsilon}\mathrm{d}\omega$.
Orthonormal condition: $\langle\Psi_i(x')|\Psi_j(x)\rangle = \delta_{ij}\delta(x'-x)$.

Discrete states are determined by $\eta(z) = 0$, analytically continued to different Riemann sheets.

$$\eta(x) = x - \omega_0 - \lambda_1^2 \int_{a_1}^{\infty} \frac{G_1(\omega)}{x - \omega \pm i\epsilon} \mathrm{d}\omega - \lambda_2^2 \int_{a_2}^{\infty} \frac{G_2(\omega)}{x - \omega \pm i\epsilon} \mathrm{d}\omega \,.$$



Pole position for small coupling

Example form factor: $G_1(\omega) = \frac{\sqrt{\omega - a_1}}{\omega + \zeta_1}$ and $G_2(\omega) = \frac{\sqrt{\omega - a_2}}{\omega + \zeta_2}$

- States from the poles of the form factor:
  near $G_1(x)$ poles, Virtual states on the II and III sheet.
  near $G_2(x)$ poles, Virtual states on the III and IV sheet.
- States generated from the bare states, $\omega_0 < a_1$:
  Bound states on I, and virtual states on II, III, IV sheets.

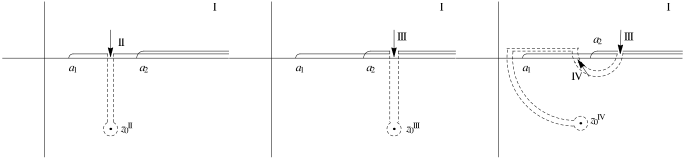
States generated from the bare discrete states

- $a_1 < \omega_0 < a_2$:
  Turn on λ_2 first, then turn on λ_1, $I \to I, II$; $II \to III, IV$:
  Bound state $\to II$ sheet resonances.
  Virtual state $\to IV$ sheet resonances.
  Turn on λ_1 first, then turn on λ_2, $I \to I, IV$; $II \to II, III$:
  II sheet resonance $\to II$, III sheet resonances.

States generated from the bare discrete states

- $a_2 < \omega_0$: Turn on λ_2 first, then turn on λ_1, $I \to I, II$; $II \to III, IV$:
  Resonance $\to III, IV$ sheet resonances.
  Turn on λ_1 first, then turn on λ_2, $I \to I, IV$; $II \to II, III$:
  II sheet resonance $\to II$, III sheet resonances.

Wave function



$$|z_0^I\rangle = N^I\left(|1\rangle + \lambda_1\int_{a_1}^{\infty} d\omega \frac{f_1(\omega)}{z_0^I-\omega}|\omega\rangle_1 + \lambda_2\int_{a_2}^{\infty} d\omega \frac{f_2(\omega)}{z_0^I-\omega}|\omega\rangle_2\right),$$

$$|z_0^{II}\rangle = N^{II}\left(|1\rangle + \lambda_1\int_{a_1}^{\infty} d\omega \frac{f_1(\omega)}{[z_0^{II}-\omega]_+}|\omega\rangle_1 + \lambda_2\int_{a_2}^{\infty} d\omega \frac{f_2(\omega)}{z_0^{II}-\omega}|\omega\rangle_2\right),$$

$$|z_0^{III}\rangle = N^{III}\left(|1\rangle + \lambda_1\int_{a_1}^{\infty} d\omega \frac{f_1(\omega)}{[z_0^{III}-\omega]_+}|\omega\rangle_1 + \lambda_2\int_{a_2}^{\infty} d\omega \frac{f_2(\omega)}{[z_0^{III}-\omega]_+}|\omega\rangle_2\right),$$

$$|z_0^{IV}\rangle = N^{IV}\left(|1\rangle + \lambda_1\int_{a_1}^{\infty} d\omega \frac{f_1(\omega)}{z_0^{IV}-\omega}|\omega\rangle_1 + \lambda_2\int_{a_2}^{\infty} d\omega \frac{f_2(\omega)}{[z_0^{IV}-\omega]_+}|\omega\rangle_2\right),$$

Completeness relation

Continuous state:

$$|\Psi^d_{i\pm}(x)\rangle = |x\rangle_i + \frac{\lambda_i f_i^*(x)}{\eta_d^\pm(x)}\Big[|1\rangle + \sum_{j=1,2}\lambda_j\int_{a_j}^\infty \mathrm{d}\omega\frac{f_j(\omega)}{x-\omega\pm i\epsilon}|\omega\rangle_j\Big]$$

$$\langle\tilde{\Psi}_{i\pm}(x)| = {}_i\langle x| + \frac{\lambda_i f_i(x)}{\eta^\mp(x)}\Big[\langle 1| + \sum_{j=1,2}\lambda_j\int_{a_j}^\infty \mathrm{d}\omega\frac{f_j^*(\omega)}{x-\omega\mp i\epsilon}\ {}_j\langle\omega|\Big]$$

$$\eta_d^\pm(\omega) \equiv \eta^\pm(\omega)\prod_{J=II,III,IV}\prod_{i=1}^{N_J}\frac{\omega - z_i^J}{[\omega - z_i^J]_\pm}\,.$$

$$\sum_{i=1,2}\int_{a_i}^\infty dx|\Psi_i^d(x)\rangle\langle\tilde{\Psi}_i(x)| + \sum_{J,i}|z_{0,i}^J\rangle\langle\tilde{z}_{0,i}^J| = \mathbf{1}$$

Summary:

- Friedrichs model in single channel and coupled channel: exactly solvable model.
- Wave function for bound state, virtual state, and Resonances.
- Dynamically generated poles and generated from bare states.
- Completeness relation.
- Probability explanation?