Hadronic Light-by-Light Scattering Contribution to the Muon $g-2$

Andreas Nyffeler

Harish-Chandra Research Institute, Allahabad, India

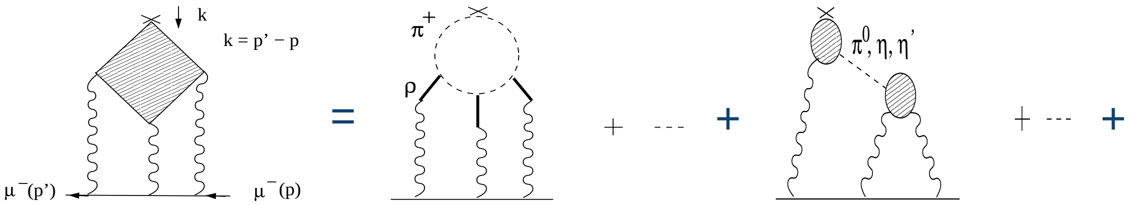
Largely based on:

- Nyffeler, Hadronic light-by-light scattering in the muon $g-2$: a new short-distance constraint on pion exchange, Phys. Rev. D 79, 073012 (2009), arXiv:0901.1172 [hep-ph]
- Jegerlehner, Nyffeler, The Muon $g-2$, Phys. Rept. 477, 1 (2009), arXiv:0902.3360 [hep-ph]

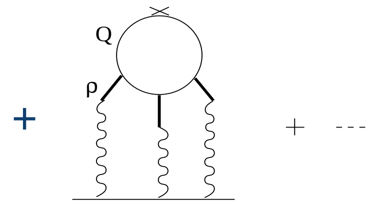
International Workshop on e^+e^- collisions from Phi to Psi
Institute of High Energy Physics, Chinese Academy of Sciences
Beijing, China
October 13-16, 2009

Had. LbyL scattering: Overview

Classification of contributions (de Rafael '94):



Exchange of other resonances $(f_0, a_1, \ldots)$



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

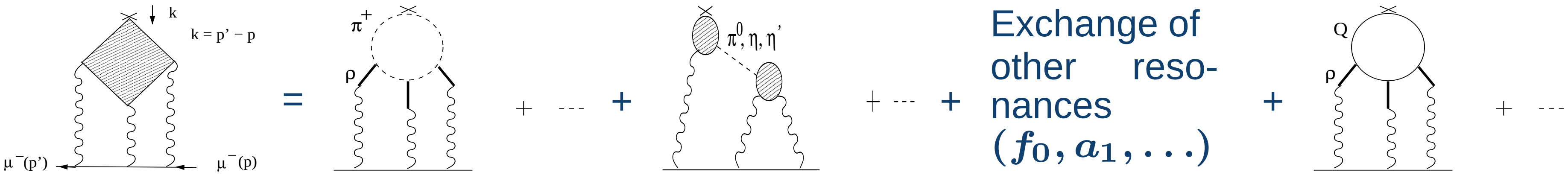
Relevant scales in $\langle VVVV \rangle$ (off-shell !): $\sim m_\mu - 2$ GeV. No direct relation to exp. data, in contrast to hadronic vacuum polarization in $g-2 \longrightarrow$ need hadronic (resonance) model

de Rafael '94: last term can be interpreted as irreducible contribution to 4-point function $\langle VVVV \rangle$. Appears as short-distance complement of low-energy hadronic models.

Reduce model dependence by imposing exp. and theor. constraints on form factors, e.g. from QCD short-distances (OPE) to get better matching with perturbative QCD for high momenta

Had. LbyL scattering: Overview

Classification of contributions (de Rafael '94):



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

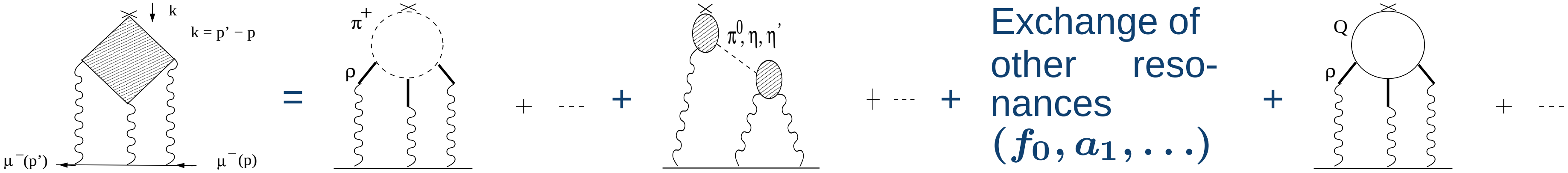
Relevant scales in $\langle VVVV \rangle$ (off-shell !): $\sim m_\mu - 2$ GeV. No direct relation to exp. data, in contrast to hadronic vacuum polarization in $g-2 \longrightarrow$ need hadronic (resonance) model

de Rafael '94: last term can be interpreted as irreducible contribution to 4-point function $\langle VVVV \rangle$. Appears as short-distance complement of low-energy hadronic models.

Reduce model dependence by imposing exp. and theor. constraints on form factors, e.g. from QCD short-distances (OPE) to get better matching with perturbative QCD for high momenta

- Evaluations of full had. LbyL scattering contribution:
  - Bijnens, Pallante, Prades '95, '96, '02
    Use mainly Extended Nambu-Jona-Lasinio (ENJL) model
  - Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02
    Use mainly Hidden Local Symmetry (HLS) model; often HLS = VMD
- Selected partial evaluations:
  - Knecht, Nyffeler '02: Use large-N_C QCD
  - Melnikov, Vainshtein '04: Use large-N_C QCD

Had. LbyL scattering: Summary of results



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

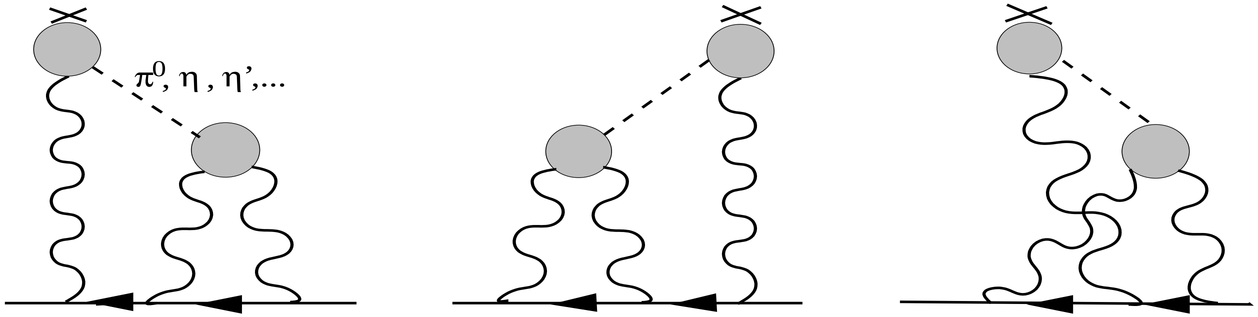
BPP: +83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
HKS: +90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) $[a_1]$	+10 (11)
KN: +80 (40)		+83 (12)		
MV: +136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007: +110 (40)				
PdRV:+105 (26)	-19 (19)	+114 (13)	+8 (12) $[f_0, a_1]$	2.3 [c-quark]
N,JN: +116 (40)	-19 (13)	+99 (16)	+15 (7) $[f_0, a_1]$	+21 (3)
	ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09; N,JN = Nyffeler '09; Jegerlehner, Nyffeler '09

- Prades, de Rafael, Vainshtein '09: New combination of existing results, added errors in quadrature. No dressed light quark loops ! Assumed to be taken into account by short-distance constraint of MV on pseudoscalar-pole contribution. Why should this be the case ?
- Nyffeler '09; Jegerlehner, Nyffeler '09: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on pion-exchange. Combined with MV (for axial-vectors) + BPP (rest of contributions). Added errors linearly. Too conservative ?

Pseudoscalar-exchange contribution to had. LbyL scattering



- Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$
- Numerically dominant contribution to had. LbyL scattering
- Exchange of lightest state π^0 yields largest contribution ⟶ warrants special attention
- Following Bijnens, Pallante, Prades '95, '96; Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, we can define off-shell form-factor for π^0 as follows:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2\cdot y)}\, \langle 0|T\{j_\mu(x) j_\nu(y) P^3(0)\}|0\rangle$$

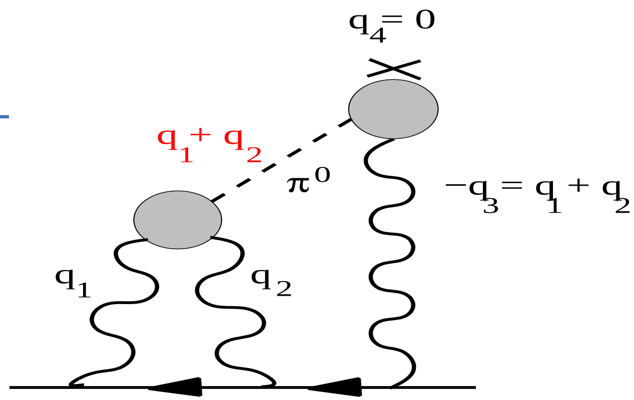
$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \frac{i\langle\overline{\psi}\psi\rangle}{F_\pi}\, \frac{i}{(q_1+q_2)^2 - m_\pi^2}\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

j_μ = light quark part of the electromagnetic current: $j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$, $\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $\hat{Q} = \mathrm{diag}(2, -1, -1)/3$

$P^3 = \overline{\psi} i\gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d\right)/2$, $\langle\overline{\psi}\psi\rangle$ = single flavor quark condensate

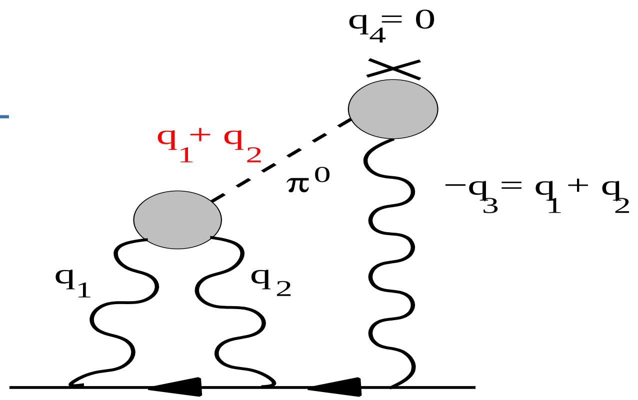
Off-shell versus on-shell form factors



- Off-shell form factors have been used to evaluate the pion-exchange contribution in BPP '96, HKS '96, HK '98, but this seems to have been forgotten later. "Rediscovered" by Jegerlehner in '07. Consider diagram:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

Off-shell versus on-shell form factors



- Off-shell form factors have been used to evaluate the pion-exchange contribution in BPP '96, HKS '96, HK '98, but this seems to have been forgotten later. "Rediscovered" by Jegerlehner in '07. Consider diagram:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

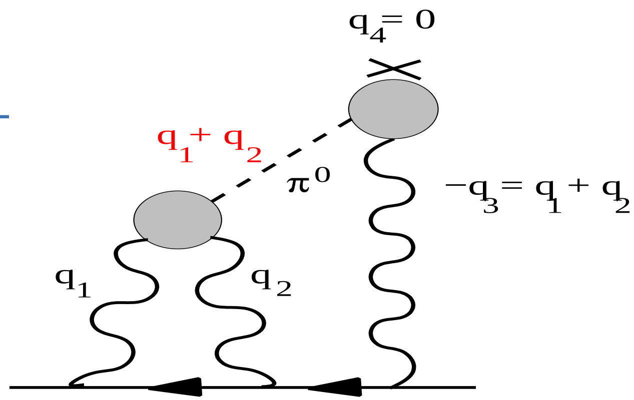
- On the other hand, Bijnens, Persson '01, Knecht, Nyffeler '02 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$

Off-shell versus on-shell form factors

- Off-shell form factors have been used to evaluate the pion-exchange contribution in BPP '96, HKS '96, HK '98, but this seems to have been forgotten later. “Rediscovered” by Jegerlehner in '07. Consider diagram:



$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

- On the other hand, Bijnens, Persson '01, Knecht, Nyffeler '02 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$

- Melnikov, Vainshtein '04 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the Wess-Zumino-Witten term

- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution ! In general, off-shell form factors will enter at both vertices.

- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell. Only in some specific model where pions appear as propagating fields can one identify the contribution from off-shell pions.

New short-distance constraint on form factor at external vertex

- Knecht, Nyffeler, EPJC '01: analysis of various short-distance constraints on $\langle VVP \rangle$ (chiral limit, octet symmetry), in particular:

$$\langle V \underbrace{VP}_{\text{OPE}} \rangle \to \langle VT \rangle \qquad \text{Vector-Tensor two-point function}$$

$$\delta^{ab} (\Pi_{\rm VT})_{\mu\rho\sigma}(p) = \int d^4x \, e^{ip\cdot x} \, \langle 0 | T\{ V_\mu^a(x) \, (\overline{\psi} \, \sigma_{\rho\sigma} \frac{\lambda^b}{2} \, \psi)(0) \} | 0 \rangle, \qquad \sigma_{\rho\sigma} = \frac{i}{2} [\gamma_\rho, \gamma_\sigma]$$

- New short-distance constraint on the off-shell form factor at the external vertex (Nyffeler '09):

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3} \chi + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad (*)$$

  where χ is the quark condensate magnetic susceptibility of QCD in the presence of a constant external electromagnetic field (Ioffe, Smilga '84):

$$\langle 0 | \bar{q} \sigma_{\mu\nu} q | 0 \rangle_F = e \, e_q \, \chi \, \langle \overline{\psi}\psi \rangle_0 \, F_{\mu\nu}, \qquad e_u = 2/3, \; e_d = -1/3$$

- Note that there is no falloff in OPE in $(*)$, unless χ vanishes !
- Corrections of $\mathcal{O}(\alpha_s)$ in OPE $\Rightarrow \chi$ depends on renormalization scale μ
- Unfortunately there is no agreement in the literature what the value of $\chi(\mu)$ should be ! Range of values from $\chi(\mu \sim 0.5 \text{ GeV}) \approx -9 \text{ GeV}^{-2}$ (Ioffe, Smilga '84; Vainshtein '03, ..., Narison '08) to $\chi(\mu \sim 1 \text{ GeV}) \approx -3 \text{ GeV}^{-2}$ (Balitsky, Yung '83; Ball et al. '03; ...; Ioffe '09). Running with μ cannot explain such a difference.

New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all QCD short-distance (OPE) constraints
- Reproduces Brodsky-Lepage behavior (confirmed by CLEO, but not by recent BABAR data):

$$\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$$

- Normalized to decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.6)$ eV

New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all QCD short-distance (OPE) constraints
- Reproduces Brodsky-Lepage behavior (confirmed by CLEO, but not by recent BABAR data):

$$\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$$

- Normalized to decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.6)$ eV

Off-shell LMD+V form factor (Knecht, Nyffeler, EPJC '01):

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)}$$

$$P_H^V(q_1^2, q_2^2, q_3^2) = h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4 + h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7, \qquad q_3^2 = (q_1 + q_2)^2$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV

We view our evaluation as being a part of a full calculation of the hadronic light-by-light scattering contribution using a resonance Lagrangian along the lines of the Resonance Chiral Theory (Ecker et al. '89, ...), which also fulfills all the relevant QCD short-distance constraints.

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}^{\rm LMD+V}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}^{\rm LMD+V}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4 / (4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to recent BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2, $h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$
with χ^2/dof $= 15.0/15 = 1.0$

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to recent BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2, $h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$
with χ^2/dof $= 15.0/15 = 1.0$

h_3, h_4, h_6 are unknown / less constrained:

- New short-distance constraint $\Rightarrow h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$ $(*)$
  LMD ansatz for $\langle VT \rangle \Rightarrow \chi^{\mathrm{LMD}} = -2/M_V^2 = -3.3$ GeV^{-2} (Balitsky, Yung '83)
  Close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} (Ball et al. '03)
  Assume large-N_C (LMD/LMD+V) framework is self-consistent
  $\Rightarrow \chi = -(3.3 \pm 1.1)$ GeV^{-2}
  $\Rightarrow$ vary $h_3 = (0 \pm 10)$ GeV^2 and determine h_4 from relation $(*)$ and vice versa
- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to h_6
  Assume that LMD/LMD+V estimates of low-energy constants from chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$ are self-consistent. Assume 100% error on estimate for the relevant, presumably small low-energy constant $\Rightarrow h_6 = (5 \pm 5)$ GeV^4

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i

- The h_i enter the LMD+V form factor linearly in the numerator, therefore (Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i\, \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\, \tilde{h}_i\, \tilde{h}_j\right]$$

  with dimensionless coefficients $c_i, c_{ij} \sim 10^{-4}$ (see Nyffeler '09 for the values), if we measure the h_i in appropriate units of GeV $\longrightarrow \tilde{h}_i$.

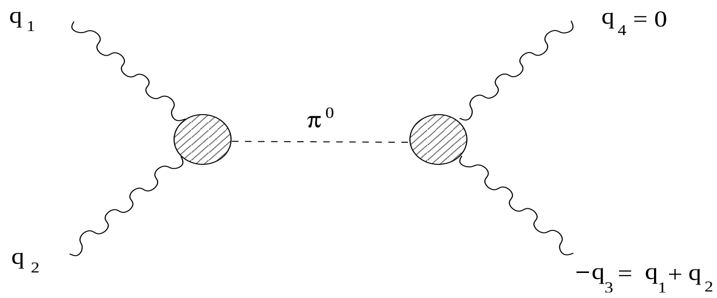
  h_1, h_3, h_4 not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2\, M_{V_2}^2\, \chi$, because of the new short-distance constraint.

- h_1, h_2, h_5, h_7 are quite well known $\longrightarrow$ can write down a simplified expression with only h_3, h_4, h_6 as free parameters (up to constraint):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \Big[\, 503.3764 - 6.5223\, \tilde{h}_3 - 5.0962\, \tilde{h}_4 + 7.8557\, \tilde{h}_6$$
$$+0.3017\, \tilde{h}_3^2 + 0.5683\, \tilde{h}_3\, \tilde{h}_4 - 0.1747\, \tilde{h}_3\, \tilde{h}_6$$
$$+0.2672\, \tilde{h}_4^2 - 0.1411\, \tilde{h}_4\, \tilde{h}_6 + 0.0642\, \tilde{h}_6^2 \,\Big] \times 10^{-4}$$

The short-distance constraint by Melnikov and Vainshtein

- Melnikov, Vainshtein ’04 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \stackrel{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Longrightarrow} \langle AV|\gamma\rangle$$

- From this they deduced for the LbyL scattering amplitude (for finite q_1^2, q_2^2):

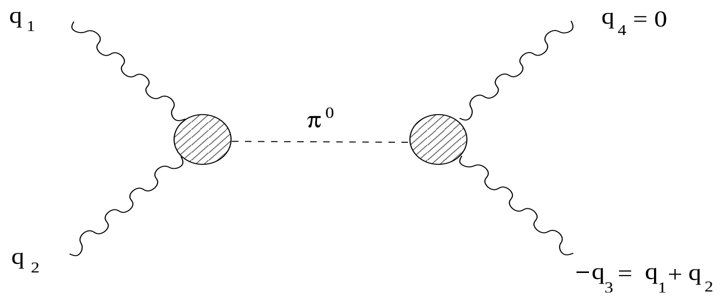
$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu} \tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ = field strength tensors of internal photons with polarization vectors ϵ_i, for external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$.

- From the expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ it is again obvious that Melnikov and Vainshtein only consider the pion-pole contribution !
- No 2nd form factor at ext. vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by constant $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0)$!

The short-distance constraint by Melnikov and Vainshtein

- Melnikov, Vainshtein '04 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \stackrel{q_1^2\sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

- From this they deduced for the LbyL scattering amplitude (for finite q_1^2, q_2^2):

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu} \tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ = field strength tensors of internal photons with polarization vectors ϵ_i, for external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$.

- From the expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ it is again obvious that Melnikov and Vainshtein only consider the pion-pole contribution !
- No 2nd form factor at ext. vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by constant $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0)$!
- Overall $1/q_3^2$ behavior for large q_3^2 (apart from $f_3^{\sigma\rho}$). MV '04: agrees with quark-loop !
- For our off-shell LMD+V form factor at external vertex we get for large q_3^2:

$$\frac{3}{F_\pi} \mathcal{F}^{\text{LMD+V}}_{\pi^{0*}\gamma^*\gamma}(q_3^2, q_3^2, 0) \stackrel{q_3^2\to\infty}{\to} \frac{h_1 + h_3 + h_4}{M_{V_1}^2 M_{V_2}^2} = \frac{2c_{VT}}{M_{V_1}^2 M_{V_2}^2} = \chi$$

With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with MV '04 !
Note: for large-N_C only the sum of all resonance exchanges has to match with quark-loop !

New estimate for pseudoscalar-exchange contribution

- π^0
  - Our new estimate (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

    With off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ which obeys new short-distance constraint.
  - Largest uncertainty from $h_6 = (5 \pm 5)$ GeV^4 $\Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

    If we would vary $h_6 = (0 \pm 10)$ GeV^4 $\Rightarrow \pm 12 \times 10^{-11}$!
  - Varying $\chi = -(3.3 \pm 1.1)$ GeV^{-2} $\Rightarrow \pm 2.1 \times 10^{-11}$

    Exact value of χ not that important, but range does not include Vainshtein's estimate $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$ GeV^{-2}
  - Varying $h_3 = (0 \pm 10)$ GeV^2 $\Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$)
  - With h_1, h_5 from fit to recent BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11} \longrightarrow$ result unchanged !

New estimate for pseudoscalar-exchange contribution

- π^0
  - Our new estimate (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

    With off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ which obeys new short-distance constraint.
  - Largest uncertainty from $h_6 = (5 \pm 5)$ GeV^4 $\Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

    If we would vary $h_6 = (0 \pm 10)$ GeV^4 $\Rightarrow \pm 12 \times 10^{-11}$!
  - Varying $\chi = -(3.3 \pm 1.1)$ GeV^{-2} $\Rightarrow \pm 2.1 \times 10^{-11}$

    Exact value of χ not that important, but range does not include Vainshtein's estimate $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$ GeV^{-2}
  - Varying $h_3 = (0 \pm 10)$ GeV^2 $\Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$)
  - With h_1, h_5 from fit to recent BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11}$ $\longrightarrow$ result unchanged !
- η, η'
  - Short-distance analysis of LMD+V form factor in Knecht, Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
  - Simplified approach: VMD form factors normalized to decay width Γ(PS $\longrightarrow \gamma\gamma$).

$$\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}^{\mathrm{VMD}}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

  - $\Rightarrow a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$

    Not taking pole-approximation as done in Melnikov, Vainshtein '04 !

    Note: VMD form factor has too strong damping at large momenta $\longrightarrow$ values might be a bit too small !
- Our estimate for the sum of all light pseudoscalars (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

Pseudoscalar exchanges: results in the literature

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
modified ENJL (off-shell) [BPP]	59(9)	85(13)
VMD / HLS (off-shell) [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	—
LMD+V (off-shell) [N]	72(12)	99(16)
AdS/QCD (off-shell ?) [HoK]	69	107
[PdRV]	—	114(13)
[JN]	72(12)	99(16)

BPP = Bijnens, Pallante, Prades '95, '96, '02 (ENJL = Extended Nambu-Jona-Lasinio model); HK(S) = Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02 (HLS = Hidden Local Symmetry model); KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; DB = Dorokhov, Broniowski '08 (χQM = Chiral Quark Model); N = Nyffeler '09; HoK = Hong, Kim '09; PdRV = Prades, de Rafael, Vainshtein '09; JN = Jegerlehner, Nyffeler '09

- BPP use rescaled VMD result for η, η'. Also all LMD+V evaluations use VMD for η, η' !
- Off-shell form factors used in BPP, HKS presumably do not fulfill new short-distance constraint at external vertex and might have too strong damping $\longrightarrow$ smaller values.
- Our result for pion with off-shell form factors at both vertices is not too far from value given by MV '04, but this is pure coincidence ! Approaches not comparable ! MV '04 evaluate pion-pole contribution and use on-shell form factors (constant form factor at external vertex).
  Note: Following MV '04 and using $h_2 = -10$ GeV^2 we obtain 79.8×10^{-11} for the pion-pole contribution, close to 79.6×10^{-11} given in Bijnens, Prades '07 and 79.7×10^{-11} in DB '08
- Nonlocal χQM: strong damping for off-shell pions. AdS/QCD: error estimated to be $< 30\%$.

Axial-vector exchanges

Model for $\mathcal{F}_{A^*\gamma^*\gamma^*}$	$a_\mu(a_1) \times 10^{11}$	$a_\mu(a_1, f_1, f_1') \times 10^{11}$
ENJL-VMD [BPP] (nonet symmetry)	2.5(1.0)	—
ENJL-like [HKS,HK] (nonet symmetry)	1.7(1.7)	—
LMD [MV] (f_1 pure octet, f_1' pure singlet)	5.7	17
LMD [MV] (ideal mixing)	5.7	22(5)
[PdRV]	—	15(10)
[JN]	—	22(5)

- MV ’04: derived QCD short-distance constraint for axial-vector pole contribution with on-shell form factor $\mathcal{F}_{A\gamma^*\gamma^*}$ at both vertices
- Simple VMD ansatz: short-distance constraints forbids form factor at external vertex. Assuming all axial-vectors in the nonet have same mass M leads to

$$a_\mu^{AV} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{M^2} N_C \mathrm{Tr}\left[\hat{Q}^4\right] \left(\frac{71}{192} + \frac{81}{16} S_2 - \frac{7\pi^2}{144}\right) + \ldots \approx 1010 \frac{m_\mu^2}{M^2} \times 10^{-11}$$

($\hat{Q} = \mathrm{diag}(2/3, -1/3, -1/3), \quad S_2 = 0.26043$)

Strong dependence on mass M:

$M = 1300$ MeV: $a_\mu^{AV} = 7 \times 10^{-11}$, $\qquad M = M_\rho$: $a_\mu^{AV} = 28 \times 10^{-11}$ (with $+\ldots$)

- More sophisticated LMD ansatz (Czarnecki, Marciano, Vainshtein ’03): see Table. Now there is form factor at external vertex. Dressing leads to lower effective mass M. Furthermore f_1, f_1' have large coupling to photons $\longrightarrow$ huge enhancement compared to BPP, HKS !

Scalar exchanges

Model for $\mathcal{F}_{S^*\gamma^*\gamma^*}$	a_μ(scalars) $\times\ 10^{11}$
Point coupling	$-\infty$
ENJL [BPP]	–7(2)
[PdRV]	–7(7)
[JN]	–7(2)

- Within ENJL model: scalar exchange contribution related by Ward identities to (constituent) quark loop $\longrightarrow$ HK argued that effect of (broad) scalar resonances below several hundred MeV might already be included in sum of (dressed) quark loops and (dressed) $\pi + K$ loops !
- Potential double-counting is definitely an issue for the broad sigma meson $f_0(600)$ ($\leftrightarrow \pi^+\pi^-; \pi^0\pi^0$). Ongoing debate whether the scalar resonances $f_0(980), a_0(980)$ are two-quark or four-quark states.
- It is not clear which scalar resonances are described by ENJL model. Model parameters fixed by fitting various low-energy observables and resonance parameters, among them $M_S = 980$ MeV. However, model then yields $M_S^{\mathrm{ENJL}} = 620$ MeV.
- Can the usually broad scalar resonances be described by a simple resonance Lagrangian which works best in large-N_C limit, i.e. for very narrow states ?

Charged pion and kaon loops

Model $\pi^+\pi^-\gamma^*(\gamma^*)$	$a_\mu(\pi^\pm)\times 10^{11}$	$a_\mu(\pi^\pm, K^\pm)\times 10^{11}$
Point coupling (scalar QED)	-45.3	-49.8
VMD [KNO, HKS]	-16	$-$
full VMD [BPP]	$-18(13)$	$-19(13)$
HLS [HKS,HK]	-4.45	$-4.5(8.1)$
[MV] (all N_C^0 terms !)	$-$	$0(10)$
[PdRV]	$-$	$-19(19)$
[JN]	$-$	$-19(13)$

- Dressing leads to a rather huge suppression compared to scalar QED ! Very model dependent.
- MV '04 studied HLS model via expansion in $(m_\pi/M_\rho)^2$ and $(m_\mu - m_\pi)/m_\pi$:

$$a_{\mu;\mathrm{HLS}}^{\mathrm{LbL};\pi^\pm} = \left(\frac{\alpha}{\pi}\right)^3 \sum_{i=0}^{\infty} f_i\left[\frac{m_\mu - m_\pi}{m_\pi}, \ln\left(\frac{M_\rho}{m_\pi}\right)\right]\left(\frac{m_\pi^2}{M_\rho^2}\right)^i = \left(\frac{\alpha}{\pi}\right)^3 (-0.0058)$$
$$= (-46.37 + 35.46 + 10.98 - 4.70 - 0.3 + \ldots) \times 10^{-11} = -4.9(3) \times 10^{-11}$$

- Large cancellation between first three terms in series. Expansion converges only very slowly. Main reason: typical momenta in the loop integral are of order $\mu = 4m_\pi \approx 550$ MeV and the effective expansion parameter is μ/M_ρ, not m_π/M_ρ.
- MV '04: Final result is very likely suppressed, but also very model dependent $\longrightarrow$ chiral expansion looses predictive power $\longrightarrow$ lumped together all terms subleading in N_C.

Dressed quark loops

Model	$a_\mu(\text{quarks}) \times 10^{11}$
Point coupling	$62(3)$
ENJL + bare heavy quark [BPP]	$21(3)$
VMD [HKS, HK]	$9.7(11.1)$
[PdRV] (Bare c-quark only !)	2.3
[JN]	$21(3)$

- de Rafael '94: dressed quark loops can be interpreted as irreducible contribution to the 4-point function $\langle VVVV \rangle$. They also appear as short-distance complement of low-energy hadronic models.
- Quark-hadron duality: the quark loops also model contributions from exchanges and loops of heavier hadronic states, like $\pi', a_0', f_0', p, n, \ldots$.
- Again very large model-dependent effect of the dressing (form factors).
- Recently, PdRV '09 argued that the dressed light-quark loops should not be included as separate contribution. They assume them to be already covered by using the short-distance constraint from MV '04 for the pseudoscalar-pole contribution. Why should this be the case ?

Conclusions

- Jegerlehner '07: one should use off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ to evaluate pion-exchange contribution. As done in earlier papers by BPP, HKS, HK !
  Prescription by Melnikov, Vainshtein '04 to use a constant WZW form factor at the external vertex only yields pion-pole contribution with on-shell form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$.
- New short-distance constraint on off-shell form factor at external vertex (Nyffeler '09):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3}\chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

[$\chi =$ chiral condensate magnetic susceptibility]

- New evaluation of pion-exchange contribution within large-N_C approximation using off-shell LMD+V form factor that fulfills all QCD short-distance constraints (Nyffeler '09):

$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

[BPP: 59 ± 9; HKS: 57 ± 4; KN: 58 ± 10; MV: 77 ± 7 in units of 10^{-11}]

- Updated values for η and η' (using simple VMD form factors):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

[BPP: 85 ± 13; HKS: 83 ± 6; KN: 83 ± 12; MV: 114 ± 10 in units of 10^{-11}]

- Combined with evaluations of the other contributions we get:

$$a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11}$$

[PdRV: $(105 \pm 26) \times 10^{-11}$]

- Corresponding contributions for the electron (Nyffeler '09, Jegerlehner, Nyffeler '09):

$$a_e^{\mathrm{LbyL};\pi^0} = (2.98 \pm 0.34) \times 10^{-14}, \quad a_e^{\mathrm{LbyL};\eta} = 0.49 \times 10^{-14}, \quad a_e^{\mathrm{LbyL};\eta'} = 0.39 \times 10^{-14}$$

$$a_e^{\mathrm{LbyL;PS}} = (3.9 \pm 0.5) \times 10^{-14}$$

$$a_e^{\mathrm{LbyL;had}} = (3.9 \pm 1.3) \times 10^{-14}$$

[Guesstimate ! Jegerlehner, Nyffeler '09; agrees with $(3.5 \pm 1.0) \times 10^{-14}$ by PdRV]

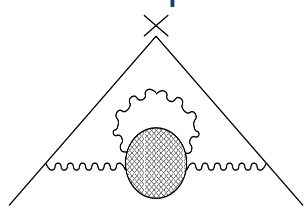
Note: naive rescaling would yield a too small result: $a_e^{\mathrm{LbyL};\pi^0}(\mathrm{rescaled}) = (m_e/m_\mu)^2 \, a_\mu^{\mathrm{LbyL};\pi^0} = 1.7 \times 10^{-14}$!

Outlook on had. LbyL scattering

- If we want to fully profit from a potential future $g - 2$ experiment with error of $\sim 15 \times 10^{-11}$, we need to better control the hadronic LbyL scattering contribution !
- Some progress made in recent years for pseudoscalars and axial-vector contributions, implementing many experimental and theoretical constraints. More work needed for η, η' !
- More uncertainty for exchanges of scalars (and heavier resonances) and for (dressed) pion + kaon loop and (dressed) quark loops. Furthermore, there are some cancellations.
- Soon results from Lattice QCD ? $\langle VVVV \rangle$ needs to be integrated over phase space of 3 off-shell photons $\longrightarrow$ much more complicated than hadronic vacuum polarization !

Suggested way forward in the meantime:

- Important to have unified consistent framework (model) which deals with all contributions.
- Purely phenomenological approach: resonance Lagrangian where all couplings are fixed from experiment. Non-renormalizable Lagrangian: how to achieve matching with pQCD ?
- Large-N_C framework: matching Green's functions with QCD short-distance constraints.
- In both approaches: experimental information on various on-shell and off-shell hadronic form factors would be very helpful. e^+e^- colliders running around 1-2 GeV could help to measure some of these hadronic form factors.
- Test models for had. LbyL scattering by comparison with exp. results for higher order contributions to had. vacuum polarization:



Backup slides

Further results for the pion-exchange contribution (Nyffeler ’09)

$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor:

	$h_6 = 0$ GeV^4	$h_6 = 5$ GeV^4	$h_6 = 10$ GeV^4
$h_3 = -10$ GeV^2	68.4	74.1	80.2
$h_3 = 0$ GeV^2	66.4	71.9	77.8
$h_3 = 10$ GeV^2	64.4	69.7	75.4
$h_4 = -10$ GeV^2	65.3	70.7	76.4
$h_4 = 0$ GeV^2	67.3	72.8	78.8
$h_4 = 10$ GeV^2	69.2	75.0	81.2

$\chi = -3.3$ GeV^{-2}, $h_1 = 0$ GeV^2, $h_2 = -10.63$ GeV^2 and $h_5 = 6.93$ GeV$^4 - h_3 m_\pi^2$

When varying h_3 (upper half of table), h_4 is fixed by constraint $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$.
In the lower half the procedure is reversed.

Within scanned region:
Minimal value: 63.2×10^{-11} [$\chi = -2.2$ GeV^{-2}, $h_3 = 10$ GeV^2, $h_6 = 0$ GeV^4]
Maximum value: 83.3×10^{-11} [$\chi = -4.4$ GeV^{-2}, $h_4 = 10$ GeV^2, $h_6 = 10$ GeV^4]

Take average of results for $h_6 = 5$ GeV^4 for $h_3 = 0$ GeV^2 and $h_4 = 0$ GeV^2 as estimate:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

Added errors from χ, h_3 (or h_4) and h_6 linearly. Do not follow Gaussian distribution !

Hadronic light-by-light scattering in the muon $g-2$

Some selected results for the various contributions to $a_\mu^{\mathrm{LbyL;had}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	$-$	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	$-$	22 ± 5	$-$	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	$-$	$-$	$-$	$-$	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	$-$	$-$	$-$	-19 ± 19	-19 ± 13
π, K loops $+$subl. N_C	$-$	$-$	$-$	0 ± 10	$-$	$-$	$-$
quark loops	21 ± 3	9.7 ± 11.1	$-$	$-$	$-$	2.3	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09

- Pseudoscalar-exchange contribution dominates numerically. But other contributions are not negligible. Note cancellation between π, K-loops and quark loops !
- $(80 \pm 40) \times 10^{-11}$ not in KN '02; estimate used by Marseille group before MV '04.
- PdRV: Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '04 on pseudoscalar-pole contribution. Why should this be the case ?
  Added all errors in quadrature ! Like HK(S). Too optimistic ?
- N, JN: Evaluation of the axial vectors by MV '04 is definitely some improvement over earlier calculations. It seems, however, again to be only the axial-vector pole contribution.
  Added all errors linearly. Like BPP, MV, BP, MdRR. Too pessimistic ?

Integral representation for pion-exchange contribution

Projection onto the muon $g-2$ leads to (Knecht, Nyffeler '02):

$$a_\mu^{\mathrm{LbyL};\pi^0} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

$$\times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2 - m_\pi^2} T_1(q_1, q_2; p) \right.$$

$$\left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} T_2(q_1, q_2; p) \right]$$

$$T_1(q_1, q_2; p) = \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2$$
$$- \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + 8(p\cdot q_2) q_1^2 q_2^2 - \frac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2$$
$$+ \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1\cdot q_2)^2$$

$$T_2(q_1, q_2; p) = \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_1)^2 q_2^2$$
$$+ \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + \frac{8}{3}(p\cdot q_1) q_1^2 q_2^2$$
$$+ \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1\cdot q_2)^2$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

Jegerlehner, Nyffeler '09: could perform non-trivial integrations over angles $P\cdot Q_1, P\cdot Q_2$ (in Euclidean space) $\longrightarrow$ 3-dimensional integral representation for general form factors !
Integration variables: Q_1^2, Q_2^2 and angle θ between Q_1 and Q_2: $Q_1\cdot Q_2 = |Q_1||Q_2|\cos\theta$

Estimates for the quark condensate magnetic susceptibility χ

Authors	Method	$\chi(\mu)$ [GeV]$^{-2}$	Footnote
Ioffe, Smilga '84	QCD sum rules	$\chi(\mu = 0.5 \text{ GeV}) = -\left(8.16^{+2.95}_{-1.91}\right)$	[1]
Narison '08	QCD sum rules	$\chi = -(8.5 \pm 1.0)$	[2]
Vainshtein '03	OPE for $\langle VVA \rangle$	$\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$	[3]
Gorsky, Krikun '09	AdS/QCD	$\chi = -(2.15 N_C)/(8\pi^2 F_\pi^2) = -9.6$	[4]
Dorokhov '05	Instanton liquid model	$\chi(\mu \sim 0.5 - 0.6 \text{ GeV}) = -4.32$	[5]
Ioffe '09	Zero-modes of Dirac operator	$\chi(\mu \sim 1 \text{ GeV}) = -3.52\ (\pm 30 - 50\%)$	[6]
Buividovich et al. '09	Lattice	$\chi = -1.547(6)$	[7]
Balitsky, Yung '83	LMD for $\langle VT \rangle$	$\chi = -2/M_V^2 = -3.3$	[8]
Belyaev, Kogan '84	QCD sum rules for $\langle VT \rangle$	$\chi(0.5 \text{ GeV}) = -(5.7 \pm 0.6)$	[9]
Balitsky et al. '85	QCD sum rules for $\langle VT \rangle$	$\chi(1 \text{ GeV}) = -(4.4 \pm 0.4)$	[9]
Ball et al. '03	QCD sum rules for $\langle VT \rangle$	$\chi(1 \text{ GeV}) = -(3.15 \pm 0.30)$	[9]

[1]: QCD sum rule evalation of nucleon magnetic moments.

[2]: Recent reanalysis of these sum rules for nucleon magnetic moments. At which scale μ ?

[3]: Probably at low scale $\mu \sim 0.5$ GeV, since pion dominance was assumed in derivation.

[4]: From derivation in holographic model it is not clear what is the relevant scale μ.

[5]: The scale is set by the inverse average instanton size ρ^{-1}.

[6]: Study of zero-mode solutions of Dirac equation in presence of arbitrary gluon fields (à la Banks-Casher).

[7]: Again à la Banks-Casher. Quenched lattice calculation for $SU(2)$. μ dependence is not taken into account. Lattice spacing corresponds to 2 GeV.

[8]: The leading short-distance behavior of Π_{VT} is given by (Craigie, Stern '81)

$$\lim_{\lambda\to\infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2}\frac{\langle\overline{\psi}\psi\rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

Assuming that the two-point function Π_{VT} is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies this OPE constraint leads to the ansatz (Balitsky, Yung '83, Belyaev, Kogan '84, Knecht, Nyffeler, EPJC '01)

$$\Pi_{VT}^{LMD}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{1}{p^2 - M_V^2} \quad \Rightarrow \quad \chi^{LMD} = -\frac{2}{M_V^2} = -3.3 \text{ GeV}^{-2}$$

Not obvious at which scale. Maybe $\mu = M_V$ as for low-energy constants in ChPT.

[9]: LMD estimate later improved by taking more resonance states $\rho', \rho'', \ldots$ in QCD sum rule analysis of $\langle VT \rangle$.

Note that the last value by Ball et al. is very close to original LMD estimate !

Constraining the LMD+V model parameter h_6

- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to value of h_6. We can get some indirect information on size and sign of h_6 as follows.
- Estimates of low-energy constants in chiral Lagrangians via exchange of resonances work quite well. However, we may get some corrections, if we consider the exchange of heavier resonances as well. Typically, a large-N_C error of 30% can be expected.
- In $\langle VVP \rangle$ appear 2 combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$, denoted by A_{V,p^2} and $A_{V,(p+q)^2}$ in Knecht, Nyffeler, EPJC '01.

$$A_{V,p^2}^{\mathrm{LMD}} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11 \, \frac{10^{-4}}{F_\pi^2}$$

$$A_{V,p^2}^{\mathrm{LMD+V}} = \frac{F_\pi^2}{8M_{V_1}^4}\frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2}\left(1 + \frac{M_{V_1}^2}{M_{V_2}^2}\right) = -1.36 \, \frac{10^{-4}}{F_\pi^2}$$

The relative change is only about 20%, well within expected large-N_C uncertainty !

$$A_{V,(p+q)^2}^{\mathrm{LMD}} = -\frac{F_\pi^2}{8M_V^4} = -0.26 \, \frac{10^{-4}}{F_\pi^2}, \qquad A_{V,(p+q)^2}^{\mathrm{LMD+V}} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4} h_6$$

Note that $A_{V,(p+q)^2}^{\mathrm{LMD}}$ is "small" compared to A_{V,p^2}^{LMD}. About same size as absolute value of the shift in A_{V,p^2} when going from LMD to LMD+V !

- Assuming that LMD/LMD+V framework is self-consistent, but allowing for a 100% uncertainty of $A_{V,(p+q)^2}^{\mathrm{LMD}}$, we get the range $h_6 = (5 \pm 5)$ GeV^4

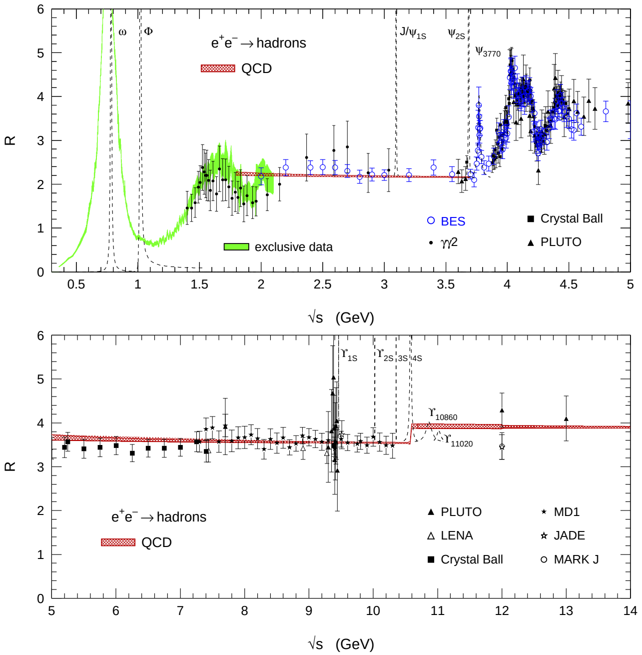
The large-N_C QCD world

Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

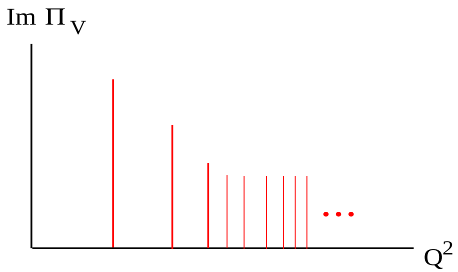
- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of Vector Meson Dominance (VMD))

Example: 2-point function $\langle VV \rangle \longrightarrow$ spectral function Im$\Pi_V \sim \sigma(e^+e^- \longrightarrow$ hadrons)

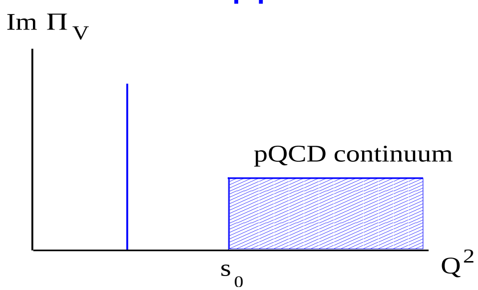
Real world (Davier et al., '03)



Large-N_C QCD ('t Hooft '74)



Minimal Hadronic Approximation



Scale s_0 fixed by the OPE