Hydrodynamics in Heavy Ion Collisions

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BROOKHAVEN
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December 2016

Collective Flows and Hydrodynamics in High Energy Nuclear Collisions
Department of Modern Physics, University of Science and Technology of China, Anhui Hefei

Introduction, Outline, Literature

- Introduction to heavy-ion collisions
  and relativistic viscous hydrodynamics

  Motiviation and deriving the relativistic equations relevant for heavy-ion collisions

  P. Romatschke *New Developments in Relativistic Viscous Hydrodynamics* (arXiv:0902.3663)

  S. Jeon, U. Heinz *Introduction to Hydrodynamics*, Int.J.Mod.Phys. E24 (2015) 10, 1530010

  C. Gale, S. Jeon, B. Schenke *Hydrodynamic Modeling of Heavy-Ion Collisions*, Int.J.Mod.Phys. A28 (2013) 1340011

- Hydrodynamic description of heavy-ion collisions

  Relevant references will appear in the lecture.

Motivation

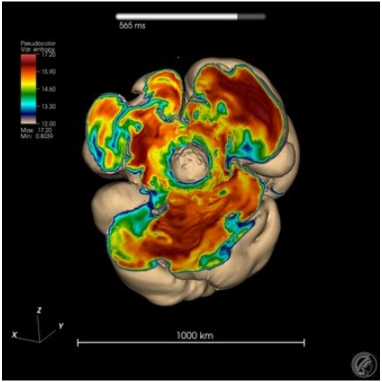
The main application of viscous relativistic hydrodynamics is the description of the time evolution of heavy ion collisions

Other applications of (usually ideal) relativistic hydrodynamics include

1. The collapse of the core of a massive star in the course of a supernova explosion: During its fall, the matter can reach velocities up to 40 percent of the speed of light.
2. Jets - situations in which matter flows onto a compact body, and some of the matter is flung away in a pair of tightly focused beams
3. Gamma-ray bursts: Theoretical models estimate that the matter responsible for the gamma-ray burst emission must be travelling at more than 99.99% of the speed of light

I will focus on heavy ion collisions because they demand the most detailed understanding of relativistic hydrodynamics, in particular the inclusion of viscosities

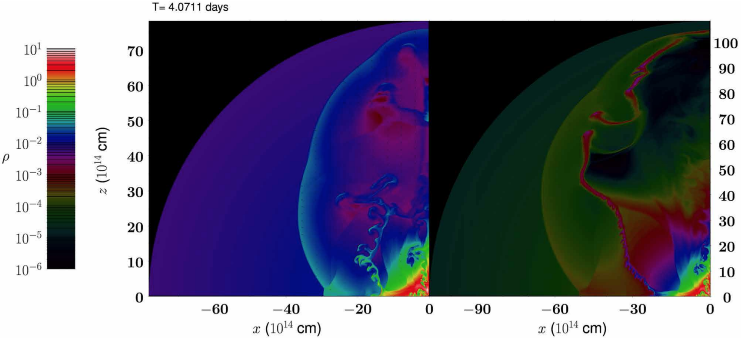
Motivation



Neutrino-driven explosion of a massive star

Simulation: F. Hanke, A. Marek, B. Müller, H.-Th. Janka (MPI for Astrophysics)
Simulation Code: PROMETHEUS (3D Hydrodynamics)
`http://www.mpcdf.mpg.de/services/visualization/rzgprojects`

Motivation

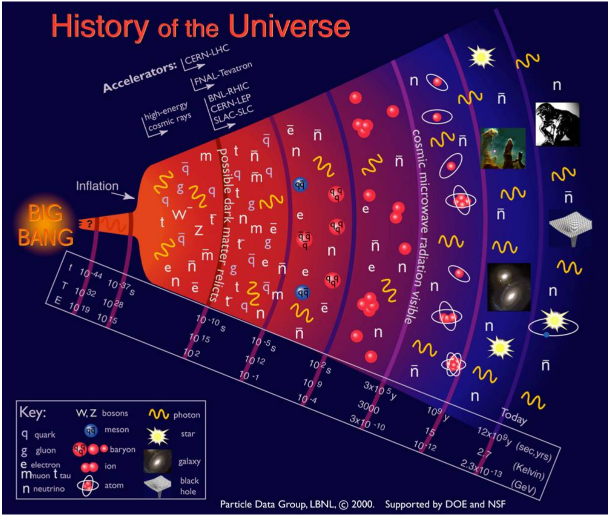


Simulation: C. Cuesta-Martinez, M. A. Aloy, and P. Mimica

`http://arxiv.org/pdf/1408.1305v2.pdf`

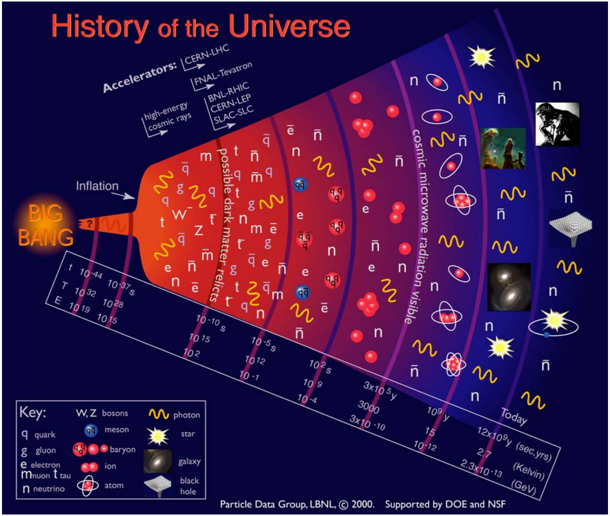
Gamma-ray burst - distribution of the rest-mass density

Evolution of the Universe



The universe gets cooler!

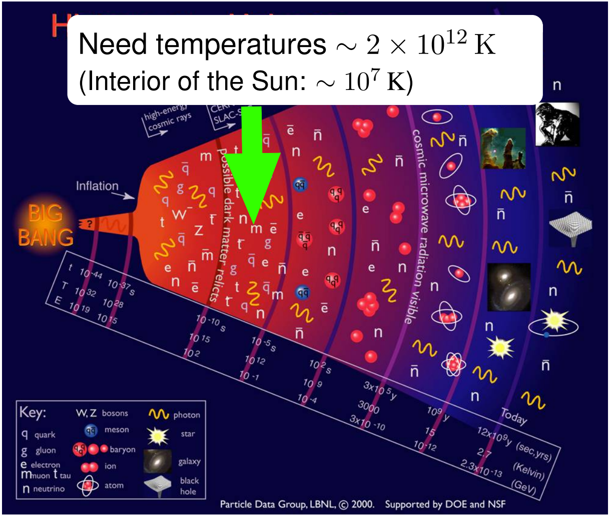
Evolution of the Universe



Reheating matter?

Evolution of the Universe

Need temperatures $\sim 2 \times 10^{12}\,\mathrm{K}$
(Interior of the Sun: $\sim 10^{7}\,\mathrm{K}$)



Reheating matter?

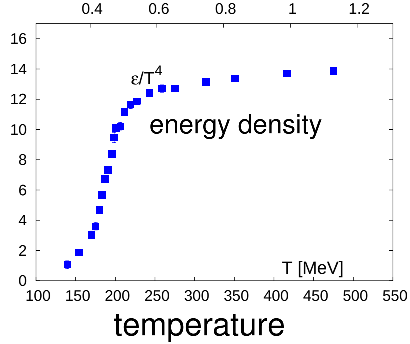
Recreating the big bang on earth (almost)

- We want to recreate the quark-gluon plasma that existed $10^{-6}\,\mathrm{s}$ after the big bang
- Test the theory of Quantum-Chromo-Dynamics (QCD) and understand fundamental properties of matter
- Application to early universe studies

Build a bridge between fundamental theory and experiment
to learn about fundamental matter and its interactions.

QCD tells us: High temperature → deconfined quarks and gluons

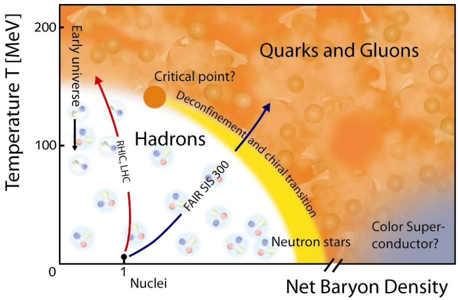
Lattice calculation - rapid crossover



HotQCD collaboration

Nucl.Phys.A830:725C-728C (2009)

Phase diagram of strongly interacting matter



GSI: www.gsi.de

How to create a deconfined state of matter

Remember, we need to create temperatures of the order of $2 \times 10^{12}\,\mathrm{K}$
The speed of light is very large ($3 \cdot 10^{8}\,\mathrm{m/s}$), so let's use

$$E = mc^2$$

Neither fusion nor fission are enough... (hydrogen bomb: $4.5 \times 10^{7} K$)
What to do?
When *not* at rest, we have

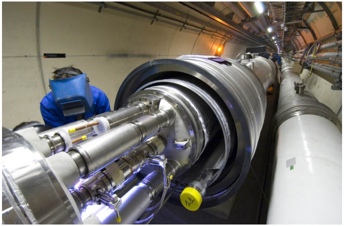
$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \gg mc^2 \qquad \text{if } v \sim c$$

ACCELERATE!

Little bang machines

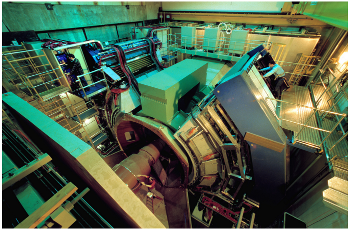
RHIC

LHC

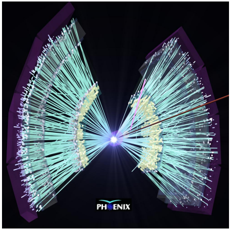


How to see the little bang (RHIC)

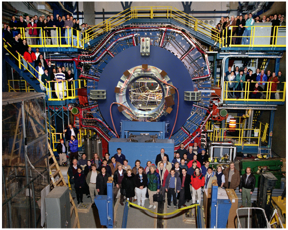
PHENIX detector



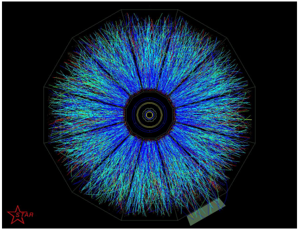
Event in PHENIX



STAR detector

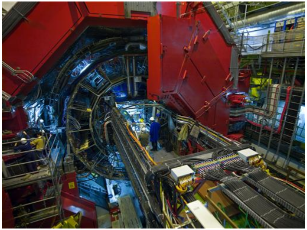


Event in STAR

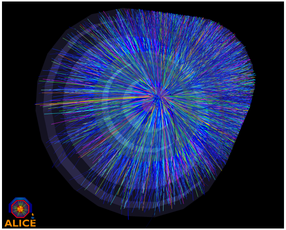


How to see the little bang (LHC)

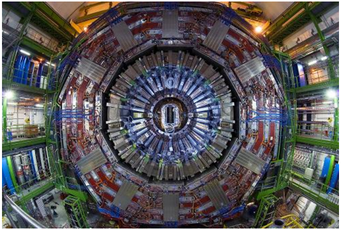
ALICE detector



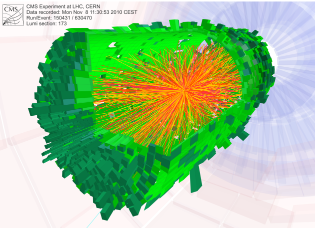
Event in ALICE



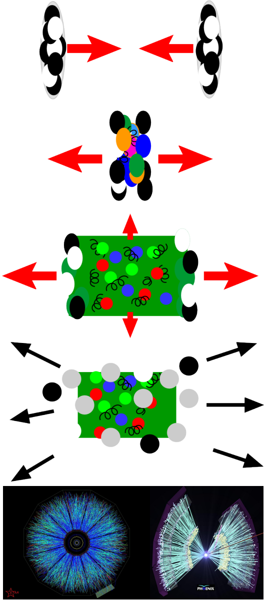
CMS detector



Event in CMS



A heavy-ion collision



before collision — initial state (e.g. color glass condensate)

0 fm/c

pre-equilibrium — thermalization (glasma state)

~ 0.5 fm/c

quark-gluon-plasma — Hydrodynamics, Jet quenching, ...

$\sim 3 - 5$ fm/c

hadronization — Hydrodynamics

hadr.rescattering — Hadronic transport

~ 10 fm/c

freeze-out — compare theory to experiment

detection

Why use hydrodynamics?

Observations at the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) have shown that the system created in heavy-ion collisions behaves like a nearly perfect fluid (not like a gas, as expected).

A perfect fluid is described by ideal hydrodynamics.
A nearly perfect fluid is described by viscous hydrodynamics with a small viscosity to entropy density ratio.

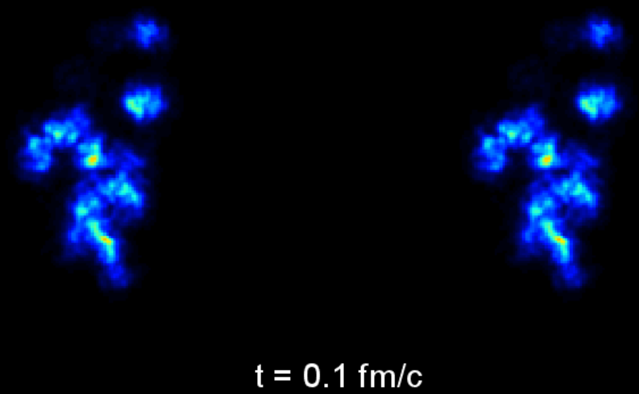
Comparison of ideal relativistic hydrodynamics with experimental data showed very good agreement.

Hydrodynamics

- Hydrodynamics is the conservation of energy and momentum.
- The mean free path has to be shorter than the lengthscales of interest.
- To describe a significant part of the system with hydrodynamics, it needs to be strongly coupled.

Weakly coupled vs. strongly coupled system

shown are energy density distributions

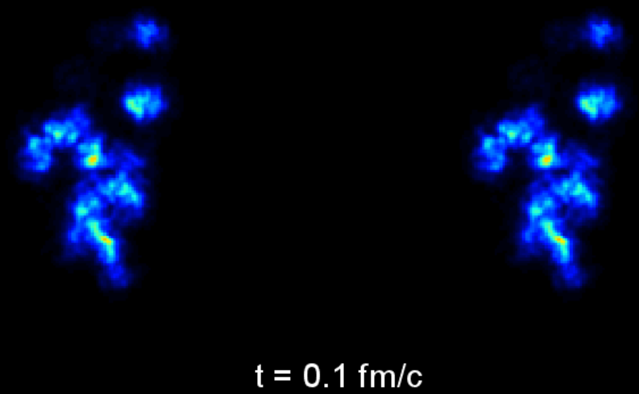


2+1D CYM
(weakly coupled at late times)

Hydro
after $\tau = 0.2\,\mathrm{fm}/c$ (CYM before)

Weakly coupled vs. strongly coupled system

shown are energy density distributions



2+1D CYM
(weakly coupled at late times)

Hydro
after $\tau = 0.2\,\mathrm{fm}/c$ (CYM before)

Ideal hydrodynamics

(Pasi will do viscous)

A little thermodynamics

Before we derive the equations of hydrodynamics, let's remind ourselves of some basic thermodynamics which will be useful:
The differential of the internal energy of a system is given by:

$$dU = -PdV + TdS + \mu dN$$

work and heat transferred to the system

- P: pressure
- V: volume
- T: temperature
- μ: chemical potential
- S: entropy
- N: nonrel. system: number of particles
  rel. system: e.g. net-baryon number

A little thermodynamics

An **extensive property** is a property that changes when the size of the sample changes.
Examples are mass, volume, length, and total charge

An **intensive property** is a bulk property and doesn't change when you take away some of the sample.
Examples include temperature, refractive index, and density

A little thermodynamics

$$dU = -PdV + TdS + \mu dN$$

For a non-viscous fluid, the mechanical work done on the system may be related to the pressure P and volume V.
The pressure is the intensive generalized force, while the volume is the extensive generalized displacement:

$$\delta W = -PdV$$

This defines the direction of work, W, to be energy flow from the working system to the surroundings, indicated by a negative term.
Taking the direction of heat transfer Q to be into the working fluid and assuming a reversible process, the heat is

$$\delta Q = TdS$$

A little thermodynamics

Energy U is an extensive function of the extensive quantities V, S, N:

$$U(\lambda V, \lambda S, \lambda N) = \lambda U(V, S, N)$$

Differentiate with respect to λ:

$$\frac{dU}{d(\lambda V)}\frac{d(\lambda V)}{d\lambda} + \frac{dU}{d(\lambda S)}\frac{d(\lambda S)}{d\lambda} + \frac{dU}{d(\lambda N)}\frac{d(\lambda N)}{d\lambda} = U$$
$$\frac{dU}{dV}V + \frac{dU}{dS}S + \frac{dU}{dN}N = U$$

λ was set to one along the way...
use $dU = -PdV + TdS + \mu dN$ and get

$$\Rightarrow U = -PV + TS + \mu N$$

A little thermodynamics

Differentiating

$$U = -PV + TS + \mu N$$

gives

$$dU = -PdV - VdP + TdS + SdT + \mu dN + Nd\mu$$

using $dU = -PdV + TdS + \mu dN$ yields the Gibbs-Duhem relation:

$$VdP = SdT + Nd\mu$$

In hydrodynamics, densities (intensive quantities) are more useful:

- $\varepsilon = U/V$: energy density
- $s = S/V$: entropy density
- $n = N/V$: baryon density

We obtain

$$\varepsilon = -P + Ts + \mu n$$

and

$$dP = sdT + nd\mu$$

$$\Rightarrow d\varepsilon = Tds + \mu dn$$

Remember these

$$\varepsilon + P = Ts + \mu n$$

and

$$d\varepsilon = Tds + \mu dn$$

These relations will be used when showing that ideal hydrodynamics conserves entropy and can be used when deriving the equations of viscous relativistic fluid dynamics.

Equations of fluid dynamics

The fluid approximation:

- Treat an ensemble of particles as a single fluid.
- Compute ensemble's mean velocity at each point $\mathbf{u}(\mathbf{r}, t)$.
- Lost information about the spread of velocities around that mean: But if we have *local thermodynamic equilibrium* (LTE), that spread is described by the temperature $T(\mathbf{r}, t)$
- LTE requires that particles are in equilibrium locally: *Mean free path* needs to be smaller than any length scale of interest.

Note: Shocks violate the fluid assumption: relevant length scales become shorter than the mean free path.

Equations of fluid dynamics

Conservation of mass:

Variation of mass in the volume V is due to in- and out-flow through the surface ∂V:

$$\frac{\partial}{\partial t}\int \rho dV = -\int_{\partial V} \rho \mathbf{u}\cdot \mathbf{n} dA$$

Gauss' theorem:

$$\frac{\partial}{\partial t}\int \rho dV = -\int_{V} \boldsymbol{\nabla}\cdot (\rho \mathbf{u}) dV$$

is true for all V, so:

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$$

This is the continuity equation.

In the comoving frame this can be expressed with the Lagrange derivative $\frac{d\rho}{dt} = \partial_t\rho + \mathbf{u}\cdot\boldsymbol{\nabla}\rho$ to read

$$\frac{d\rho}{dt} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u}$$

Equations of fluid dynamics

Conservation of momentum:

Momentum density is $\rho\mathbf{u}$. Same procedure as for the mass, but also include the force from the outside fluid on the surface ∂V (pressure p times area element $\mathbf{n}dA$):

$$\frac{\partial}{\partial t}\int(\rho\mathbf{u})dV = -\int_{\partial V}(\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}dA - \int_{\partial V} p\mathbf{I}\cdot\mathbf{n}dA$$

Gauss' theorem

$$\frac{\partial}{\partial t}\int(\rho\mathbf{u})dV = -\int_{V}\boldsymbol{\nabla}\cdot((\rho\mathbf{u})\mathbf{u} + p\mathbf{I})dV$$

is true for all V:

$$\partial_t(\rho\mathbf{u}) + \boldsymbol{\nabla}\cdot\rho\mathbf{u}\mathbf{u} + \boldsymbol{\nabla}p = 0$$

Using the continuity equation we can write

$$\partial_t\mathbf{u} + \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{u}) = -\frac{1}{\rho}\boldsymbol{\nabla}p$$

This is the Euler equation. Lagrange form: $\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla}p$

Relativistic hydrodynamics

In a relativistic system, mass density is not a good degree of freedom: It does not account for kinetic energy which can be large for motions close to the speed of light c.
So replace ρ by the total energy density ε.
Also, $\mathbf{u}$ does not transform correctly under Lorentz transformations and should be replaced by the Lorentz four-vector

$$u^\mu = \frac{dx^\mu}{d\mathcal{T}}$$

We use $g^{\mu\nu} = \mathrm{diag}(+ - --)$, $d\mathcal{T}$ is the proper time increment:

$$\begin{aligned}(d\mathcal{T})^2 &= g_{\mu\nu}dx^\mu dx^\nu = (dt)^2 - (d\mathbf{x})^2 \\ &= (dt)^2\left[1 - \left(\frac{d\mathbf{x}}{dt}\right)^2\right] = (dt)^2[1 - \mathbf{u}^2]\end{aligned}$$

where here and in the following $c = k_B = \hbar = 1$.

Relativistic hydrodynamics - flow velocity

$$u^\mu = \frac{dx^\mu}{d\mathcal{T}} = \frac{dt}{d\mathcal{T}}\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1-\mathbf{u}^2}}\begin{pmatrix}1\\ \mathbf{u}\end{pmatrix} = \gamma(\mathbf{u})\begin{pmatrix}1\\ \mathbf{u}\end{pmatrix}$$

Local rest frame: $u^\mu = (1, \mathbf{0})$.

u^μ obeys the relation:

$$u^2 = u^\mu g_{\mu\nu} u^\nu = \gamma^2(\mathbf{u})(1-\mathbf{u}^2) = 1$$

So, no need for an additional equation when replacing $\mathbf{u}$ by the u^μ.

Ideal energy momentum tensor (no viscosity) has to be built from the pressure p, the energy density ε, and u^μ, as well as $g^{\mu\nu}$.

Properties: symmetric, transforms like a Lorentz-tensor.
So the most general form is

$$T^{\mu\nu} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu)$$

Constraints:
$T^{00} = \varepsilon$ and $T^{0i} = 0$ and $T^{ij} = \delta^{ij} p$ in the local rest frame.

Relativistic hydrodynamics - energy-momentum tensor

$$T^{\mu\nu} = \epsilon(c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p(c_2 g^{\mu\nu} + c_3 u^\mu u^\nu)$$

$T^{00} = \varepsilon$ and $T^{0i} = 0$ and $T^{ij} = \delta^{ij} p$ in the local rest frame.

$$T^{00} = \varepsilon(c_0 + c_1) + p(c_2 + c_3) = \varepsilon$$
$$\Rightarrow c_0 = 1 - c_1 \text{ and } c_2 = -c_3$$

$$T^{ij} = -\varepsilon c_0 \delta^{ij} - p c_2 \delta^{ij} = \delta^{ij} p$$
$$\Rightarrow c_0 = 0 \text{ and } c_2 = -1$$
$$\Rightarrow c_1 = 1 \text{ and } c_3 = 1$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu)$$

Relativistic hydrodynamics

Introduce $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ - projector to the space orthogonal to u^μ. Then

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu}$$

No external sources: $T^{\mu\nu}$ is conserved:

$$\partial_\mu T^{\mu\nu} = 0$$

Relativistic hydrodynamics

To identify equations of motion analogous to the non-relativistic ones, project on directions parallel and perpendicular to u^μ.
Parallel (show):

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &= u^\mu \partial_\mu \varepsilon + \varepsilon(\partial_\mu u^\mu) + \varepsilon u_\nu u^\mu \partial_\mu u^\nu - p u_\nu \partial_\mu \Delta^{\mu\nu} \\ &= (\varepsilon + p)\partial_\mu u^\mu + u^\mu \partial_\mu \varepsilon = 0 \end{aligned}$$

Perpendicular:

$$\begin{aligned} \Delta^\alpha_\nu \partial_\mu T^{\mu\nu} &= \varepsilon u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu - \Delta^{\mu\alpha}(\partial_\mu p) + p u^\mu \Delta^\alpha_\nu \partial_\mu u^\nu \\ &= (\varepsilon + p) u^\mu \partial_\mu u^\alpha - \Delta^{\mu\alpha}\partial_\mu p = 0 \end{aligned}$$

Introducing $D = u^\mu \partial_\mu$ and $\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu$, we can write

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0$$
$$(\varepsilon + p) D u^\alpha - \nabla^\alpha p = 0$$

Fundamental equations for relativistic fluid dynamics

In the non-relativistic limit

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu = 0$$
$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p = 0$$

can be related to the Euler and continuity equations.
If $|\mathbf{u}| \ll 1$ one finds

$$D = u^\mu \partial_\mu \simeq \partial_t + \mathbf{u} \cdot \boldsymbol{\nabla} + \mathcal{O}(\mathbf{u}^2)$$
$$\nabla^i = \Delta^{i\mu}\partial_\mu \simeq \partial^i + \mathcal{O}(|\mathbf{u}|)$$

Imposing also a non-relativistic equation of state, where $p \ll \varepsilon$ and assume that energy density is dominated by mass density $\varepsilon \simeq \rho$, we get

$$\partial_t \rho + \mathbf{u} \cdot \boldsymbol{\nabla}\rho + \rho\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$
$$\rho\partial_t \mathbf{u} + \rho\mathbf{u}\boldsymbol{\nabla} \cdot \mathbf{u} - \boldsymbol{\nabla} p = 0$$

... remember?

Entropy conservation

In ideal hydrodynamics entropy is conserved.
Show that from $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu(nu^\mu) = 0$:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu((\varepsilon+p)u^\mu u^\nu) - \partial_\mu(pg^{\mu\nu}) = 0 \\
&\Leftrightarrow \partial_\mu((\varepsilon+p)u^\mu)u^\nu + (\varepsilon+p)u^\mu\partial_\mu u^\nu - \partial^\nu p = 0 \\
&\overset{\times u_\nu}{\Leftrightarrow} \partial_\mu((\varepsilon+p)u^\mu) + (\varepsilon+p)u^\mu \underbrace{u_\nu\partial_\mu u^\nu}_{=0} - u_\nu\partial^\nu p = 0 \\
&\Leftrightarrow u^\mu\partial_\mu(\varepsilon+p) + (\varepsilon+p)\partial_\mu u^\mu - u_\mu\partial^\mu p = 0 \\
&\Leftrightarrow u^\mu\partial_\mu\varepsilon + (\varepsilon+p)\partial_\mu u^\mu = 0
\end{aligned}$$

Also

$$u^\mu\partial_\mu(n) + n\partial_\mu u^\mu = 0| \times \mu$$
$$u^\mu\mu\partial_\mu(n) + \mu n\partial_\mu u^\mu = 0$$

Entropy conservation

Now subtract

$$u^\mu \mu \partial_\mu (n) + \mu n \partial_\mu u^\mu = 0$$

from

$$u^\mu \partial_\mu \varepsilon + (\varepsilon + p) \partial_\mu u^\mu = 0$$

$$\Rightarrow u^\mu \partial_\mu \varepsilon + \underbrace{(\varepsilon + p)\partial_\mu u^\mu - \mu n \partial_\mu u^\mu}_{Ts\partial_\mu u^\mu} - u^\mu \mu \partial_\mu n = 0$$

$$\Leftrightarrow Ts\partial_\mu u^\mu + u^\mu \partial_\mu \varepsilon - u^\mu \mu \partial_\mu n = 0$$

$$\overset{d\varepsilon - \mu dn = Tds}{\Leftrightarrow} Ts\partial_\mu u^\mu + Tu^\mu \partial_\mu s = 0 | \div T$$

$$\Leftrightarrow s\partial_\mu u^\mu + u^\mu \partial_\mu s = \partial_\mu (s u^\mu) = 0\,,$$

where su^μ is the entropy current.

Relativistic viscous hydrodynamics

Including dissipative (viscous) effects, we write

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \Pi^{\mu\nu}$$

where $T^{\mu\nu}_{(0)}$ is the ideal part that we considered before.
$\Pi^{\mu\nu}$ is the viscous stress tensor.

In a system without conserved charges (or at $\mu_B = 0$), all momentum density is due to flow of energy density:

$$u_\mu T^{\mu\nu} = \varepsilon u^\nu \rightarrow u_\mu \Pi^{\mu\nu} = 0$$

In a more general system this corresponds to choosing the
Landau-Lifshitz frame
(local frame where the energy density is at rest)
alternativley, the Eckart frame is the frame where charge density (if there is one) is at rest.

The equations of motion are obtained by projection of $\partial_\mu T^{\mu\nu}$ as in the ideal case:

$$u_\nu \partial_\mu T^{\mu\nu} = D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu + u_\nu \partial_\mu \Pi^{\mu\nu} = 0$$
$$\Delta^\alpha_\nu \partial_\mu T^{\mu\nu} = (\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

We can simplify the first equation using

- $u_\nu \partial_\mu \Pi^{\mu\nu} = \partial_\mu(u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu}\partial_\mu u_\nu = \partial_\mu(u_\nu \Pi^{\mu\nu}) - \Pi^{\mu\nu}\partial_{(\mu} u_{\nu)}$
- $u_\nu \Pi^{\mu\nu} = 0$ (frame)
- $\partial_\mu = u_\mu D + \nabla_\mu$

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu} u_{\nu)} = 0$$
$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0$$

Fundamental equations for relativistic viscous fluid dynamics

Symmetrization $A_{(\mu} B_{\nu)} = \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$ not necessary but helpful later. Is ok here because $\Pi^{\mu\nu}$ is symmetric.

I will end here and leave the detailed discussion of viscous relativistic hydrodynamics to Pasi.
In my next lecture I will focus on the initial state for hydrodynamic calculations in heavy ion collision scenarios.

BACKUP

$\Pi^{\mu\nu}$ from thermodynamics

$\Pi^{\mu\nu}$ has not been specified yet.
We can use the second law of thermodynamics to get it:
Entropy must always increase locally.
Basic relations for $\mu_B = 0$:

$$\varepsilon + p = Ts\,, Tds = d\varepsilon$$

In covariant form the second law reads:

$$\partial_\mu s^\mu \geq 0\,,$$

with $s^\mu = su^\mu$. So,

$$\partial_\mu s^\mu = Ds + s\partial_\mu u^\mu = \frac{1}{T}D\varepsilon + \frac{\varepsilon + p}{T}\partial_\mu u^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \geq 0$$

using $D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$

$\Pi^{\mu\nu}$ from thermodynamics

We split $\Pi^{\mu\nu}$ into a traceless part and the rest:

$$\Pi^{\mu\nu} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi.$$

Also introducing the traceless part of $\nabla_{(\mu}u_{\nu)}$,

$$\nabla_{<\mu}u_{\nu>} = 2\nabla_{(\mu}u_{\nu)} - \frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha$$

the second law becomes (see next slide):

$$\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0$$

$$\Delta_{\mu\nu}\Delta^{\mu\nu} = 3$$
$$\Delta_{\mu\nu}\pi^{\mu\nu} = \pi^\mu_\mu - u_\mu\pi^{\mu\nu}u_\nu = 0$$

This is fulfilled by

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>},\ \Pi = -\zeta\nabla_\alpha u^\alpha,\ \eta \geq 0,\ \zeta \geq 0$$

because then we have a positive sum of squares.

$\Pi^{\mu\nu}$ from thermodynamics

We now show that

$$\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha \geq 0$$

First

$$\begin{aligned}\frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} &= \frac{1}{2T}\Pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} + \frac{1}{3T}\Pi^{\mu\nu}\Delta_{\mu\nu}\nabla_\alpha u^\alpha \\ &= \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{2T}\Pi\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} \\ &\quad + \underbrace{\frac{1}{3T}\pi^{\mu\nu}\Delta_{\mu\nu}\nabla_\alpha u^\alpha}_{=0} - \frac{1}{3T}\Delta^{\mu\nu}\Delta_{\mu\nu}\Pi\nabla_\alpha u^\alpha \\ &= \frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{2T}\Pi\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T}\Pi\nabla_\alpha u^\alpha\end{aligned}$$

because

$$\Delta_{\mu\nu}\Delta^{\mu\nu} = (g_{\mu\nu} - u_\mu u_\nu)(g^{\mu\nu} - u^\mu u^\nu) = 4 - 1 - 1 + 1 = 3$$

$\Pi^{\mu\nu}$ from thermodynamics

It remains to be shown that

$$\frac{1}{2T}\Pi\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>}=0$$

$$\begin{aligned}\Delta^{\mu\nu}\nabla_{<\mu}u_{\nu>} &= \Delta^{\mu\nu}(2\nabla_{(\mu}u_{\nu)}-\frac{2}{3}\Delta_{\mu\nu}\nabla_\alpha u^\alpha) \\ &= 2\Delta^{\mu\nu}\nabla_{(\mu}u_{\nu)}-2\nabla_\alpha u^\alpha \\ &= \Delta^{\mu\nu}(\nabla_\mu u_\nu+\nabla_\nu u_\mu)-2\nabla_\alpha u^\alpha \\ &= (g^{\mu\nu}-u^\mu u^\nu)(\nabla_\mu u_\nu+\nabla_\nu u_\mu)-2\nabla_\alpha u^\alpha \\ &= \nabla^\nu u_\nu+\nabla^\mu u_\mu-2\nabla_\alpha u^\alpha=0\end{aligned}$$

Thus

$$\partial_\mu s^\mu=\frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}=\frac{1}{2T}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>}-\frac{1}{T}\Pi\nabla_\alpha u^\alpha\geq 0$$

Relativistic Navier-Stokes equations

The non-relativistic Navier-Stokes equation is of the form

$$\frac{\partial u^i}{\partial t} + u^k \frac{\partial u^i}{\partial x^k} = -\frac{1}{\rho}\frac{\partial p}{\partial x^i} - \frac{1}{\rho}\frac{\partial \Pi^{ki}}{\partial x^k}$$

$$\Pi^{ki} = -\eta\left(\frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} - \frac{2}{3}\delta^{ki}\frac{\partial u^l}{\partial x^l}\right) - \zeta\delta^{ki}\frac{\partial u^l}{\partial x^l}$$

with the coefficients for shear viscosity η and bulk viscosity ζ.
So we can identify the equations

$$D\varepsilon + (\varepsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0$$
$$(\varepsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0\,,$$

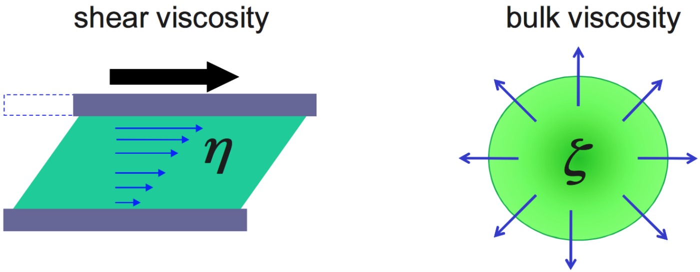
$$\Pi^{\mu\nu} = \pi^{\mu\nu} - \Delta^{\mu\nu}\Pi\,,$$

and

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>},\ \Pi = -\zeta\nabla_\alpha u^\alpha,\ \eta \geq 0,\ \zeta \geq 0$$

as the relativistic Navier-Stokes equations.

Shear and bulk viscosity



Shear viscosity describes the resistance to flow
Bulk viscosity describes the resistance to expansion

Acausality problem of the rel. Navier-Stokes equations

Unlike the non-relativistic Navier-Stokes equations, the relativistic ones exhibit acausal propagation.
Consider small perturbations to an equilibrium system at rest:

$$\varepsilon = \varepsilon_0 + \delta\varepsilon(t,x), \qquad u^\mu = (1,\mathbf{0}) + \delta u^\mu(t,x)$$

(perturbation only depends on one space variable)
For $\alpha = y$ (transverse), the Navier-Stokes equation gives:

$$(\varepsilon + p)Du^y - \nabla^y p + \Delta^y{}_\nu \partial_\mu \Pi^{\mu\nu} = (\varepsilon_0 + p_0)\partial_t \delta u^y + \partial_x \Pi^{xy} + \mathcal{O}(\delta^2) = 0$$

$$\Pi^{xy} = \eta(\nabla^x u^y + \nabla^y u^x) - \left(\zeta + \frac{2}{3}\eta\right)\Delta^{xy}\nabla_\alpha u^\alpha = -\eta\partial_x \delta u^y + \mathcal{O}(\delta^2)$$

Together, that gives

$$\partial_t \delta u^y - \frac{\eta}{\varepsilon_0 + p_0}\partial_x^2 \delta u^y = \mathcal{O}(\delta^2)$$

Problem with relativistic Navier-Stokes equations

To investigate the individual modes of the diffusion process

$$\partial_t \delta u^y - \frac{\eta}{\varepsilon_0 + p_0} \partial_x^2 \delta u^y = \mathcal{O}(\delta^2)$$

we use a mixed Laplace-Fourier wave ansatz:

$$\delta u^y(t,x) = e^{-\omega t + ikx} f_{\omega,k}$$

This leads to the “dispersion relation”

$$\omega = \frac{\eta}{\varepsilon_0 + p_0} k^2$$

We get as the speed of diffusion of a mode with wave number k:

$$v_T(k) = \frac{d\omega}{dk} = 2\frac{\eta}{\varepsilon_0 + p_0} k$$

As k grows, v_T grows, eventually exceeding the speed of light.
Violates causality.

Problem with relativistic Navier-Stokes equations

So what? Hydrodynamics is supposed to be an effective theory for long wavelength modes ($k \to 0$) anyway.
We could just not care about what happens at $k \gg 1$.

However, numerically the high k modes lead to instabilities:

- Modes that travel faster than light in one Lorentz frame, travel backwards in time in another.
- Hydrodynamics is an initial value problem and requires well defined set of initial conditions.
- If modes travel backwards in time, the initial conditions cannot be freely given. So one cannot solve the relativistic Navier-Stokes equations numerically.

The diffusion speed exceeding the speed of light is a hint but no proof of causality violation.
See appendix in Romatschke, Int.J.Mod.Phys. E19 (2010) 1-53 for a proof.

Fixing the relativistic Navier-Stokes eqs

One way to regulate the theory is to introduce a relaxation time τ_π, yielding the "Maxwell-Cattaneo law":

$$\tau_\pi \partial_t \Pi^{xy} + \Pi^{xy} = -\eta \partial^x \delta u^y$$

which replaces

$$\Pi^{xy} = -\eta \partial^x \delta u^y$$

This is useful because it leads to the modified dispersion relation

$$\omega = \frac{\eta}{\varepsilon_0 + p_0} \frac{k^2}{1 - \omega \tau_\pi}$$

For $\omega, k \to 0$, this equals the original dispersion relation.
For $k \gg 1$ v_T is finite (show):

$$v_T^{\max} = \lim_{k\to\infty} \frac{d|\omega|}{dk} = \sqrt{\frac{\eta}{(\varepsilon_0 + p_0)\tau_\pi}} \leq 1 \text{ for all known fluids}$$

Works great. But introduced by hand. Unsatisfactory.

Second order viscous hydrodynamics

When we derived the relativistic Navier-Stokes equations using

$$\partial_\mu s^\mu \geq 0$$

we used the equilibirum entropy current $s^\mu = s u^\mu$. But there are dissipative corrections: use of the equilibrium s^μ may not be good approximation.

Including corrections to the entropy current

$$s^\mu = s u^\mu - \frac{\beta_0}{2T} u^\mu \Pi^2 - \frac{\beta_2}{2T} u^\mu \pi_{\alpha\beta}\pi^{\alpha\beta} + \mathcal{O}(\Pi^3)$$

with coefficients β_0 and β_2, one gets

$$\pi_{\alpha\beta} = \eta \left(\nabla_{<\alpha} u_{\beta>} - \pi_{\alpha\beta} T D \left(\frac{\beta_2}{T}\right) - 2\beta_2 D\pi_{\alpha\beta} - \beta_2 \pi_{\alpha\beta} \partial_\mu u^\mu \right)$$

$$\Pi = \zeta \left(\nabla_\alpha u^\alpha - \frac{1}{2} \Pi T D \left(\frac{\beta_0}{T}\right) - \beta_0 D\Pi - \frac{1}{2}\beta_0 \Pi \partial_\mu u^\mu \right)$$

from the condition $\partial_\mu s^\mu \geq 0$.

Derivation from kinetic theory

Alternatively, second order viscous hydrodynamics can be derived from kinetic theory.

Short recap of kinetic theory:

The evolution of the one-particle distribution $f(\mathbf{p}, \mathbf{x}, t)$ follows from Liouville's theorem (conservation of density in phase-space):

$$\frac{df}{d\mathcal{T}} = \frac{dt}{d\mathcal{T}}\partial_t f + \frac{d\mathbf{x}}{d\mathcal{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f = 0$$

Using $m\frac{dt}{d\mathcal{T}} = m\gamma(\mathbf{v}) = p^0$ and $m\frac{d\mathbf{x}}{d\mathcal{T}} = m\mathbf{v}\gamma(\mathbf{v}) = \mathbf{p}$ we get

$$p^\mu \partial_\mu f = 0$$

with $p_\mu p^\mu = m^2$.

Now, with collisions one gets the Boltzmann equation

$$p^\mu \partial_\mu f = -\mathcal{C}[f] \leftarrow \text{functional of f}$$

Derivation from kinetic theory

In equilibirum $f = f_0(\mathbf{p})$ and

$$p^\mu \partial_\mu f_0 = 0 = -\mathcal{C}[f_0]$$

so $\mathcal{C}[f_0] = 0$.

Hydrodynamics corresponds to the limit where $\mathcal{C}$ is large (short mean free path) and drives the system towards equilibrium.

Now, the relation between $T^{\mu\nu}$ and f is given by

$$T^{\mu\nu} = \int \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\mu p_\mu - m^2) 2\theta(p^0) f(p,x)$$

Derivation of ideal hydro from kinetic theory

Ultrarelativistic limit: $m \to 0$

Taking the first moment of the Boltzmann equation one finds

$$\int d\chi p^\nu p^\mu \partial_\mu f(p^\mu, x^\mu) = -\int d\chi p^\nu \mathcal{C}[f]$$

$$= \partial_\mu \int d\chi p^\nu p^\mu f(p,x) = \partial_\mu T^{\mu\nu}$$

here we use $\int d\chi = \int \frac{d^4p}{(2\pi)^3}\delta(p^\mu p_\mu) 2\theta(p^0) = \int \frac{d^3p}{(2\pi)^3 E}$

When $\mathcal{C}$ conserves energy and momentum

$$\int d\chi p^\nu \mathcal{C}[f] = 0$$

If $T^{\mu\nu}$ can be interpreted as a fluid's energy-momentum tensor (like it can in equilibrium), this means that the first moment of the Boltzmann equation corresponds to the fundamental equations of fluid dynamics:

$$\partial_\mu T^{\mu\nu} = 0$$

Derivation of ideal hydro from kinetic theory

In the relativistic case it is better to write $f_{\rm eq}(p^\mu u_\mu/T)$ instead of $f_0(\mathbf{p})$ (Lorentz invariance).

With that we can write

$$T_0^{\mu\nu} = \int d\chi p^\mu p^\nu f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right) = a_{20}u^\mu u^\nu + a_{21}\Delta^{\mu\nu}$$

So a_{20} corresponds to energy density ε and $-a_{21}$ to pressure p.

They can be computed by contraction of above expression with $u^\mu u^\nu$ and $\Delta^{\mu\nu}$ respectively.

Their exact values depend on $f_{\rm eq}$.
See calculation for Boltzmann statistics $f_{\rm eq}(p_\mu u^\mu/T) = \exp(-p_\mu u^\mu/T)$

Viscous hydro from kinetic theory

Small deviation from equilibrium:

$$f(p^\mu, x^\mu) = f_{\rm eq}(p^\mu u_\mu/T)(1 + \delta f(p^\mu, x^\mu))$$

with $\delta f \ll 1$. So one can identify

$$T^{\mu\nu} = T_0^{\mu\nu} + \int d\chi p^\mu p^\nu f_{\rm eq} \delta f = T_0^{\mu\nu} + \pi^{\mu\nu}$$

Momentum dependence of δf can be expressed in a Taylor series

$$\delta f(p^\mu, x^\mu) = c + p^\alpha c_\alpha + p^\alpha p^\beta c_{\alpha\beta} + \mathcal{O}(p^3)$$

and is an algebraic function of $\varepsilon, p, u^\mu, g^{\mu\nu}$, and $\pi^{\mu\nu}$.
δf vanishes in equilibrium $\rightarrow c = 0, c_\alpha = 0, c_{\alpha\beta} = c_2 \pi_{\alpha\beta}$ so

$$\pi^{\mu\nu} = \pi_{\alpha\beta} c_2 I^{\mu\nu\alpha\beta}$$

with $I^{\mu_1\mu_2\cdots\mu_n} = \int d\chi p^{\mu_1} p^{\mu_2} \dots p^{\mu_n} f_{\rm eq}$

Viscous hydro from kinetic theory

$I^{\mu\nu\alpha\beta}$ can be decomposed into Lorentz tensors:

$$I^{\mu\nu\alpha\beta} = a_{40}u^\mu u^\nu u^\alpha u^\beta + a_{41}(u^\mu u^\nu \Delta^{\alpha\beta} + \text{perm.}) + a_{42}(\Delta^{\mu\nu}\Delta^{\alpha\beta} + \Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha})$$

Because $u_\mu \pi^{\mu\nu} = 0$ and $\pi^\mu_\mu = 0,\ (\Delta_{\mu\nu}\pi^{\mu\nu} = 0)$, a_{40} and a_{41} vanish.

Contracting the indices on the RHS of

$$\pi^{\mu\nu} = \pi_{\alpha\beta} c_2 I^{\mu\nu\alpha\beta}$$

we find

$$c_2 = \frac{1}{2a_{42}}$$

and finally

$$f(p^\mu, x^\mu) = f_{\text{eq}}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \overbrace{\frac{1}{2a_{42}} p^\alpha p^\beta \pi_{\alpha\beta}}^{\delta f}\right]$$

Viscous hydro from kinetic theory

$$f(p^\mu, x^\mu) = f_{\rm eq}\left(\frac{p^\mu u_\mu}{T}\right)\left[1 + \frac{1}{2a_{42}} p^\alpha p^\beta \pi_{\alpha\beta}\right]$$

for a Boltzmann gas $f_{\rm eq}(x) = e^{-x}$ and

$$a_{42} = (\varepsilon + \mathcal{P})T^2$$

which follows from a calculation analogous to that of a_{20}.

Viscous hydro from kinetic theory

The first moment of the Boltzmann equation gave conservation of $T^{\mu\nu}$.

The integral

$$\int d\chi p^\alpha p^\beta \mathcal{C}$$

does not trivially vanish (except in equilibrium).

So, the second moment of the Boltzmann equation

$$\int d\chi p^\alpha p^\beta p^\mu \partial_\mu f = -\int d\chi p^\alpha p^\beta \mathcal{C}[f]$$

will carry information on the non-equilibrium (viscous) dynamics.

From our earlier expansion of f, we find for the LHS

$$\int d\chi p^\alpha p^\beta p^\mu \partial_\mu f = \partial_\mu \left(I^{\alpha\beta\mu} + \frac{\pi_{\gamma\delta}}{2a_{42}} I^{\alpha\beta\mu\gamma\delta} \right)$$

Using the second moment was a choice by Israel and Stewart (1979). Other moments could be used, leading to some ambiguity.
For analysis and improvements on this see e.g. Denicol, Molnar, Niemi, and Rischke, arXiv:1206.1554

Viscous hydro from kinetic theory

Now, projecting on the part that is symmetric and traceless with

$$P^{\mu\nu}_{\alpha\beta} = \Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta}\Delta^{\mu}_{\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$$

and doing the same for the RHS (assuming Boltzmann statistics) it follows from the second moment of the Boltzmann equation that

$$\pi^{\mu\nu} + \frac{a_{52}T\eta}{a_{42}^2}\left[\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}D\pi^{\alpha\beta} + P^{\mu\nu}_{\alpha\beta}\pi^{\phi\beta}\nabla_{\phi}u^{\alpha} + \frac{2}{3}\pi^{\mu\nu}\partial_{\alpha}u^{\alpha}\right] = \eta\nabla^{<\mu}u^{\nu>}$$

The expression $P^{\mu\nu}_{\alpha\beta}\pi^{\phi\beta}\nabla_{\phi}u^{\alpha}$ can be rewritten when introducing the fluid vorticity

$$\Omega_{\alpha\beta} = \nabla_{[\alpha}u_{\beta]}$$

where $A_{[\mu}B_{\nu]} = \frac{1}{2}(A_{\mu}B_{\nu} - A_{\nu}B_{\mu})$ is anti-symmetrization

Viscous hydro from kinetic theory

One finds

$$P^{\mu\nu}_{\alpha\beta}\pi^{\phi\beta}\nabla_\phi u^\alpha = P^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\gamma}\pi^{\phi\beta}\left[\Omega_{\phi\gamma} + \frac{1}{2}\nabla_{<\phi}u_{\gamma>} + \frac{1}{3}\Delta_{\phi\gamma}\nabla_\delta u^\delta\right]$$
$$= -2\pi^{\phi(\mu}\Omega^{\nu)}_{\phi} + \frac{\pi^{\phi<\mu}\pi^{\nu>}_{\phi}}{2\eta} + \frac{2}{3}\pi^{\mu\nu}\nabla_\delta u^\delta + \mathcal{O}(\delta^3)$$

Finally we get

$$\pi^{\mu\nu}+\overbrace{\frac{a_{52}T\eta}{a_{42}^2}}^{\tau_\pi}\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta}+\frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha-2\pi^{\phi(\mu}\Omega^{\nu)}_{\phi}+\frac{\pi^{\phi<\mu}\pi^{\nu>}_{\phi}}{2\eta}\right]=\eta\nabla^{<\mu}u^{\nu>}$$
$$+\mathcal{O}(\delta^2)$$

where τ_π is the second order transport coefficient “relaxation time”.

Viscous hydro from kinetic theory

Again, for $k \gg 1$ the "dispersion relation" for the diffusion equation becomes

$$\omega \approx \frac{\eta}{\varepsilon + p} \frac{k^2}{\omega \tau_\pi}$$

We don't know τ_π if the underlying theory is unknown.
For a massless Boltzmann gas we get

$$\tau_\pi = \frac{a_{52} T \eta}{a_{42}^2} = \frac{3}{2} \pi^2 \frac{\eta}{T^4}$$

So

$$\omega \approx \sqrt{\frac{2}{3} \frac{T^4}{\pi^2 (\varepsilon + p)}}\, k = \sqrt{\frac{1}{6}}\, k$$

such that $v_T^{\max} = \sqrt{\frac{1}{6}}$.

Studying long. velocity perturbations (sound) one finds $v_L^{\max} = \sqrt{\frac{5}{9}}$.
Bose-Einstein statistics lead to only small numerical modifications.

Viscous hydro from kinetic theory

The final result

$$\pi^{\mu\nu}+\tau_\pi\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta}+\frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha-2\pi^{\phi(\mu}\Omega^{\nu)}_\phi+\frac{\pi^{\phi<\mu}\pi^{\nu>}_\phi}{2\eta}\right]=\eta\nabla^{<\mu}u^{\nu>} + \mathcal{O}(\delta^2)$$

Müller (1976), Israel and Stewart (1979)

is different from what we found from the second law of thermodynamics

$$\pi_{\alpha\beta} = \eta\left(\nabla_{<\alpha}u_{\beta>} - \pi_{\alpha\beta}TD\left(\frac{\beta_2}{T}\right) - 2\beta_2 D\pi_{\alpha\beta} - \beta_2\pi_{\alpha\beta}\partial_\mu u^\mu\right)$$

which for a Boltzmann gas ($\beta_2 = \frac{\tau_\pi}{2\eta} = \frac{3}{4p}$) reads

$$\pi^{\mu\nu} + \tau_\pi\left[D\pi^{\mu\nu} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha\right] = \eta\nabla^{<\mu}u^{\nu>} + \mathcal{O}(\delta^2)$$

where we used $\tau_\pi/(\eta T) \sim T^{-5}$ and
$D\ln T = D\ln(\varepsilon^{1/4}) = -\frac{1}{3}\nabla_\alpha u^\alpha + \mathcal{O}(\delta^2)$ (from Navier-Stokes).

Viscous hydro from kinetic theory

Differences between the two equations vanish when contracted with $\pi_{\mu\nu}$, hence do not contribute to entropy production.

recall $s^\mu = su^\mu - \frac{\beta_0}{2T}u^\mu\Pi^2 - \frac{\beta_2}{2T}u^\mu\pi_{\alpha\beta}\pi^{\alpha\beta}$

So the entropy-wise derivation could not capture the terms.

However, the terms are important:

Contraction with u_μ gives zero for the kinetic theory result.

But leads to an unphysical constraint $u_\mu D\pi^{\mu\nu} = 0$ for the entropy res.

$\Rightarrow$ kinetic theory result is superior.

But

$$\pi^{\mu\nu}+\tau_\pi\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta}+\frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha-2\pi^{\phi(\mu}\Omega^{\nu)}_\phi+\frac{\pi^{\phi<\mu}\pi^{\nu>}_\phi}{2\eta}\right]=\eta\nabla^{<\mu}u^{\nu>} + \mathcal{O}(\delta^2)$$

misses terms of second order in gradients (because we don't know the collision term).

Consistent gradient expansion

- ideal hydrodynamics: no gradients (0th order)

$$\pi^{\mu\nu} = 0$$

- Navier-Stokes equation: first order gradients

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>}$$

- Müller-Israel-Stewart theory: second order gradients

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} + \tau_\pi[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta}\dots] + \mathcal{O}(\delta^2)$$

1. Zeroth order gradient expansion is complete (hydrodynamic energy-momentum tensor is most general structure allowed by symmetry)
2. First order is complete (see the following)
3. Second order Israel-Stewart theory is not complete.

Consistent gradient expansion: first order

First order: independent gradients are $\partial_\mu u^\alpha$ and $\partial_\mu \varepsilon$.
(pressure is linked to ε by the equation of state)

Using $\partial_\mu = \nabla_\mu + u_\mu D$, all Du^α, $D\varepsilon$ can be expressed in terms of space-like ∇_μ. $\Rightarrow$ Only the ∇_μ are independent.

So the complete shear-stress tensor should have the structure

$$\pi^{\mu\nu} = c_4 \nabla^{(\mu} u^{\nu)} + c_5 \Delta^{\mu\nu} \nabla_\alpha u^\alpha + c_6 u^{(\mu} \nabla^{\nu)} \varepsilon$$

- Landau-Lifshitz frame condition $u_\mu \pi^{\mu\nu} = 0 \to c_6 = 0$ (next slide)
- No bulk viscosity: traceless $\pi^{\mu\nu} \to c_5 = -\frac{1}{3} c_4$ (next slide)
- Choosing $c_4 = 2\eta$ one finds $\pi^{\mu\nu} = \eta \nabla^{<\mu} u^{\nu>}$ (next slide)

Navier-Stokes equation is complete first order gradient expansion.

Consistent gradient expansion: first order

$$\pi^{\mu\nu} = c_4 \nabla^{(\mu} u^{\nu)} + c_5 \Delta^{\mu\nu} \nabla_\alpha u^\alpha + c_6 u^{(\mu} \nabla^{\nu)} \varepsilon$$

- Landau-Lifshitz frame condition $u_\mu \pi^{\mu\nu} = 0$:
  First term: $u_\mu \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) = \frac{1}{2} u_\mu \nabla^\mu u^\nu + \frac{1}{2} u_\mu \nabla^\nu u^\mu$
  $u_\mu \nabla^\mu u^\nu = 0$ because $u^\mu \perp \nabla^\mu$ and $u_\mu \nabla^\nu u^\mu = 0$ because
  $u_\mu \nabla^\nu u^\mu = \frac{1}{2} \nabla^\nu (u_\mu u^\mu) = 0$
  Second term: $u_\mu \Delta^{\mu\nu} = u_\mu (g^{\mu\nu} - u^\mu u^\nu) = u^\nu - u^\nu = 0$
  Third term: does not vanish $\rightarrow c_6 = 0$
- No bulk viscosity: traceless $\pi^{\mu\nu}$:
  $c_4 \frac{1}{2}(\nabla^\mu u_\mu + \nabla_\mu u^\mu) + c_5 \Delta^\mu_\mu \nabla_\alpha u^\alpha = 0$
  $\Leftrightarrow c_4 \nabla_\mu u^\mu + c_5 (3) \nabla_\mu u^\mu = 0$
  $\rightarrow c_5 = -\frac{1}{3} c_4$
- Choosing $c_4 = 2\eta$ one finds
  $\pi^{\mu\nu} = 2\eta \nabla^{(\mu} u^{\nu)} - \frac{2}{3} \Delta^{\mu\nu} \nabla_\alpha u^\alpha = \eta \nabla^{<\mu} u^{\nu>}$

Consistent gradient expansion: second order

Second order: obviously more terms.
For now: Restrict number by assuming conformal symmetry.

In curved space there are 8 contributions of second order that obey $\pi^\mu_\mu = 0$ and $u_\mu \pi^{\mu\nu} = 0$.

Only 5 of them transform homogeneously under Weyl rescalings (obey conformal symmetry).

Baier, Romatschke, Son, Starinets, Stephanov, JHEP 0804, 100 (2008)

Consistent gradient expansion: second order

The most general expression to second order in gradients in curved space for a conformal theory is

$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} - \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}(\nabla_\alpha u^\alpha)\right] + \frac{\kappa}{2}\left[R^{<\mu\nu>} + 2u_\alpha R^{\alpha<\mu\nu>\beta}u_\beta\right] - \frac{\lambda_1}{2\eta^2}\pi^{<\mu}{}_\lambda \pi^{\nu>\lambda} - \frac{\lambda_2}{2\eta}\pi^{<\mu}{}_\lambda \Omega^{\nu>\lambda} - \frac{\lambda_3}{2}\Omega^{<\mu}_\lambda \Omega^{\nu>\lambda}$$

where $R^{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R^{\mu\nu}$ the Ricci tensor, and τ_π, κ, $\lambda_1, \lambda_2, \lambda_3$ are five independent second order transport coefficients.

Baier, Romatschke, Son, Starinets, Stephanov, JHEP 0804, 100 (2008)

When are these additional terms relevant?

Exact formalism for relativistic causal viscous hydrodynamics is not settled.

More terms.

And that was just the shear part.
Additional equations for heat flow and bulk (volume) viscosity in a non-conformal fluid.
see Betz, Henkel and Rischke J.Phys.G G36, 064029 (2009)

See Denicol, Koide, Rischke, Phys.Rev.Lett. 105, 162501 (2010) for an alternative way of deriving second order viscous hydrodynamics from kinetic theory (leads to different coefficients).

What is often used in simulations of heavy-ion collisions is

$$\pi^{\mu\nu} + \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha \right] = \eta \nabla^{<\mu} u^{\nu>}$$

but more terms should be included and studied (and are).