Observational role of dark matter in f(R) models for structure formation

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Main points

1. f(R) Gravity
2. Field Equations in f(R) Gravity with dark matter
3. Phase space analysis of Field Equations with dark matter
4. Thermodynamic connection
5. Conclusion

Outline

The fixed points for the dynamical system in the phase space have been calculated with dark matter in the f(R) gravity models. The stability conditions of these fixed points are obtained in the ongoing accelerated phase of the universe, and the values Hubble parameter and Ricci scalar are obtained for various evolutionary stages of the universe. We present a range of some modifications of general relativistic action consistent with the lambda-CDM model. We elaborate upon the fact that the upcoming cosmological observations would further constrain the bounds on the possible forms of f(R) with greater precision that could in turn constrain the search for dark matter in colliders.

f(R) Gravity

- f(R)gravity is one of the simplest modified gravity models in which 4-dimensional action is given by some general function f(R) of the Ricci scalar R.
- There are two approaches to derive field equations in f(R) gravity (i) metric formalism and (ii)The Palatini formalism.
- The action in $f(R)$ gravity is given by-

$$A = \int d^4x\sqrt{-g}[\frac{1}{2\kappa^2}f(R) + L_{rad} + L_m] \tag{1}$$

where κ^2 = 8πG while G is a bare gravitational constant,f(R) is some general function of the Ricci scalar R, and L_{rad}, L_m are the Lagrangian densities of radiation and dustlike matter, respectively.

- In both formalisms action is varied w.r.t. $g_{\mu\nu}$ but $\Gamma^{\alpha}_{\beta\gamma}$ and $g_{\mu\nu}$ are treated as independent variables in the Palatini formalism.

The Set-up of metric formalism

We consider the field equations in the background of spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime with a metric

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2}$$

where $a(t)$ is time dependent scale factor. For this metric the Ricci scalar R is given by

$$R = 6(2H^2 + \dot{H}) \tag{3}$$

where H is the Hubble parameter and the overdot represents derivative w.r.t. time.

Metric formalism continued...

The field equations obtained by varying the action with respect to $g_{\mu\nu}$ are:

$$F(R)R_{\mu\nu}(g) - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa^2 T_{\mu\nu} \tag{4}$$

where $F(R) \equiv \frac{\partial f}{\partial R}$ and $T_{\mu\nu}$ is the matter energy-momentum tensor.
The [00] and [11] components of the above equation, respectively, are:

$$3FH^2 = \kappa^2(\rho_m + \rho_r) + \frac{(FR - f)}{2} - 3H\dot{F} \tag{5}$$

$$-2F\dot{H} = \kappa^2(\rho_m + \frac{4}{3}\rho_r) + \ddot{F} - H\dot{F} \tag{6}$$

Phase space analysis of $f(R)$ gravity models

Let us consider four dimensionless variables defined as

$$x_1 \equiv -\frac{\dot{F}}{FH} \tag{7}$$

$$x_2 \equiv -\frac{f}{6FH^2} \tag{8}$$

$$x_3 \equiv \frac{R}{6H^2} \tag{9}$$

$$x_4 \equiv \frac{\kappa^2 \rho_r}{3FH^2} \tag{10}$$

and two quantities defined as

$$m \equiv \frac{d\log F}{d\log R} = \frac{Rf_{,RR}}{f_{,R}}, \tag{11}$$

$$q \equiv -\frac{d\log f}{d\log R} = -\frac{Rf_{,R}}{f} = \frac{x_3}{x_2} \tag{12}$$

...continued

The fixed points of dynamical system are obtained by differentiating the variables (x_1, x_2, x_3, x_4) w.r.t. $N = \ln a$ and equating to zero. They are :

- $P_1 : (x_1, x_2, x_3, x_4) = (0, -1, 2, 0), \Omega_m = 0, w_{eff} = -1$
- $P_2 : (x_1, x_2, x_3, x_4) = (-1, 0, 0, 0), \Omega_m = 2, w_{eff} = \frac{1}{3}$
- $P_3 : (x_1, x_2, x_3, x_4) = (1, 0, 0, 0), \Omega_m = 0, w_{eff} = \frac{1}{3}$
- $P_4 : (x_1, x_2, x_3, x_4) = (-4, 5, 0, 0), \Omega_m = 0, w_{eff} = \frac{1}{3}$
- $P_5 : (x_1, x_2, x_3, x_4) = (\frac{3m}{1+m}, -\frac{1+4m}{2(1+m)^2}, \frac{1+4m}{2(1+m)}, 0), \Omega_m = 1 - \frac{m(7+10m)}{2(1+m)^2}, w_{eff} = -\frac{m}{(1+m)}$
- $P_6 : (x_1, x_2, x_3, x_4) = (\frac{2(1-m)}{1+2m}, \frac{1-4m}{m(1+2m)}, -\frac{(1-4m)(1+m)}{m(1+2m)}, 0), \Omega_m = 0, w_{eff} = \frac{2-5m-6m^2}{3m(1+2m)}$
- $P_7 : (x_1, x_2, x_3, x_4) = (0, 0, 0, 1), \Omega_m = 0, w_{eff} = \frac{1}{3}$
- $P_8 : (x_1, x_2, x_3, x_4) = (\frac{4m}{1+m}, -\frac{2m}{(1+m)^2}, \frac{2m}{1+m}, \frac{1-2m-5m^2}{(1+m)^2}), \Omega_m = 0, w_{eff} = \frac{1-3m}{3+3m}$

Stability of fixed points without radiation

We consider the properties and stability of these fixed points in the absence of radiation. For stability about the fixed points (x_1, x_2, x_3) we take linear perturbations $\delta x_i (i = 1, 2, 3)$ around the points. Linearization of the equations gives first order differential equations

$$\frac{d}{dN}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix} = M \begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{pmatrix}, \tag{13}$$

where M is a 3×3 matrix whose components depend upon x_1, x_2 and x_3. Stability of each fixed point depends upon the eigenvalues of the matrix M obtained by taking linear perturbations around that specific point. In the absence of radiation, we have only six fixed points $P_1 - P_6$ as below.

...stability continued

(1) Point $P_1 : (0, -1, 2)$ corresponds to de-Sitter point. Here $w_{eff} = -1$ and eigenvalues corresponding to this point are

$$-3, -\frac{3}{2} \pm \frac{\sqrt{25 - \frac{16}{m}}}{2} \tag{14}$$

P_1 is stable when real parts of all the eigenvalues is negative. Hence condition for stability is $0 < m(q = -2) < 1$ otherwise it is a saddle point. So this point can be taken as an acceleration point.

(2) Point $P_2 : (-1, 0, 0)$ is denoted by ϕ-matter-dominated (ϕ MDE) epoch. The eigenvalues of the 3×3 matrix of perturbations about P_2 are given by

$$-2, \frac{1}{2}[7 + \frac{1}{m} - \frac{m'}{m^2}q(1+q) \mp \sqrt{\left(7 + \frac{1}{m} - \frac{m'}{m^2}q(1+q)\right)^2 - 4\left(12 + \frac{3}{m} - \frac{m'}{m^2}q(3+4q)\right)}], \tag{15}$$

where m' is derivative of m w.r.t. q. If m is constant, then eigenvalues are $-2, 3, 4 + \frac{1}{m}$. In this case P_2 is a saddle point because eigenvalues are negative and positive

...continued

(3) Point $P_3 : (1,0,0)$ is the kinetic point. The eigenvalues corresponding to this point are

$$2, \frac{1}{2}[9 + \frac{1}{m} - \frac{m'}{m^2} q(1+q) \mp \sqrt{\left(9 - \frac{1}{m} + \frac{m'}{m^2} q(1+q)\right)^2 - 4\left(20 - \frac{5}{m} - \frac{m'}{m^2} q(5+4q)\right)}]. \quad (16)$$

If m is constant, the eigenvalues are $2, 5, 4 - \frac{1}{m}$. In this case P_3 is unstable for $m < 0$ and $m > \frac{1}{4}$ and a saddle otherwise.

(4) Point $P_4 : (-4, 5, 0)$ has eigenvalues:

$$-5, -3, 4(1 + \frac{1}{m}) \quad (17)$$

It is stable for $-1 < m < 0$ and saddle otherwise. This point cannot be used as a radiation or a matter dominated point.

...continued

(5) Point P_5 : $\left(\frac{3m}{1+m}, -\frac{1+4m}{2(1+m)^2}, \frac{1+4m}{2(1+m)}\right)$ can be regarded as a standard matter point in the limit $m \to 0$. In this limit $\Omega_m = 1$ and $a \propto t^{\frac{2}{3}}$. Hence, the necessary condition for this point to be a standard matter point is

$$m(q = -1) = 0. \tag{18}$$

Eigenvalues corresponding to point P_5 are given by

$$\begin{aligned} 3(1+m'), \\ \frac{-3m \pm \sqrt{m(256m^3 + 160m^2 - 31m - 16)}}{4m(m+1)} \end{aligned} \tag{19}$$

For a cosmologically viable trajectory, we want a saddle matter point. Hence, the condition for a saddle matter epoch is given by

$$\begin{aligned} m(q \leq -1) > 0, m'(q \leq -1) > -1, \\ m(q = -1) = 0 \end{aligned} \tag{20}$$

...continued

(6) Point P_6 : $\left(\frac{2(1-m)}{1+2m}, \frac{1-4m}{m(1+2m)}, -\frac{(1-4m)(1+m)}{m(1+2m)}\right)$ can also be an acceleration dominated point. The eigenvalues corresponding to this point are:

$$-4+\frac{1}{m}, \frac{2-3m-8m^2}{m(1+2m)}, -\frac{2(m^2-1)(1+m')}{m(1+2m)} \tag{21}$$

Stability of this point depends on both m and m'. The condition of acceleration ($w_{eff} < -\frac{1}{3}$) depends on the value of m.

Stability of fixed points with radiation

Next, we consider the radiation with other components of universe. In this case we have eight fixed points. Stability about the fixed points (x_1, x_2, x_3, x_4) is determined in the same way as in absence of radiation. Then we have (4×4) matrix of perturbations about each fixed point and four eigenvalues.

(1) Point P_1 corresponds to de-Sitter point. Here $w_{eff} = -1$ and eigenvalues corresponding to this point are

$$-4, -3, -\frac{3}{2} \pm \frac{\sqrt{25 - \frac{16}{m}}}{2} \tag{22}$$

In the presence of radiation, we have an eigenvalue -4 in addition to those in the absence of radiation. Since this eigenvalue is negative, the condition of stability is the same in both cases. P_1 is stable when $0 < m(q = -2) < 1$. This point may be taken as an acceleration point. The condition of stability for this point is same as in the case of without radiation because here we have only an extra eigenvalue -4, which is negative.

...continued

(2) Point P_2 is denoted by ϕ-matter-dominated (ϕ MDE) epoch. From eigenvalues corresponding to this point it is found that P_2 is either saddle or stable point. In this case P_2 can not be a matter point because $\Omega_m = 2$ and $w_{eff} = \frac{1}{3}$.

(3) P_3 is the kinetic point. The eigenvalues for the 4×4 matrix of perturbations about this point shows that if m is constant, the eigenvalues corresponding to this point are $2, 5, 4 - \frac{1}{m}$. In this case P_3 is unstable for $m < 0$ and $m > \frac{1}{4}$ and a saddle otherwise.

(4) Point P_4 has eigenvalues $-5, -4, -3, 4(1 + \frac{1}{m})$.
It is stable for $-1 < m < 0$ and saddle otherwise. This point cannot be used as a radiation or a matter dominated point.

(5) Point P_5 can be regarded as a standard matter point in the limit $m \to 0$. Eigenvalues for point P_5 are given by
$-1, 3(1 + m'), \frac{-3m \pm \sqrt{m(256m^3 + 160m^2 - 31m - 16)}}{4m(m+1)}$ where m' is derivative of m w.r.t. q. For a cosmologically viable trajectory, we want a saddle matter point. The condition for a saddle matter epoch is given by

...continued

(6) Point P_6 can also be an acceleration dominated point. The eigenvalues corresponding to this point are given by $-\frac{2(-1+2m+5m^2)}{m(1+2m)}, -4+\frac{1}{m}, \frac{2-3m-8m^2}{m(1+2m)}, -\frac{2(m^2-1)(1+m')}{m(1+2m)}$ Stability of this point depends on both m and m'. Condition of acceleration ($w_{eff} < -\frac{1}{3}$) depends on the value of m.

(7) Point P_7 corresponds to a standard radiation point. The eigenvalues of P_7 for constant m are $4, 4, 1, -1$. Thus, P_7 is a saddle point.

(8) Point P_8 also is a radiation point. In this case dark energy is non-zero, therefore P_8 is acceptable as a radiation point. The eigenvalues of P_8 are given by $1, 4(1+m'), \frac{m-1\pm\sqrt{81m^2+30m-15}}{2(m+1)}$. Point P_8 is a saddle point in the limit $m \to 0$. The acceptable radiation dominated point P_8 lies at point $(0, -1)$ in the (m, q) plane.

Dynamics of radiation dominated phase

For radiation dominated era, phase space analysis shows that we can find a radiation point in the limit $m \to 0$ at point P_8. This point lies on the line $m = -q - 1$ in the (m, q) plane. Hence, the necessary condition for this point to exist as an exact standard radiation point is given by

$$m(q = -1) \approx 0. \tag{24}$$

From definition of q and the above condition, the form of $f(R)$ for radiation dominated era is given by

$$f(R) = \alpha R \tag{25}$$

where α is an integration constant. Standard radiation point is obtained by substitution of $m \approx 0$ in the radiation point of $m(q)$ curve. In this condition, the effective equation of state is

$$w_{eff} = \frac{1}{3} \tag{26}$$

The Hubble parameter is given by

$$H(t) = \frac{1}{(2t - c_1)} \tag{27}$$

where c_1 is an integration constant. The scale factor $a(t)$ for this era is given by

$$a(t) = c_2(2t - c_1)^{\frac{1}{2}} \tag{28}$$

where c_2 is another integration constant. In radiation dominated phase we notice that the scale factor $a(t) \propto t^{\frac{1}{2}}$, which is same as in the case of standard model. The Ricci scalar R for radiation dominated era is given by

$$R = 0 \tag{29}$$

Dynamics of matter dominated era

From the field equations we obtain the following equations

$$-\frac{\kappa^2\rho_r}{3} + 3FH^2 + F\dot{H} - \frac{f}{2} - 2H\dot{F} - \ddot{F} = 0 \tag{30}$$

In phase space analysis of dynamical system, there is a point P_5 which represents a standard matter era in the limit $m \to 0$. In matter dominated phase of the Universe

$$m(q = -1) \approx 0 \tag{31}$$

Using the definition of q or m, the form of $f(R)$ is given by-

$$f(R) = \beta R \tag{32}$$

where β is a integration constant. Thus, in matter dominated phase the form of $f(R)$ is similar as in the case of radiation dominated phase.

The parameters...

In matter dominated phase, we neglect the energy density of radiation i.e. $\rho_r = 0$. For $f(R) = \beta R$, $F = \beta$ and therefore $\dot{F} = 0$. The Hubble parameter is given as

$$H(t) = \frac{1}{(\frac{3}{2}t - c_3)} \tag{33}$$

where c_3 is an integration constant. Scale factor in this phase is given by the expression

$$a(t) = c_4 \left(\frac{3}{2}t - c_3\right)^{\frac{2}{3}} \tag{34}$$

The Ricci scalar in matter dominated phase is given by

$$R = \frac{3}{(\frac{3}{2}t - c_3)^2} \tag{35}$$

Dynamics of accelerated expansion dominated phase

In the phase space analysis, there is a point P_1, for which effective equation of state is

$$w_{eff} = -1 \qquad (36)$$

This point is a de Sitter point. If we take de Sitter expansion, this point is stable when $0 < m < 1$ at $q = -2$. Now from the definition of q, the form of $f(R)$ in this phase is given by

$$f(R) = \alpha R^2 \qquad (37)$$

We have the effective equation of state

$$w_{eff} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} \qquad (38)$$

Parameters of accelerating phase

- The Hubble parameter in this phase is given by

$$H(t) = c_5 \tag{39}$$

where c_5 is an integration constant.

- Ricci scalar in this phase is given by

$$R = 12c_5^2 \tag{40}$$

- Using the expression of Hubble parameter $H(t)$, the scale factor is given by

$$a(t) = e^{c_5 t + c_6} \tag{41}$$

where c_6 is another integration constant.

Therefore, this seems to be similar to the ΛCDM approach to the late-time acceleration.

- The evolution of the universe through the fixed points P_8, P_5 and P_1 may define an arrow of time for structure formation.
- Thus P_1 would be a stable point signifying some equilibrium where R has a non-zero positive value that sustains the acceleration.
- That may mean an ever-accelerating universe!
- Time itself may be an emergent feature!

Conclusion

- We carried out the phase space analysis of $f(R)$ gravity models with dark matter.
- Determined the stability conditions of the fixed points in the presence of radiation with other components of the universe.
- Applying the conditions of stability on fixed points we derived different parameters of the dynamics in all phases of the universe.
- We have calculated the expression for Ricci scalar in all phases. Using the connection between thermodynamic parameter like entropy and the Ricci scalar we can do the ordering of different phases of the universe evolving to an equilibrium state.

Thank you for the kind attention !