Review of 3+1D Hydordynamics Evolution

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$t - z$ coordinates $\to$ $\tau - \eta$ coordinates

$$\begin{cases} t \text{ (laboratory time)} \\ z \text{ (beam direction)} \end{cases} \Longrightarrow \begin{cases} \tau = \sqrt{t^2 - z^2} \text{ (longitudinal proper time)} \\ \eta = \tanh^{-1}(z/t) \text{ (space-time rapidity)} \end{cases}$$

- Puzzling part

  Covariant derivative in curvilinear coordinates

Covariant derivative in curvilinear coordinates

A vector **a** expend by the local frame by point $M(x^1, x^2, \cdots, x^n)$ move to another point $N(x^1 + dx, x^2 + dx^2, \cdots, x^n + dx^n)$, where $d\mathbf{a} = 0$,thus

$$\begin{aligned} 0 &= \mathbf{x}_\alpha da^\alpha + a^\alpha d\mathbf{x}_\alpha \\ &= \mathbf{x}_\alpha da^\alpha + a^\beta \frac{\partial^2 \mathbf{x}}{\partial x^\beta \partial x^\gamma} dx^\gamma \end{aligned} \tag{1}$$

we change the dummy index of the second term, and $a^\beta \frac{\partial^2 \mathbf{x}}{\partial x^\beta \partial x^\gamma}$ can be expended by the local frame at point M,

$$\frac{\partial^2 \mathbf{x}}{\partial x^\beta \partial x^\gamma} = \Gamma^\alpha_{\beta\gamma} \mathbf{x}_\alpha$$

put it into the formula (1)

$$(da^\alpha)\mathbf{x}_\alpha = -(\Gamma^\alpha_{\beta\gamma} a^\beta dx^\gamma)\mathbf{x}_\alpha$$

Covariant derivative in curvilinear coordinates

as the $\mathbf{x}_\alpha$ are independent with each other,

$$da^\alpha = -\Gamma^\alpha_{\beta\gamma} a^\beta$$

The Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(\frac{\partial g_{\lambda\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\lambda}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\lambda} \right)$$

Now we can derive the covariant derivative. Consider a vector field $\mathbf{a}(M)$, whose covariant components are $a_\alpha(x^1, x^2, \cdots, x^n)$. When coordinates transform, a_α changes to,

$$a_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} a^\alpha$$

take the derivative with respect to $x^{\beta'}$,

$$\frac{\partial a_{\alpha'}}{\partial x^{\beta'}} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial a_\alpha}{\partial x^\beta} + \frac{\partial^2 x^\alpha}{\partial x^{\alpha'} \partial x^{\beta'}} a_\alpha$$

Covariant derivative in curvilinear coordinates

though some calculation we can derive that

$$\frac{\partial a_{\gamma'}}{\partial x^{\beta'}} - \Gamma^{\alpha'}_{\beta'\gamma'} a_{\alpha'} = \left(\frac{\partial a_{\gamma}}{\partial x^{\beta}} - \Gamma^{\alpha}_{\beta\gamma} a_{\alpha} \right) \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}$$

from above ,we can see that the expression with two freedom index $\beta\gamma$

$$\frac{\partial a_{\gamma}}{\partial x^{\beta}} - \Gamma^{\alpha}_{\beta\gamma} a_{\alpha} = \left(\frac{\partial}{\partial x^{\beta}} \delta^{\alpha}_{\gamma} - \Gamma^{\alpha}_{\beta\gamma} \right) a_{\alpha}$$

follow the second order covariant tensor's transformational rules.Then we find **Covariant derivative** of the covariant tensor field,

$$\nabla_{\beta} a_{\gamma} = \left(\frac{\partial}{\partial x^{\beta}} \delta^{\alpha}_{\gamma} \mp \Gamma^{\alpha}_{\beta\gamma} \right) a_{\alpha}$$

where " $-$ " for covariant tensor, " $+$ " for contravariant tensor. This can be generalized to arbitrary high order.

$$\nabla_s T^{lk}_{ij} = \partial_s T^{lk}_{ij} - \Gamma^{r}_{is} T^{lk}_{rj} - \Gamma^{r}_{js} T^{lk}_{ir} + \Gamma^{l}_{rs} T^{rk}_{ij} + \Gamma^{k}_{rs} T^{lr}_{ij}$$

The conservation law in the new coordinate system

$$\begin{cases} \partial\mu J^{\mu} = 0 \\ \partial\mu T^{\mu\nu} = 0 \end{cases}$$

transformed into

$$\begin{aligned} T^{\tau\tau}_{,\tau} + T^{\tau x}_{,x} + T^{\tau y}_{,y} + T^{\tau\eta}_{,\eta} + \frac{1}{\tau} T^{\tau\tau} + \tau T^{\eta\eta} &= 0 \\ T^{\tau x}_{,\tau} + T^{xx}_{,x} + T^{xy}_{,y} + T^{\eta x}_{,\eta} + \frac{1}{\tau} T^{\tau x} &= 0 \\ T^{\tau y}_{,\tau} + T^{xy}_{,x} + T^{yy}_{,y} + T^{\eta y}_{,\eta} + \frac{1}{\tau} T^{\tau y} &= 0 \\ T^{\tau\eta}_{,\tau} + T^{\eta x}_{,x} + T^{\eta y}_{,y} + T^{\eta\eta}_{,\eta} + \frac{3}{\tau} T^{\tau\eta} &= 0 \\ J^{\tau}_{,\tau} + J^{x}_{,x} + J^{y}_{,y} + J^{\eta}_{,\eta} + \frac{1}{\tau} J^{\tau} &= 0 \end{aligned}$$

The conservation law in the new coordinate system

By making use of the relations $T^{\tau i} = \bar{v}_i T^{\tau\tau} + v_i P, T^{\eta\eta} = \frac{P}{\tau^2}$, the energy-momentum conservation equations turn into

$$T^{\tau\tau}_{,\tau} + (v_x T^{\tau\tau})_{,x} + (v_y T^{\tau\tau})_{,y} + (\frac{v_\eta}{\tau} T^{\tau\tau}) + \frac{1}{\tau} T^{\tau\tau}$$

$$+(v_x P)_{,x} + (v_y P)_{,y} + (\frac{v_\eta}{\tau} P)_{,\eta} + \frac{v_\eta^2}{\tau}(T^{\tau\tau} + P) + \frac{P}{\tau} = 0$$

$$T^{\tau x}_{,\tau} + (v_x T^{\tau x})_{,x} + (v_y T^{\tau x})_{,y} + (\frac{v_\eta}{\tau} T^{\tau x})_{,\eta} + \frac{1}{\tau} T^{\tau x} + P_{,x} = 0$$

$$T^{\tau y}_{,\tau} + (v_x T^{\tau y})_{,x} + (v_y T^{\tau y})_{,y} + (\frac{v_\eta}{\tau} T^{\tau y})_{,\eta} + \frac{1}{\tau} T^{\tau y} + P_{,y} = 0$$

$$T^{\tau\eta}_{,\tau} + (v_x T^{\tau\eta})_{,x} + (v_y T\tau\eta)_{,y} + (\frac{v_\eta}{\tau} T^{\tau\eta}_{,\eta} + (\frac{P}{\tau^2})_{,\eta} + \frac{3}{\tau} T^{\tau\eta} = 0$$

$$J^{\tau}_{,\tau} + (v_x J^{\tau})_{,x} + (v_y J^{\tau})_{,y} + (\frac{v_\eta}{\tau} J^{\tau})_{,\eta} + \frac{1}{\tau} J^{\tau} = 0$$

Evolution

- FCT-SHASTA Algorithm

  We can use the SHASTA(SHarp And Smooth Transport Algorithm) algorithm to solve partial differential equations with the form

  $$\partial_t (T) + \partial_i (v_i T) = S$$

  to solve the hydrodynamic equations.

- Transport stage in SHASTA
- Calculation for Hydrodynamics evolution equations

Transport stage in SHASTA

$$\partial_t \rho + \partial_x (v\rho) = 0$$

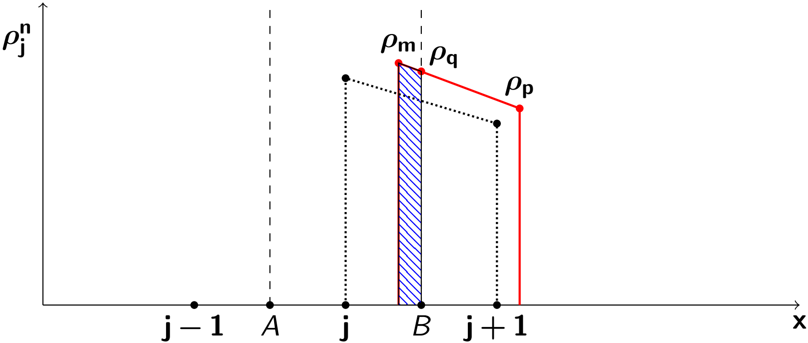
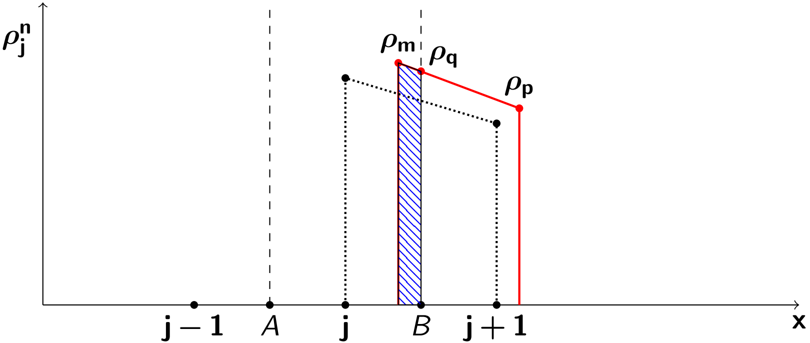


Figure : The geometric explanation of SHASTA algorithm

Transport stage in SHASTA



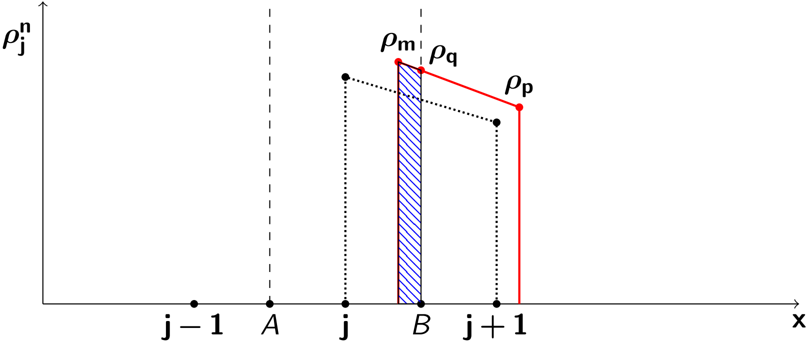
$$\frac{1}{2}(\rho_j^0 + \rho_{j+1}^0)\delta x = \frac{1}{2}(\rho_m + \rho_p)\left[\delta x + \left(v_{j+1}^{1/2} - v_j^{1/2}\right)\delta t\right]$$

If we consider the two sides if the solid trapezoid vary in the same rate, we get

$$\rho_p = \rho_{j+1}^0 \delta x / \left[\delta x + \left(v_{j+1}^{1/2} - v_j^{1/2}\right)\delta t\right]$$

$$\rho_m = \rho_j^0 \delta x \left[\delta x + \left(v_{j+1}^{1/2} - v_j^{1/2}\right)\delta t\right]$$

Transport stage in SHASTA

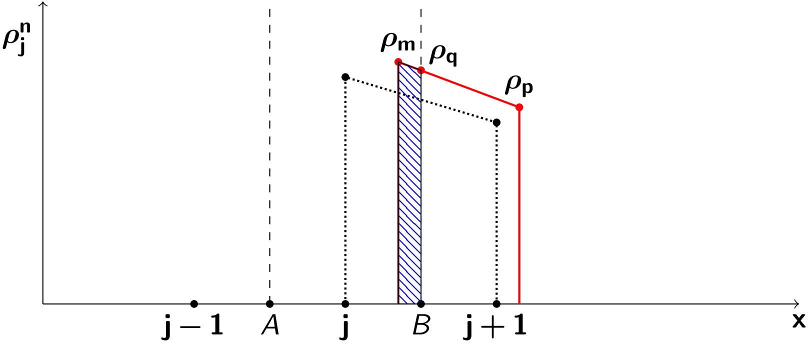


$$\rho_q = \rho_p + (\rho_m - \rho_p)\left(\delta x/2 + v_{j+1}^{1/2}\delta t\right)\left[\delta x + \left(v_{j+1}^{1/2} - v_j^{1/2}\right)\delta t\right]$$

The residual mass is given by the the blue shadow area,

$$\Delta m_{re,j}^n = (\rho_m + \rho_q)\left(\delta x/2 - v_j^{1/2}\delta t\right)$$
$$= \delta x\left[\frac{1}{2}Q_+^2\left(\rho_{j+1}^n - \rho_j^n\right) + Q_+\rho_j^n\right]$$

Transport stage in SHASTA

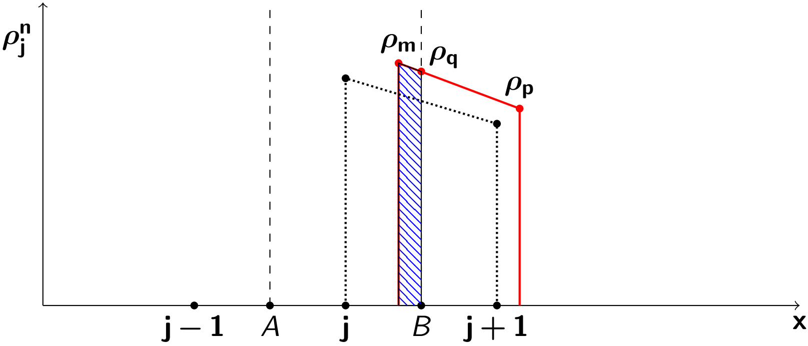


The another part of the element's mass (in cell$j+1$) is $\Delta m^n_{an,j}$,we can easily calculate

$$\Delta m^n_{an,j-1} = \delta x \left[\frac{1}{2} Q_-^2 \left(\rho^n_{j-1} - \rho^n_j \right) + Q_- \rho^n_j \right]$$

$$Q_\pm = \left(1/2 \mp v_j^{1/2} \delta t / \delta x \right) / \left[1 \pm \left(v_{j\pm1}^{1/2} - v_j^{1/2} \right) \delta t / \delta x \right]$$

Transport stage in SHASTA



The mass density at grid point j and time step $n+1$ is

$$\rho_j^{n+1} = \left(\Delta m_{an,j-1}^n + \Delta m_{re,j}^n\right)/\delta x$$
$$= \frac{1}{2}Q_-^2\left(\rho_{j-1}^n - \rho_j^n\right) + \frac{1}{2}Q_+^2\left(\rho_{j+1}^n - \rho_j^n\right) + \left(Q_+ + Q_-\right)\rho_j^n$$

Transport stage in SHASTA

If we consider a uniform velocity, the above equation will be simplified to

$$\rho_j^{n+1} = \rho_j^n - \frac{\varepsilon}{2}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) + \left(\frac{1}{8} + \frac{\varepsilon^2}{2}\right)\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right)$$

where $\varepsilon = v\delta t/\delta x$. If $v = 0$,

$$\rho_j^{n+1} = \frac{1}{8}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right)$$

Anti-diffusion stage

Now,we can see that the solution is conservative and positivity but has a large zero order diffusion.To correct the diffusion, we used an anti-diffusion form

$$\bar{\rho}_j^{n+1} = \frac{1}{8}\left(\rho_{j+1}^{n+1} - 2\rho_j^{n+1} + \rho_{j-1}^{n+1}\right)$$

but it will destroy the positivity. So the anti-diffusion terms are written in mass flux form

$$\bar{\rho}_j^{n+1} = \rho_j^{n+1} - f_{j+1/2}^{n+1} + f_{j-1/2}^{n+1}$$

where the mass flux is

$$f_{j\pm1/2}^{n+1} = \pm\frac{1}{8}\left(\rho_{j\pm1}^{n+1} - \rho_j^{n+1}\right)$$

Anti-diffusion stage

and the flux is corrected as

$$f_{j+1/2}^{(c)n+1} = \sigma \max\left\{0, \min\left\{\sigma\Delta_{j-1/2}, \frac{1}{8}\Delta_{j+1/2}, \sigma\Delta_{j+3/2}\right\}\right\}$$

where $\Delta_{j+1/2} = \rho_{j+1}^{n+1} - \rho_j^{n+1}$ and $\sigma = \text{sgn}\Delta_{j+1/2}$.
The final mass density at grid j and time step$n+1$ after corrected anti-diffusion stage is

$$\bar{\rho}_j^{n+1} = \rho_j^{(c)n+1} - f_{j+1/2}^{n+1} + f_{j-1/2}^{(c)n+1}$$

Multi-dimensional algorithm

In this multi-dimensional algorithm the 1D FCT-SHASTA algorithm with time-splitting is used along one direction at a split-time step, while a $x \to y \to \eta \to y \to x$ rotation is used to extend the FCT-SHASTA algorithm to multi-dimensions and to suppress the numerical eccentricity produced in transverse direction during the hydrodynamic evolution.

Evolution

At first, we will change conservation equations to the form as

$$\tilde{T}^{\tau\tau}_{,\tau} + \left(v_x \tilde{T}^{\tau\tau}\right)_{,x} + \left(v_y \tilde{T}^{\tau\tau}\right)_{,y} + \left(v'_\eta \tilde{T}^{\tau\eta}\right)_{,\eta} = S^\tau$$

$$\tilde{T}^{\tau x}_{,\tau} + \left(v_x \tilde{T}^{\tau x}\right)_{,x} + \left(v_y \tilde{T}^{\tau x}\right)_{,y} + \left(v'_\eta \tilde{T}^{\tau x}\right)_{,\eta} = S^x$$

$$\tilde{T}^{\tau y}_{,\tau} + \left(v_x \tilde{T}^{\tau y}\right)_{,x} + \left(v_y \tilde{T}^{\tau y}\right)_{,y} + \left(v'_\eta \tilde{T}^{\tau y}\right)_{,\eta} = S^y$$

$$\tilde{T}^{\tau\eta}_{,\tau} + \left(v_x \tilde{T}^{\tau\eta}\right)_{,x} + \left(v_y \tilde{T}^{\tau\eta}\right)_{,y} + \left(v'_\eta \tilde{T}^{\tau\eta}\right)_{,\eta} = S^\eta$$

$$\tilde{J}^{\tau}_{,\tau} + \left(v_x \tilde{J}^{\tau}\right)_{,x} + \left(v_y \tilde{J}^{\tau}\right)_{,y} + \left(v'_\eta \tilde{J}^{\tau}\right)_{,\eta} = 0$$

where the tilde term $\tilde{X} = \tau X$, $v'_\eta = v_\eta / \tau$, and the source terms are

$$\begin{pmatrix} S^\tau \\ S^x \\ S^y \\ S^\eta \end{pmatrix} = \begin{pmatrix} -\nabla \cdot \left(\mathbf{v}\tilde{P}\right) - \tau {v'_\eta}^2 \left(\tilde{T}^{\tau\tau} + \tilde{P}\right) - \tilde{P}/\tau \\ -\tilde{P}_{,x} \\ -\tilde{P}_{,y} \\ -\left(\frac{\tilde{P}}{\tau^2}\right)_{,\eta} - \frac{2}{\tau}\tilde{T}^{\tau\eta} \end{pmatrix}$$

Evolution

Then the numerical method and procedures of a combined FCT-SHASTA and the 2nd order Runge-Kutta algorithm in solving hydrodynamic equations with a simplified conservation as below

$$\partial_\tau \mathscr{T} + \partial_i \left(v_i \mathscr{T} \right) = S$$

where $\mathscr{T} := \tau T^{\tau v}$ denotes one component of the energy-momentum tensor.

Evolution

1. Calculate source term at time step $n : S = S\left(\tau^n, \varepsilon^n, v_i^n, \mathscr{T}^n\right)$.
2. Get $\mathscr{T}'^{n+1/2}$ by using SHASTA algorithm through $\partial_\tau \mathscr{T} + \partial_i \left(v_i \mathscr{T}\right) = 0$ within time step $n + 1/2$.
3. Get $\mathscr{T}^{n+1/2} = \mathscr{T}'^{n+1/2} + \frac{1}{2}\Delta\tau S\left(\tau^n, \varepsilon^n, v_i^n, \mathscr{T}^n\right)$ and use the root-finding method to calculate the energy density and velocity $\varepsilon^{n+1/2}$, $v_i^{n+1/2}$.
4. Calculate source term at half-time step$n + 1/2 : S = S\left(\tau^{n+1/2}, \varepsilon^{n+1/2}, v_i^{n+1/2}, \mathscr{T}^{n+1/2}\right)$.
5. Get $\mathscr{T}'^{n+1}$ by using SHASTA algorithm through $\partial_\tau \mathscr{T} + \partial_i \left(v_i \mathscr{T}\right) = 0$ with $v_i^{n+1/2}$
6. Get $\mathscr{T}^{n+1} = \mathscr{T}'^{n+1} + \frac{1}{2}\Delta\tau S\left(\tau^{n+1/2}, \varepsilon^{n+1/2}, v_i^{n+1/2}, \mathscr{T}^{n+1/2}\right)$ and use the root-finding method to calculate the energy density and velocity ε^{n+1}, v_i^{n+1}.

Root-finding

The energy density is determined from $T^{\tau\nu}$ through a root finding method by iterating the equation

$$\varepsilon = T^{\tau\tau} - \frac{M^2}{T^{\tau\tau} + P(\varepsilon)}$$

with the iteration value of ε for the iteration is approximated by $\varepsilon = T^{\tau\tau}$. Where $M^2 = \left(T^{\tau\perp}\right)^2 + \left(\tau T^{\tau\eta}\right)^2$. And the flow velocity is given by

$$\vec{v}_\perp = \vec{T}^{\tau\perp} / \left[T^{\tau\tau} + P(\varepsilon)\right]$$
$$v_\eta = \tau T^{\tau\eta} / \left[T^{\tau\tau} + P(\varepsilon)\right]$$


תודה
Dankie
Gracias
Спасибо
شكراً
Merci
Takk
Köszönjük
Terima kasih
Grazie
Dziękujemy
Dėkojame
Ďakujeme
Vielen Dank
Paldies
Kiitos
Täname teid
谢谢
Thank You
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ขอบคุณ
Bedankt
Děkujeme vám
ありがとうございます
Tack