Microscopic Foundations of Relativistic Fluid Dynamics

Contents

Units : $\hbar = c = k_B = 1$

metric tensor: $g^{\mu\nu} = \mathrm{diag}(+,-,-,-)$

Introduction to relativistic fluid dynamics

I. ~~Fluid-dynamical variables~~

1. Particle 4-current

The 4-vector associated to the current of particles is given by

$$N^\mu = (N^0, \vec{N})^T ,$$

where N^0 is the particle density and $\vec{N}$ the particle 3-current in a given reference frame.

Now consider a frame which is moving with 4-velocity

$$u^\mu = (u^0, \vec{u})^T$$

with respect to the reference frame.

u^μ must be time-like, $u^\mu u_\mu > 0$.

~~It can be normalized~~ As a velocity, it is normalized to 1:

$$u^\mu u_\mu = +1 .$$

Then, $1 = u^\mu u_\mu = u_0^2 - \vec{u}^2$

$\Rightarrow \quad u_0 = +\sqrt{1+\vec{u}^2} = \cosh\eta \quad \Rightarrow \quad |\vec{u}| = \sqrt{\cosh^2\eta - 1} \equiv \sinh\eta$

$\equiv \gamma \equiv \dfrac{1}{\sqrt{1-\vec{v}^2}} \quad \Rightarrow \quad \vec{u} = \gamma\vec{v}$

$\Rightarrow \quad u^\mu = \gamma(1, \vec{v})^T$ with $\vec{v}$ being the 3-velocity of the frame.

One can tensor-decompose N^μ with respect to u^μ.

To this end, introduce a projector onto the 3-space orthogonal to u^μ,

$$\boxed{\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu}$$

Obviously, $\Delta^{\mu\nu} u_\nu = \underbrace{g^{\mu\nu} u_\nu}_{u^\mu} - u^\mu \underbrace{u^\nu u_\nu}_{1} \equiv 0$

Also, $\Delta^{\mu\alpha} \Delta_\alpha{}^\nu = (g^{\mu\alpha} - u^\mu u^\alpha)(g_\alpha{}^\nu - u_\alpha u^\nu)$

$$= g^{\mu\alpha} g_\alpha{}^\nu - g^{\mu\alpha} u_\alpha u^\nu - u^\mu u^\alpha g_\alpha{}^\nu + u^\mu \underbrace{u^\alpha u_\alpha}_{1} u^\nu$$

$$= g^{\mu\nu} - 2u^\mu u^\nu + u^\mu u^\nu \equiv \Delta^{\mu\nu}$$

The dimensionality of the space $\Delta^{\mu\nu}$ projects on is given by the trace of $\Delta^{\mu\nu}$:

$$\Delta^\mu_\mu = g^\mu_\mu - u^\mu u_\mu = 4 - 1 = 3$$

In the local rest frame defined by u^μ, i.e., $u^\mu_{LR} \equiv (1,\vec{0})^T$, $\Delta^{\mu\nu}_{LR} = g^{\mu\nu} - u^\mu_{LR} u^\nu_{LR} \equiv \mathrm{diag}(0,-,-,-)$

$$\Delta^\mu_{LR,\nu} = \begin{pmatrix} 0 & 0\,0\,0 \\ 0 & \\ 0 & (+\delta^{ij}) \\ 0 & \end{pmatrix}$$

Now $\boxed{N^\mu = n u^\mu + n^\mu}$

where $\boxed{n \equiv N^\mu u_\mu}$ is the particle density in the local rest frame given by u^μ: Indeed, in that frame $u^\mu_{LR} = (1,\vec{0})^T$ and

$$n_{LR} = N^\mu_{LR} u_{LR,\mu} = N^0_{LR}.$$

However, since $N^\mu u_\mu$ is a Lorentz scalar, it is frame-independent, so we dispense with the index "LR" and simply write $n \equiv n_{LR}$.

Furthermore, $\boxed{n^\mu \equiv \Delta^{\mu\nu} N_\nu}$ is the 3-current of particles in the frame defined by u^μ, so it is the flow of particles relative to u^μ. ("particle diffusion current")

Indeed, in the local rest frame of u^μ:

$$n^\mu_{LR} = \Delta^{\mu\nu}_{LR} N_{LR,\nu} = N^\mu_{LR} - u^\mu_{LR} \underbrace{u^\nu_{LR} N_{LR,\nu}}_{\equiv n_{LR} \equiv N^0_{LR}}$$

$$\equiv (N^0_{LR} - N^0_{LR}, \vec{N}_{LR})^T \equiv (0, \vec{N}_{LR})^T$$

($u^\mu_{LR} = (1,\vec{0})^T$)

Since n^μ is orthogonal to u^μ,

$$n^\mu u_\mu = \underbrace{u_\mu \Delta^{\mu\nu}}_{=0} N_\nu = 0,$$

it has only 3 independent components.

N^μ has 4 independent components.

2. Energy - momentum tensor

The energy-momentum tensor is denoted as $T^{\mu\nu}$ ~~is a symmetric rank 2 tensor~~ $T^{\mu\nu} = T^{\nu\mu}$. In matrix notation

$$T^\mu{}_\nu = \begin{pmatrix} T^{00} & -T^{0x} & -T^{0y} & -T^{0z} \\ T^{x0} & & & \\ T^{y0} & & -T^{ij} & \\ T^{z0} & & & \end{pmatrix}$$

Here, T^{00} is the energy density in the frame of reference, T^{0i} ~~$\equiv T^{i0}$~~ is the ith component of the energy flow ~~momentum density~~ in that frame, ~~and T^{ij} is the so-called stress tensor~~ T^{i0} is the ith component of momentum density in that frame and T^{ij} is the so-called momentum flow or pressure tensor

The energy-momentum tensor can be chosen to be symmetric, $T^{\mu\nu} \equiv T^{\nu\mu}$, thus energy flow and momentum density are the same, $T^{0i} \equiv T^{i0}$

Now perform a tensor decomposition of $T^{\mu\nu}$ with respect to u^μ

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu + W^\mu u^\nu + u^\mu W^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}}$$

$$\equiv (\epsilon + P) u^\mu u^\nu + 2W^{(\mu} u^{\nu)} - P g^{\mu\nu} + \pi^{\mu\nu}$$

~~Obviously, ϵ~~

where $W^\mu u_\mu \equiv 0$ ~~i.e., $W^\mu = \Delta^\mu{}_\alpha T^{\alpha\beta} u_\beta$~~

$$\pi^{\mu\nu} u_\nu = u_\mu \pi^{\mu\nu} = 0, \quad \pi^\mu{}_\mu = 0, \text{ i.e.}$$

~~$\pi^{\mu\nu} = \Delta^{\mu\nu}{}_{\alpha\beta} T^{\alpha\beta} \equiv \left[\frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$~~

$$a^{(\mu} b^{\nu)} \equiv \frac{1}{2}(a^\mu b^\nu + a^\nu b^\mu)$$

How to obtain the various quantities in this tensor decomposition?

Obviously, $T^{\mu\nu} u_\nu = \epsilon u^\mu + W^\mu$,

$$\Delta^\mu{}_\alpha T^{\alpha\beta} = \Delta^\mu{}_\alpha W^\alpha u^\beta - P\Delta^{\mu\beta} + \Delta^\mu{}_\alpha \pi^{\alpha\beta},$$

i.e.,

$$\boxed{\epsilon \equiv u_\mu T^{\mu\nu} u_\nu}$$

$$\boxed{W^\mu = \Delta^\mu{}_\alpha T^{\alpha\beta} u_\beta}$$

$$\boxed{P = -\frac{1}{3}(T^\mu{}_\mu - u_\mu T^{\mu\nu} u_\nu) \equiv -\frac{1}{3} T^{\mu\nu}\Delta_{\mu\nu}}$$

Check: $= -\frac{1}{3}(\epsilon - 3P - \epsilon) = -\frac{3}{3}P$ ✓

$$\boxed{\pi^{\mu\nu}} = \Delta^\mu{}_\alpha T^{\alpha\beta} \Delta_\beta{}^\nu + P\Delta^{\mu\nu}$$

$$= \Big[\underbrace{\frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}}_{\equiv \Delta^{\mu\nu}{}_{\alpha\beta}}\Big] T^{\alpha\beta}$$

$$\equiv \boxed{\Delta^{\mu\nu}{}_{\alpha\beta} T^{\alpha\beta}}$$

What is the interpretation of these quantities?

Again, we go to the LR frame defined by u^μ:

$$\epsilon_{LR} = u_{LR,\mu} T^{\mu\nu}_{LR} u_{LR,\nu} \equiv T^{00}_{LR},$$

so ϵ_{LR} is the energy density in the LR frame of u^μ. But since it is a Lorentz scalar, ~~it is the case~~ we may write $\epsilon \equiv \epsilon_{LR}$.

$$W^\mu_{LR} = \Delta^\mu_{LR,\alpha}\, T^{\alpha\beta}_{LR}\, u_{LR\beta}$$

$$= T^{\mu 0}_{LR} - u^\mu_{LR}\, T^{00}_{LR} = \left(0, T^{x0}_{LR}, T^{y0}_{LR}, T^{z0}_{LR}\right)^T,$$

is the energy flow or momentum density in the LR frame ~~of~~ given by u^μ.
It is the flow of energy relative to u^μ ("heat flow"). Since $W^\mu u_\mu = 0$, it has 3 independent components.

$$P_{LR} = -\frac{1}{3}\Delta^{\mu\nu}_{LR}\, T_{LR,\mu\nu} = -\frac{1}{3}\left(T_{LR}{}^{\mu}{}_{\mu} - \epsilon\right) = -\frac{1}{3}\left(T^{x}_{LR,x} + T^{y}_{LR,y} + T^{z}_{LR,z}\right)$$

$$= \frac{1}{3}\left(T^{xx}_{LR} + T^{yy}_{LR} + T^{zz}_{LR}\right)$$

is $\frac{1}{3}$ of the trace of the ~~momentum-flow~~ pressure tensor. It is the so-called isotropic pressure.
Since it is a Lorentz scalar, we may simply write $P_{LR} \equiv P$

$$\pi^{\mu\nu}_{LR} = \Delta^{\mu\alpha}_{LR}\, T_{LR,\alpha\beta}\, \Delta^{\beta\nu}_{LR} + P\, \Delta^{\mu\nu}_{LR}$$

$$= \left(T^{\mu}_{LR,\beta} - u^\mu_{LR}\, T^{0}_{LR,\beta}\right)\left(g^{\beta\nu} - u^\beta_{LR} u^\nu_{LR}\right) + P\left(g^{\mu\nu} - u^\mu_{LR} u^\nu_{LR}\right)$$

$$= T^{\mu\nu}_{LR} - T^{\mu 0}_{LR} u^\nu_{LR} - u^\mu_{LR} T^{0\nu}_{LR} + u^\mu_{LR} T^{00}_{LR} u^\nu_{LR} + P\left(g^{\mu\nu} - u^\mu_{LR} u^\nu_{LR}\right)$$

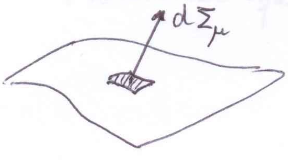
$$\pi_{LR}{}^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 \quad 0 \quad 0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & -T^{ij}_{LR} + P\,\delta^{ij} \end{pmatrix}$$

is the trace-free part of the ~~momentum-flow~~ pressure tensor. It is the so-called shear-stress tensor. Since $\pi^{\mu\nu} u_\nu = u_\mu \pi^{\mu\nu} = 0$, and $\pi^\mu{}_\mu = 0$, it has $10 - 4 - 1 = 5$ independent components.
$T^{\mu\nu}$ has $1 + 3 + 1 + 5 = 10$ independent components.

3. Conservation laws

Consider a hypersurface Σ in 4-dim. space-time. The normal vector on this hypersurface is denoted by $d\Sigma_\mu(X)$.

$d\Sigma = \sqrt{|d\Sigma_\mu\, d\Sigma^\mu|}$ is the infinitesimal hypersurface element on which $d\Sigma_\mu(X)$ is normal:



The amount of energy and momentum and of particles flowing through $d\Sigma$ is given by

$$dP^\mu \equiv T^{\mu\nu}\, d\Sigma_\nu$$
$$dN \equiv N^\mu\, d\Sigma_\mu$$

Now consider an arbitrary space-time volume V_4. This has a closed surface $\Sigma \equiv \partial V_4$. If there are neither sources nor sinks of energy, momentum, or particles inside V_4, we have

$$\oint_\Sigma d\Sigma_\nu\, T^{\nu\mu} = 0$$

$$\oint_\Sigma d\Sigma_\mu N^\mu = 0$$

Applying Gauß' theorem leads to the global conservation of energy, momentum, and particles,

$$\int_{V_4} d^4x\, \partial_\nu T^{\nu\mu} = 0$$

$$\int_{V_4} d^4x\, \partial_\mu N^\mu = 0$$

But V_4 was arbitrary, so the integrals can only vanish if the integrands vanish. This leads to local conservation of energy, momentum and particles:

$$\boxed{\begin{aligned} \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\mu N^\mu &= 0 \end{aligned}}$$

These are the equations of motion of relativistic fluid dynamics.

Note that these are 5 equations for $\cancel{14}$ $10+4=14$ unknowns, so the system of equations of motion is not closed.

Either, one makes some drastic assumptions about the form of $T^{\mu\nu}$, N^μ in order to reduce the number of unknowns ($\rightarrow$ ideal fluid dynamics).

Or one provides 9 additional eqs. ~~for the~~ ($\rightarrow$ dissipative fluid dynamics).

4. Choice of frame

So far, the 4-velocity u^μ was arbitrary. We now want to give it some physical meaning.

We could define $\boxed{u_N^\mu \equiv \frac{N^\mu}{\sqrt{N_\alpha N^\alpha}}}$

Then, u^μ is the 4-velocity of particle flow. Obviously,

$$\cancel{n = N^\mu u_{N,\mu} \equiv n\, u^\mu_{N}}$$

$$n_N^\mu = \Delta_N^{\mu\nu} N_\nu \equiv N^\mu - \underbrace{u_N^\mu}_{= \frac{N^\mu}{\sqrt{N_\alpha N^\alpha}}} \underbrace{u_N^\nu}_{=\frac{N^\nu}{\sqrt{N_\beta N^\beta}}} N_\nu = N^\mu \left(1 - \frac{u_N^\nu N_\nu}{\sqrt{N_\alpha N^\alpha}}\right) = N^\mu \left(1 - \frac{N^\nu N_\nu}{N_\alpha N_\alpha}\right) \equiv 0 .$$

There is no flow of particles relative to u_N^μ, i.e., no diffusion current.

This choice of u^μ defines the so-called Eckart frame.

We could also choose u^μ to be the time-like eigenvector of $T^{\mu\nu}$, i.e.,

$$\boxed{T^{\mu\nu} u_{E,\nu} \equiv \lambda u_E^\mu}$$

Clearly, $\lambda \equiv u_{E,\mu} T^{\mu\nu} u_{E,\nu} \equiv \epsilon$, the energy density.

Another way to write this definition of u^μ is

$$\boxed{u_E^\mu \equiv \frac{T^{\mu\nu} u_{E,\nu}}{\sqrt{u_E^\alpha T_\alpha^{\ \beta} T_\beta^{\ \gamma} u_{E,\gamma}}}},$$

since

$$= \frac{\epsilon\, u_E^\mu}{\sqrt{\epsilon\, u_E^\beta\, \epsilon\, u_{E,\beta}}} \equiv u_E^\mu .$$

~~There is no~~ Obviously,

$$W_E^\mu = \Delta_E^{\mu\alpha} T_{\alpha\beta} u_E^\beta \equiv \Delta_E^{\mu\alpha} \epsilon u_{E,\alpha} \equiv 0 ,$$ i.e.,

there is no flow of energy (heat flow) relative to u_E^μ.

This choice defines the so-called Landau frame.

5. Tensor-decomposed conservation laws

We now write the conservation eqs. in tensor-decomposed form:

$$0 = \partial_\mu N^\mu = \partial_\mu (n u^\mu + n^\mu) = u^\mu \partial_\mu n + n\, \partial_\mu u^\mu + \partial_\mu n^\mu$$

Denote $\boxed{\dot a \equiv u^\mu \partial_\mu a}$, the comoving derivative of a

(In the LR frame defined by u^μ, this is the ordinary time derivative,

$$\dot a = u^\mu_{LR}\, \partial_\mu a \equiv \partial_t a \;)$$

Denote $\boxed{\Theta \equiv \partial_\mu u^\mu}$, the expansion scalar.

Note: since $\partial_\mu \equiv u_\mu u^\nu \partial_\nu + \underbrace{\Delta_\mu^{\ \nu} \partial_\nu} \equiv u_\mu u^\nu \partial_\nu + \nabla_\mu$,

∇_μ 3-space gradients

(in LR frame: $\nabla_{LR,\mu} = (0, \vec\nabla)$).

we can also write $\Theta = \partial_\mu u^\mu = (u_\mu u^\nu \partial_\nu + \nabla_\mu) u^\mu = \underbrace{u_\mu u^\nu \partial_\nu u^\mu}_{\equiv 0,} + \nabla_\mu u^\mu = \nabla_\mu u^\mu$

since $d(u^\mu u_\mu) \equiv 0$
$= 2 u_\mu du^\mu$

Since $\partial_\mu n^\mu = \underbrace{u_\mu u^\nu \partial_\nu n^\mu} + \nabla_\mu n^\mu$

$$= u^\nu \partial_\nu (\underbrace{u_\mu n^\mu}_{=0}) - n^\mu \underbrace{u^\nu \partial_\nu u_\mu}_{\equiv \dot u_\mu}$$

$$\Rightarrow \boxed{0 = \dot n + n\Theta + \nabla_\mu n^\mu - n^\mu \dot u_\mu}$$

The energy-momentum conservation equation can be projected either onto u_μ or onto $\Delta_{\mu\nu}$

(a) $0 = u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu (\epsilon u^\mu u^\nu + W^\mu u^\nu + u^\mu W^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu})$

$$= \dot\epsilon + \epsilon\Theta + \epsilon \underbrace{u_\nu \dot u^\nu}_{=0} + \partial_\mu W^\mu + W^\mu \underbrace{u_\nu \partial_\mu u^\nu}_{=0} + \underbrace{u_\nu \dot W^\nu}_{=-W^\nu \dot u_\nu} + \underbrace{u_\nu W^\nu}_{=0} \Theta$$

$$- \underbrace{u_\nu \Delta^{\mu\nu}}_{=0} \partial_\mu P - P u_\nu \underbrace{\partial_\mu \Delta^{\mu\nu}} + \underbrace{u_\nu \partial_\mu \pi^{\mu\nu}}_{= -\pi^{\mu\nu}\partial_\mu u_\nu}$$

$$-u^\mu \partial_\mu u^\nu - u^\nu \Theta$$

$$= -\cancel{\dot u^\nu} - u^\nu \Theta$$

$$= \dot\epsilon + (\epsilon + P)\Theta + \underbrace{\partial_\mu W^\mu - W^\mu \dot u_\mu}_{\equiv \nabla_\mu W^\mu - W^\mu \dot u_\mu} - \pi^{\mu\nu} \partial_\mu u_\nu$$

Now ~~define~~ decompose (relativistic Cauchy-Stokes formula)

$$\boxed{\partial_\mu u_\nu \equiv \cancel{u_\mu u^\alpha \partial_\alpha u_\nu + \nabla_\mu u_\nu}\ u_\mu \dot{u}_\nu + \tfrac{1}{3}\Theta \Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}}$$

$$\cancel{\tfrac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) - \tfrac{1}{3}\Theta\Delta_{\mu\nu}} \quad \cancel{\Theta \equiv \nabla_\alpha u^\alpha} \quad \cancel{\Big\} \equiv \Big[\tfrac{1}{2}(\Delta^\alpha_\mu \Delta^\beta_\nu + \Delta^\alpha_\nu \Delta^\beta_\mu) - \tfrac{1}{3}\Delta_{\mu\nu}\Delta^{\alpha\beta}\Big] \nabla_\alpha u_\beta}$$

$$\cancel{+\tfrac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu) + \tfrac{1}{3}\Theta\Delta_{\mu\nu}} \quad \cancel{=}$$

$$\cancel{\equiv u_\mu \dot{u}_\nu + \Delta^{\alpha\beta}_{\mu\nu}\partial_\alpha u_\beta}$$

where

$$\boxed{\sigma_{\mu\nu} \equiv \Delta^{\alpha\beta}_{\mu\nu}\partial_\alpha u_\beta}, \text{ shear tensor}$$

$$\boxed{\omega_{\mu\nu} \equiv \tfrac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu)}, \text{ vorticity tensor}$$

<u>Proof</u>: Start from r.h.s.:

$$u_\mu \dot{u}_\nu + \tfrac{1}{3}\Theta\Delta_{\mu\nu} + \Big[\tfrac{1}{2}(\Delta^\alpha_\mu \Delta^\beta_\nu + \Delta^\alpha_\nu \Delta^\beta_\mu) - \tfrac{1}{3}\Delta_{\mu\nu}\Delta^{\alpha\beta}\Big]\partial_\alpha u_\beta + \tfrac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu) \quad \cancel{+\tfrac{1}{2}(\Delta^\alpha_\mu \partial_\alpha u_\nu - \Delta^\alpha_\nu \partial_\alpha u_\mu)}$$

$$= u_\mu \dot{u}_\nu + \cancel{\tfrac{1}{3}\nabla_\alpha u^\alpha \Delta_{\mu\nu}} + \Big[\tfrac{1}{2}(\Delta^\beta_\nu \nabla_\mu u_\beta + \Delta^\beta_\mu \nabla_\nu u_\beta) - \cancel{\tfrac{1}{3}\Delta_{\mu\nu}\nabla^\beta u_\beta}\Big]$$

$$+ \tfrac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu)$$

$$= u_\mu \dot{u}_\nu + \tfrac{1}{2}(\nabla_\mu u_\nu + \cancel{\nabla_\nu u_\mu}) + \tfrac{1}{2}(\nabla_\mu u_\nu - \cancel{\nabla_\nu u_\mu}) = u_\mu u^\alpha \partial_\alpha u_\nu + \nabla_\mu u_\nu$$

$\equiv \partial_\mu u_\nu$, q.e.d.

Since $\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$, we have

$$\pi^{\mu\nu}\partial_\mu u_\nu \underset{\uparrow}{=} T^{\alpha\beta}\Delta^{\mu\nu}_{\alpha\beta}\sigma_{\mu\nu} \equiv \pi^{\mu\nu}\sigma_{\mu\nu}$$

$$u_\mu \Delta^{\mu\nu}_{\alpha\beta} = 0, \quad \Delta_{\mu\nu}\Delta^{\mu\nu}_{\alpha\beta} = \tfrac{1}{2}(\Delta^\mu_\alpha \Delta_{\mu\beta} + \Delta^\mu_\alpha \Delta_{\mu\beta}) - \tfrac{1}{3}\cdot 3\,\Delta_{\alpha\beta} \equiv 0$$

$\omega_{\mu\nu}\Delta^{\mu\nu}_{\alpha\beta} = 0$ (symmetric ~~times~~ contracted with antisymmetric tensor)

$$\Rightarrow \boxed{0 = \dot{\epsilon} + (\epsilon + P)\Theta + \nabla_\mu W^\mu - 2W^\mu \dot{u}_\mu - \pi^{\mu\nu}\sigma_{\mu\nu}} \quad \text{energy conservation eq.}$$