Origin of the relaxation time in dissipative fluid dynamics

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with:

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based on:

S. Pu, T. Koide, DHR, PRD 81 (2010) 114039

G.S. Denicol, J. Noronha, H. Niemi, DHR, PRD 83 (2011) 074019

Israel-Stewart equation for shear stress tensor

Equation of motion for shear-stress tensor ($\Pi = q^\mu = 0$) :

$$\tau_\pi \, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = 2\,\eta\,\sigma^{\mu\nu} + 2\,\tau_\pi\,\hat{\eta}_1\,\pi_\lambda{}^{<\mu}\omega^{\nu>\lambda} - 2\,\tau_\pi\,\hat{\eta}_2\,\pi^{\mu\nu}\,\theta - 2\,\tau_\pi\,\hat{\eta}_3\,\pi_\lambda{}^{<\mu}\sigma^{\nu>\lambda} - \frac{\tau_\pi}{\eta}\,\hat{\eta}_4\,\pi_\lambda{}^{<\mu}\pi^{\nu>\lambda} - 2\,\tau_\pi\,\eta\,\hat{\eta}_5\,\omega_\lambda{}^{<\mu}\omega^{\nu>\lambda}$$

L.D. Landau, E.M. Lifshitz (Relativistic Navier-Stokes theory)

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

A. Muronga, PRC 76 (2007) 014909

B. Betz, D. Henkel, DHR, Prog. Part. Nucl. Phys. 62 (2009) 556

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ shear-stress tensor, $T^{\mu\nu}$ energy-momentum tensor

$\dot{a} \equiv u^\mu \partial_\mu a$ comoving derivative

$a^{<\mu\nu>} \equiv \Delta^{\mu\nu}_{\alpha\beta}\, a^{\alpha\beta}$, $\Delta^{\mu\nu}_{\alpha\beta} \equiv \Delta_\alpha{}^{(\mu}\Delta^{\nu)}{}_\beta - \frac{1}{3}\,\Delta^{\mu\nu}\Delta_{\alpha\beta}$ symmetrized, traceless spatial projector

$\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$ 3-space projector orthogonal to fluid 4-velocity U, $\Delta^{\mu\nu}u_\nu = 0$

$a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor $\Longrightarrow \pi^{\mu\nu}u_\mu = \pi^{\mu\nu}u_\nu = 0\,,\ \pi^\mu{}_\mu = 0$

$\sigma^{\mu\nu} = \nabla^{<\mu}u^{\nu>}$ shear tensor, $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient

$\omega^{\mu\nu} \equiv \dfrac{1}{2}\,\Delta^{\mu\alpha}\Delta^{\nu\beta}\,(\partial_\alpha u_\beta - \partial_\beta u_\alpha)$ vorticity, $\theta = \partial \cdot U$ expansion scalar

η shear-viscosity coefficient, τ_π shear relaxation time

$\hat{\eta}_1\,,\ \ldots\,,\ \hat{\eta}_5$ dimensionless coefficients

Linear stability analysis (I)

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725, PRD 35 (1987) 3723

S. Pu, T. Koide, DHR, PRD 81 (2010) 114039

Compute dispersion relations of perturbation on homogeneous, static background:

$$\epsilon = \epsilon_0 + \delta\epsilon\, e^{i(\omega t - kx)}\,, \quad \pi^{\mu\nu} = \pi_0^{\mu\nu} + \delta\pi^{\mu\nu}\, e^{i(\omega t - kx)}\,, \quad u^\mu = u_0^\mu + \delta u^\mu\, e^{i(\omega t - kx)}$$

where $\epsilon_0 = const.\,,\ \pi_0^{\mu\nu} = 0\,,\ U_0 = (1,0,0,0)$

$\Longrightarrow$ insert into fluid-dynamical equations of motion, linearize in perturbation:

$$\boxed{AX = 0}$$

$\Longrightarrow$ second and higher-order terms in equation of motion for $\pi^{\mu\nu}$ drop out !

vector of independent variables $X \equiv (\delta\epsilon, \delta u^1, \delta\pi^{11}, \delta u^2, \delta\pi^{12}, \delta u^3, \delta\pi^{13}, \delta\pi^{22}, \delta\pi^{23})\,,$

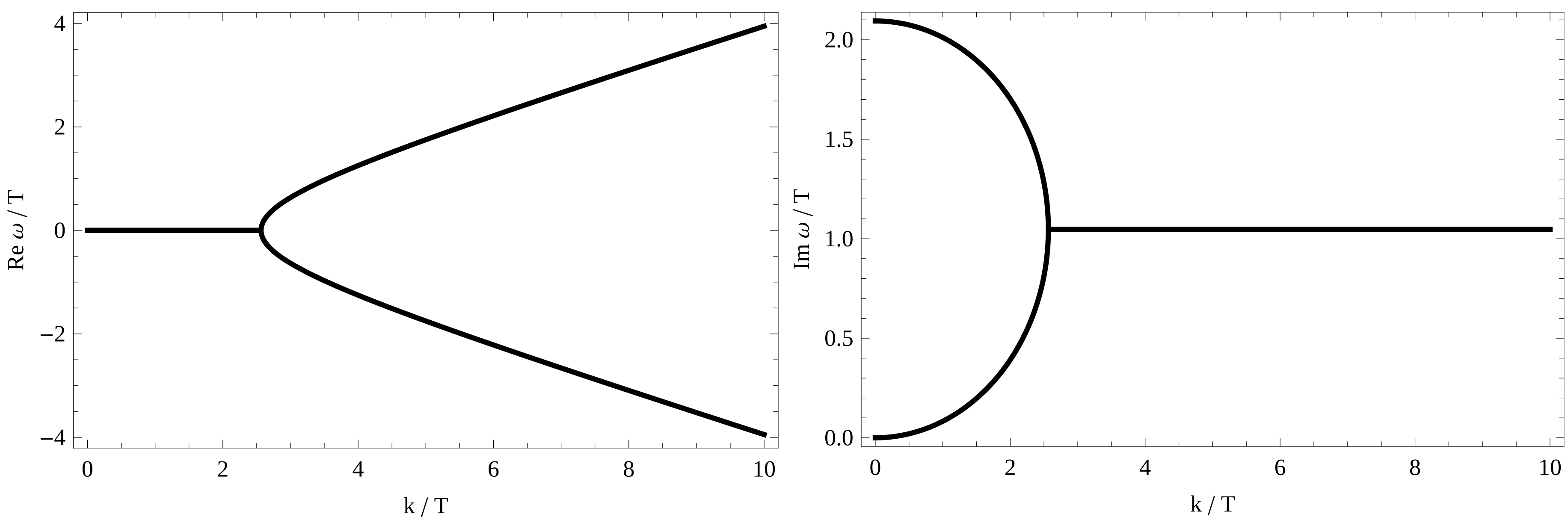
$$A = \begin{pmatrix} i\omega & -ik(\epsilon_0+p_0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -ikc_s^2 & i\omega(\epsilon_0+p_0) & -ik & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4ik\eta/3 & 1+i\omega\tau_\pi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega(\epsilon_0+p_0) & -ik & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -ik\eta & 1+i\omega\tau_\pi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\omega(\epsilon_0+p_0) & -ik & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ik\eta & 1+i\omega\tau_\pi & 0 & 0 \\ 0 & 2ik\eta/3 & 0 & 0 & 0 & 0 & 0 & 1+i\omega\tau_\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+i\omega\tau_\pi \end{pmatrix}$$

Linear stability analysis (II)

Dispersion relations follow from $\det A = 0$:

1. **two non-propagating modes:** $\omega = i/\tau_\pi$
2. **four shear modes:** $\omega = \dfrac{1}{2\tau_\pi}\left(i \pm \sqrt{\dfrac{4\eta\tau_\pi}{\epsilon_0 + p_0}\, k^2 - 1}\right) \quad (\times 2)$

$\Longrightarrow$ **propagating, if** $k \geq k_c = \sqrt{\dfrac{\epsilon_0 + p_0}{4\eta\tau_\pi}}$

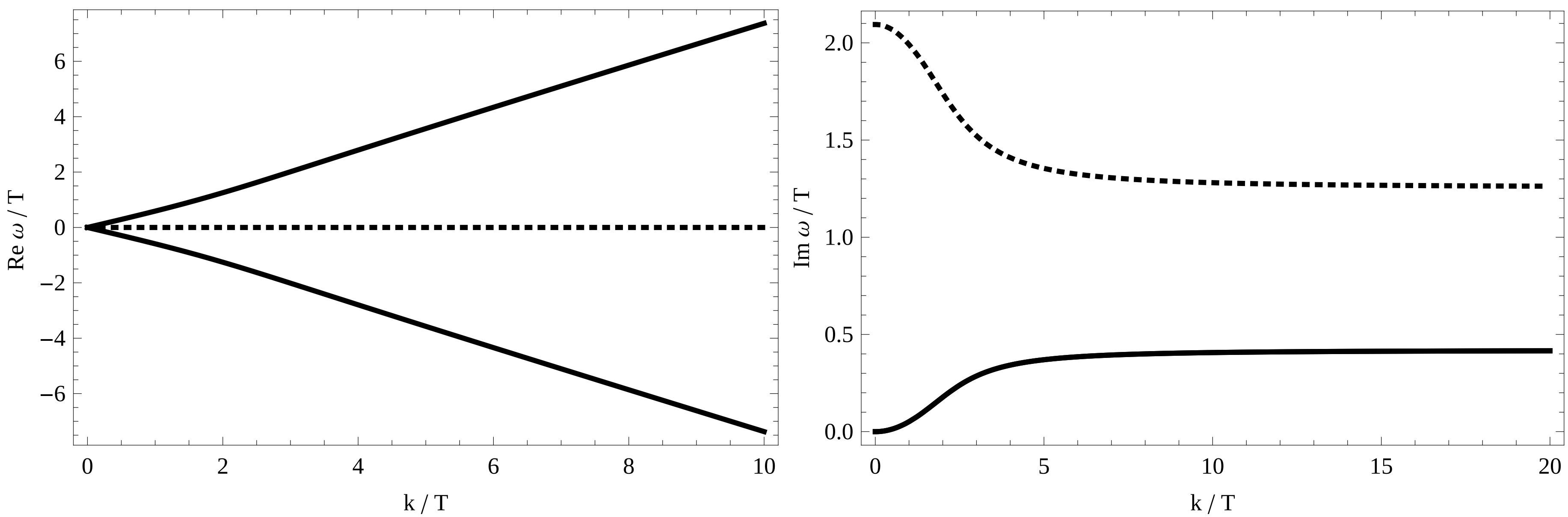


Linear stability analysis (III)

3. one non-propagating mode: $\omega \to \begin{cases} i/\tau_\pi & (k \to 0) \\ i/\tau_\pi[1+\Gamma_s/(\tau_\pi c_s^2)] & (k \to \infty) \end{cases}$

with sound attenuation length $\Gamma_s \equiv \dfrac{4\eta}{3(\epsilon_0 + p_0)}$

4. two sound modes: $\omega = \begin{cases} \pm k\, c_s + i\,\dfrac{\Gamma_s}{2}\,k^2 & (k \to 0) \\ \pm k\, c_s \sqrt{1+\dfrac{\Gamma_s}{\tau_\pi c_s^2}} + \dfrac{i}{2\tau_\pi}\left[1+\dfrac{\tau_\pi c_s^2}{\Gamma_s}\right]^{-1} & (k \to \infty) \end{cases}$



Linear stability analysis (IV)

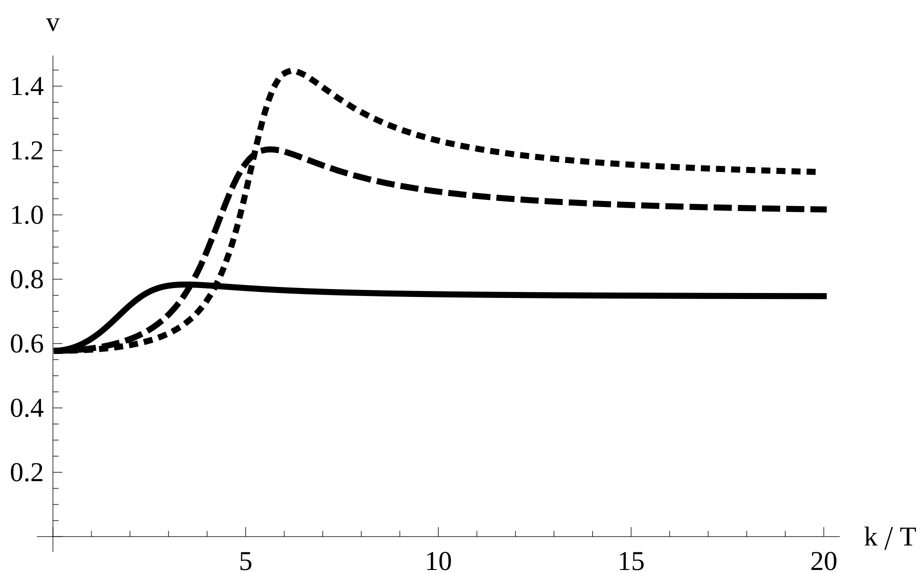
$\Longrightarrow$ for all modes: $\mathrm{Im}\,\omega \geq 0$ $\Longrightarrow$ all modes stable !

But: are they also causal? $\Longrightarrow$ consider group velocity $v_g \equiv \dfrac{\partial \mathrm{Re}\,\omega(k)}{\partial k}$

$\Longrightarrow$ employ Sommerfeld-Brillouin argument of classical electrodynamics:

theory is causal as long as asymptotic group velocity $v_g^{\mathrm{as}} \equiv \lim_{k\to\infty} v_g \leq 1$

1. sound mode: $v_g^{\mathrm{as}} = \sqrt{c_s^2 + \dfrac{\Gamma_s}{\tau_\pi}}$ $\Longrightarrow$ causal if $\dfrac{\Gamma_s}{\tau_\pi} \leq 1 - c_s^2$



asymptotic causality condition (ACC)

$\Longrightarrow$ $\dfrac{4\tau_\pi}{3\Gamma_s} \geq \dfrac{4}{3(1-c_s^2)} = 2$ for $c_s^2 = \dfrac{1}{3}$

$\dfrac{4\tau_\pi}{3\Gamma_s} = 1.5$ (short-dashed)

$= 2$ (long-dashed)

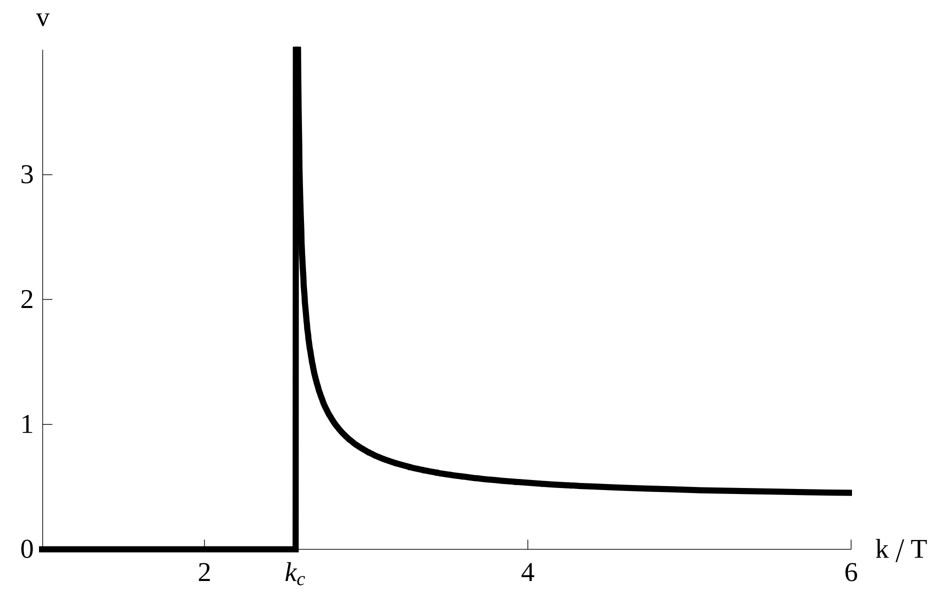
$= 6$ (solid)

Linear stability analysis (V)

2. shear mode:

$$v_g^{\rm as} = \sqrt{\frac{3\Gamma_s}{4\tau_\pi}} \leq \sqrt{\frac{3}{4}(1-c_s^2)} < 1$$

if ACC is fulfilled



S. Pu, T. Koide, DHR, PRD 81 (2010) 114039 : stability analysis in boosted frame

$\Longrightarrow$ **theory becomes unstable if ACC is violated !**

$\Longrightarrow$ **stability and causality require ACC:**

$$\boxed{\frac{\tau_\pi}{\Gamma_s} \geq \frac{1}{1-c_s^2}} \quad \Longrightarrow \quad \frac{\tau_\pi}{\eta} \text{ should not be too small !}$$

$\Longrightarrow$ **Navier-Stokes limit: $\tau_\pi \to 0$ while $\eta \neq 0$ $\Longrightarrow$ ACC is violated !**

$\Longrightarrow$ **Navier-Stokes theory is neither causal nor stable !**

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (I)

Preliminaries

G.S. Denicol, J. Noronha, H. Niemi, DHR, PRD 83 (2011) 074019

Assumption: dissipative current J ($\equiv \pi^{\mu\nu}$) is linearly proportional to thermodynamic force F($\equiv \sigma^{\mu\nu}$)

$\Longrightarrow$ particular solution to the equation of motion:

$$\boxed{J(X) = \int_Y \mathrm{d}^4Y \, G_R(X-Y)\, F(Y)}$$

$G_R(X-Y)$ retarded Green's function

$\Longrightarrow$ in Fourier space: $\boxed{\tilde{J}(Q) = \tilde{G}_R(Q)\, \tilde{F}(Q)}$ $Q = (\omega, \vec{q}\,)$

$\Longrightarrow$ $J(X) = \displaystyle\int \frac{\mathrm{d}^4Q}{(2\pi)^4}\, \tilde{G}_R(Q)\, \tilde{F}(Q)\, e^{-i\,Q\cdot X}$

$\Longrightarrow$ if $\tilde{G}_R(\omega, \vec{q}\,)$ analytic in a region around $\omega = 0$ in complex $\omega-$plane and $\tilde{F}(\omega, \vec{q}\,) \simeq 0$ outside this region

$\Longrightarrow$ Taylor expansion of $\tilde{G}_R(\omega, \vec{q}\,)$ around $\omega = 0$ is possible

$\Longrightarrow$ gradient expansion!

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (II)

Equivalency of Taylor and Gradient Expansions

If $\tilde{G}_R(\omega)$ analytic around $\omega = 0$, then

$$\tilde{G}_R(\omega) = \tilde{G}_R(0) + \partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} \omega + \frac{1}{2} \partial_\omega^2 \tilde{G}_R(\omega)\Big|_{\omega=0} \omega^2 + \mathcal{O}\left(\omega^3\right)$$

$$\Longrightarrow \quad G_R(t-t') = \tilde{G}_R(0)\, \delta(t-t') + i\, \partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} \partial_t\, \delta(t-t') - \frac{1}{2} \partial_\omega^2 \tilde{G}_R(\omega)\Big|_{\omega=0} \partial_t^2\, \delta(t-t') + \mathcal{O}\left(\partial_t^3\right)$$

$$\Longrightarrow \quad \boxed{J(t) = \bar{D}_0\, F(t) + \bar{D}_1\, \partial_t\, F(t) + \bar{D}_2\, \partial_t^2\, F(t) + \mathcal{O}\left(\partial_t^3\, F\right)}$$

with $\boxed{\bar{D}_0 = \tilde{G}_R(0)\, ,\ \bar{D}_1 = i\, \partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0}\, ,\ \bar{D}_2 = -\frac{1}{2} \partial_\omega^2 \tilde{G}_R(\omega)\Big|_{\omega=0} .}$

$\Longrightarrow$ expansion of $J(t)$ in terms of gradients of $F(t)$, gradient expansion!

Remark 1: $\partial_t F$ can be converted into $\vec{\nabla} F$ using conservation equations

Remark 2: $F\ (\equiv \sigma^{\mu\nu}) \sim \partial \sim \ell^{-1}$, ℓ macroscopic scale
$\tilde{G}_R(0) \sim \lambda$, $\partial_\omega \sim \lambda$, λ microscopic scale, scattering rate

$\Longrightarrow$ $\bar{D}_n\, \partial_t^n\, F \sim (\lambda/\ell)^{n+1}$

$\Longrightarrow$ gradient expansion equivalent to expansion in terms of Knudsen number $\mathrm{Kn} = \lambda/\ell$! $\Longrightarrow$ truncation possible, if $\mathrm{Kn} \ll 1$

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (III)

Single Simple Pole of the Retarded Green's Function (I)

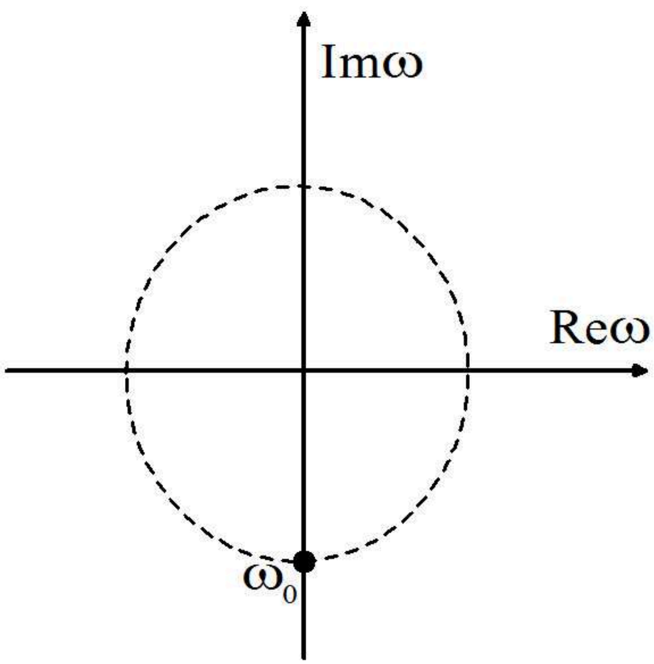
$\tilde{G}_R(\omega, \vec{q}\,)$ has singularities and $\tilde{F}(\omega, \vec{q}\,) \neq 0$ in a region around $\omega, \vec{q} = 0$

Case 1: $\tilde{G}_R(\omega)$ has single simple pole at ω_0 , $\quad \tilde{G}_R(\omega) = \dfrac{f(\omega)}{\omega - \omega_0}$, $f(\omega)$ analytic

$\Longrightarrow$ time-reversal invariance: $\mathrm{Re}\,\tilde{G}_R(\omega)$ even in $\omega \in \mathbb{R}$, $\mathrm{Im}\,\tilde{G}_R(\omega)$ odd in $\omega \in \mathbb{R}$

$\Longrightarrow$ $\mathrm{Re}\, f(\omega)$ odd in $\omega \in \mathbb{R}$, $\mathrm{Im}\, f(\omega)$ even in $\omega \in \mathbb{R}$,

$\omega_0 = -i\,\zeta\ , \quad 0 < \zeta \in \mathbb{R}$



Taylor series: radius of convergence $= |\omega_0|$

$\Longrightarrow$ use Laurent series around ω_0:

$$\tilde{G}_R(\omega) = \frac{f(\omega_0)}{\omega - \omega_0} + \partial_\omega\, f(\omega)\big|_{\omega=\omega_0} + \frac{1}{2}\,\partial^2_\omega\, f(\omega)\big|_{\omega=\omega_0} (\omega - \omega_0) + \mathcal{O}\left[(\omega - \omega_0)^2\right]$$

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (IV)

Single Simple Pole of the Retarded Green's Function (II)

Taylor series and Laurent series can be matched

$$\Longrightarrow \quad \tilde{G}_R(\omega) = \frac{f(\omega_0)}{\omega - \omega_0} + \left[\tilde{G}_R(0) + \frac{f(\omega_0)}{\omega_0}\right] + \left[\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{f(\omega_0)}{\omega_0^2}\right] \omega + \mathcal{O}(\omega^2)$$

$$\Longrightarrow \quad G_R(t-t') = -i\, f(\omega_0)\, e^{-i\,\omega_0(t-t')}\,\theta(t-t') + \left[\tilde{G}_R(0) + \frac{f(\omega_0)}{\omega_0}\right] \delta(t-t') + i\left[\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{f(\omega_0)}{\omega_0^2}\right] \partial_t\,\delta(t-t') + \mathcal{O}(\partial_t^2)$$

$\Longrightarrow$ **differential eq. for $G_R(t-t')$:**

$$\frac{1}{i\,\omega_0}\,\partial_t\, G_R(t-t') + G_R(t-t') = \tilde{G}_R(0)\,\delta(t-t') + \left[i\,\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{1}{i\,\omega_0}\tilde{G}_R(0)\right] \partial_t\,\delta(t-t') + \mathcal{O}\left(\partial_t^2\right)$$

$\Longrightarrow$ **differential eq. for $J(t)$!**

$$\tau_R\,\partial_t J + J = D_0\,F + D_1\,\partial_t\,F + \mathcal{O}\left(\partial_t^2\,F\right)$$

with

$$\tau_R = \frac{1}{i\,\omega_0} = \frac{1}{\zeta}\,, \qquad D_0 = \tilde{G}_R(0) \equiv \bar{D}_0\,,$$

$$D_1 = i\,\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + D_0\,\tau_R \equiv \bar{D}_1 + D_0\,\tau_R = \tau_R\,\partial_\omega\, f(\omega)|_{\omega=0}\,.$$

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (V)

Single Simple Pole of the Retarded Green's Function (III)

Remarks:

(i) Relaxation-type equation, similar to the ones in Israel-Stewart theories!

$\Longrightarrow$ J relaxes on a time scale τ_R to the value given by the r.h.s.

(ii) $D_0 \equiv \bar{D}_0$, while $D_i \neq \bar{D}_i$, $i \geq 1$

(iii) Pushing pole ω_0 to infinity $\Longrightarrow$ $\tau_R \to 0$

$\Longrightarrow$ $J = \bar{D}_0\, F + \bar{D}_1\, \partial_t\, F + \mathcal{O}\left(\partial_t^2\, F\right)$ $\Longrightarrow$ gradient expansion!

(Remember: taking $\tau_R\, (\equiv \tau_\pi) \to 0$, while $\bar{D}_0\, (\sim \eta) \neq 0$, violates ACC!)

(iv) If F is slowly varying on times scales τ_R

$\Longrightarrow$ for asymptotic times, $t \gg \tau_R$, J follows time dependence of r.h.s.

$\Longrightarrow$ $\tau_R\, \partial_t\, J \ll J$

$\Longrightarrow$ replace J in $\tau_R\, \partial_t\, J$ by $D_0\, F + D_1\, \partial_t\, F + \mathcal{O}\left(\partial_t^2\, F\right)$

$\Longrightarrow$ $\tau_R\, \partial_t\, (D_0\, F) + J = D_0\, F + D_1\, \partial_t\, F + \mathcal{O}\left(\partial_t^2\, F\right)$

$\Longleftrightarrow$ $J = D_0\, F + (D_1 - \tau_R\, D_0)\, \partial_t\, F + \mathcal{O}\left(\partial_t^2\, F\right) \equiv \bar{D}_0\, F + \bar{D}_1\, \partial_t\, F + \mathcal{O}\left(\partial_t^2\, F\right)$

$\Longrightarrow$ gradient expansion is asymptotic solution of differential eq. for J!

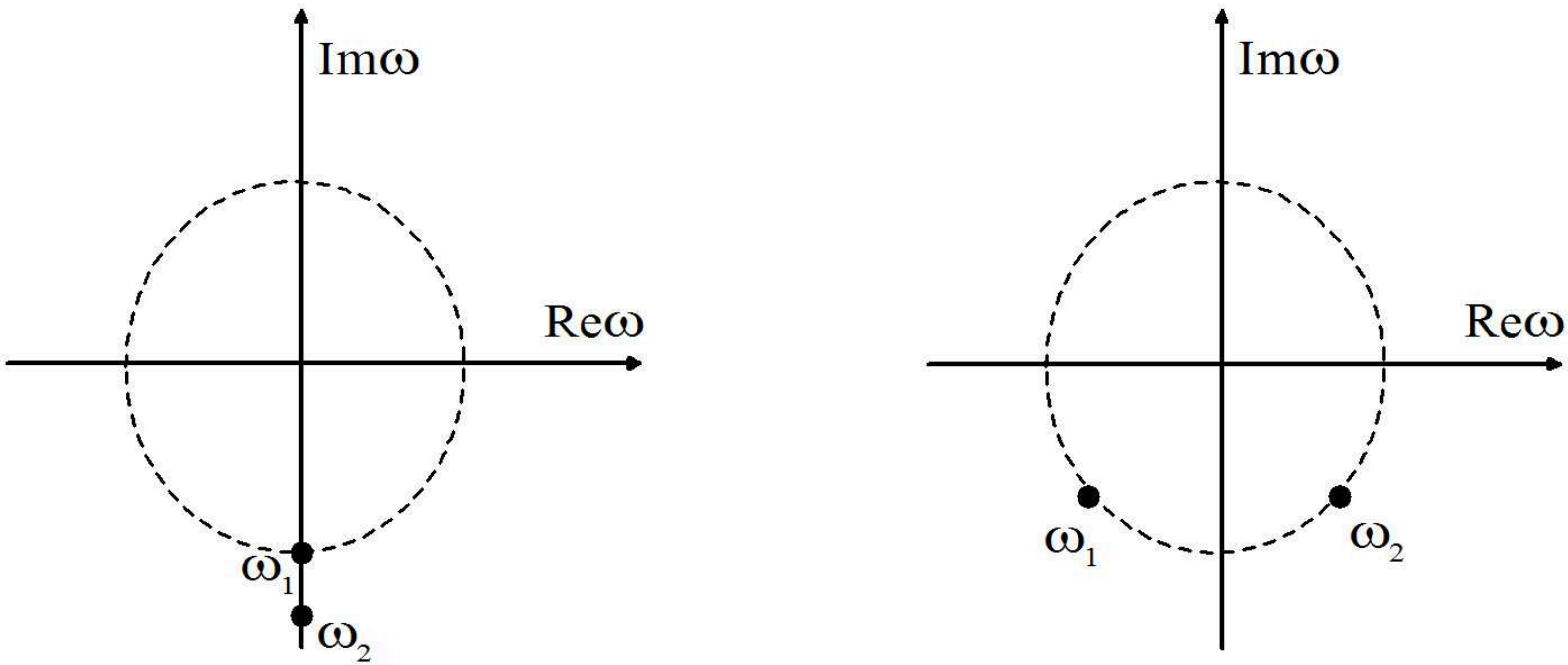
Deriving the Equation of Motion for $\pi^{\mu\nu}$ (VI)

Two Simple Poles of the Retarded Green's Function (I)

Case 2: $\tilde{G}_R(\omega)$ has two simple poles at $\omega_{1,2}$,

$$\tilde{G}_R(\omega) = \frac{f_1(\omega)}{\omega - \omega_1} + \frac{f_2(\omega)}{\omega - \omega_2} , \quad f_1(\omega) , f_2(\omega) \text{ analytic}$$

$\Longrightarrow$ time-reversal symmetry allows for two scenarios:



(A): $\omega_i = -i\zeta_i$, $\zeta_i \in \mathbb{R}$, $i = 1, 2$,
$\zeta_2 > \zeta_1 > 0$

(B): $\omega_1 = -\omega_2^*$,
$\mathrm{Im}\,\omega_1 = \mathrm{Im}\,\omega_2 = -\zeta$, $\zeta > 0$

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (VII)

Two Simple Poles of the Retarded Green's Function (II)

Matching Taylor expansion around $\omega = 0$ with Laurent expansion around ω_1 , ω_2

$$\Longrightarrow \tilde{G}_R(\omega) = \frac{f_1(\omega_1)}{\omega-\omega_1} + \frac{f_2(\omega_2)}{\omega-\omega_2} + \left[\tilde{G}_R(0) + \frac{f_1(\omega_1)}{\omega_1} + \frac{f_2(\omega_2)}{\omega_2}\right] + \left[\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{f_1(\omega_1)}{\omega_1^2} + \frac{f_2(\omega_2)}{\omega_2^2}\right]\omega$$
$$+ \left[\frac{1}{2}\partial_\omega^2 \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{f_1(\omega_1)}{\omega_1^3} + \frac{f_2(\omega_2)}{\omega_2^3}\right]\omega^2 + \mathcal{O}(\omega^3)$$

$$\Longrightarrow G_R(t-t') = -i\left[f_1(\omega_1)\, e^{-i\,\omega_1(t-t')} + f_2(\omega_2)\, e^{-i\,\omega_2(t-t')}\right]\theta(t-t')$$
$$+ \left[\tilde{G}_R(0) + \frac{f_1(\omega_1)}{\omega_1} + \frac{f_2(\omega_2)}{\omega_2}\right]\delta(t-t') + i\left[\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{f_1(\omega_1)}{\omega_1^2} + \frac{f_2(\omega_2)}{\omega_2^2}\right]\partial_t\,\delta(t-t')$$
$$- \left[\frac{1}{2}\partial_\omega^2 \tilde{G}_R(\omega)\Big|_{\omega=0} + \frac{f_1(\omega_1)}{\omega_1^3} + \frac{f_2(\omega_2)}{\omega_2^3}\right]\partial_t^2\,\delta(t-t') + \mathcal{O}(\partial_t^3)$$

$\Longrightarrow$ differential eq. for $G_R(t-t')$ is of second order in time!

$$-\frac{1}{\omega_1\,\omega_2}\partial_t^2\, G_R(t-t') + \left(\frac{1}{i\,\omega_1} + \frac{1}{i\,\omega_2}\right)\partial_t\, G_R(t-t') + G_R(t-t') = \tilde{G}_R(0)\,\delta(t-t')$$
$$+ \left[i\,\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} + \left(\frac{1}{i\,\omega_1} + \frac{1}{i\,\omega_2}\right)\tilde{G}_R(0)\right]\partial_t\,\delta(t-t')$$
$$+ \left[-\frac{1}{2}\partial_\omega^2 \tilde{G}_R(\omega)\Big|_{\omega=0} + \left(\frac{1}{i\,\omega_1} + \frac{1}{i\,\omega_2}\right) i\,\partial_\omega \tilde{G}_R(\omega)\Big|_{\omega=0} - \frac{1}{\omega_1\,\omega_2}\tilde{G}_R(0)\right]\partial_t^2\,\delta(t-t') + \mathcal{O}(\partial_t^3)$$

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (VIII)
Two Simple Poles of the Retarded Green's Function (III)

$\Longrightarrow$ **differential eq. for $J(t)$ is of second order in time!**

$$\boxed{\chi_2\,\partial_t^2\,J + \chi_1\,\partial_t\,J + J = D_0\,F + D_1\,\partial_t\,F + D_2\,\partial_t^2\,F + \mathcal{O}\left(\partial_t^3 F\right)}\ ,\quad \text{with}$$

$$\chi_2 = -\frac{1}{\omega_1\,\omega_2}\ ,\quad \chi_1 = \frac{1}{i\,\omega_1} + \frac{1}{i\,\omega_2}\ ,\quad D_0 = \tilde{G}_R(0) \equiv \bar{D}_0\ ,$$
$$D_1 = i\,\partial_\omega\,\tilde{G}_R(\omega)\Big|_{\omega=0} + D_0\,\chi_1\ ,\quad D_2 = -\frac{1}{2}\,\partial_\omega^2\,\tilde{G}_R(\omega)\Big|_{\omega=0} + D_1\,\chi_1 + D_0\left(\chi_2 - \chi_1^2\right)$$

Remarks:

(i) Both poles enter $\chi_1\,,\ \chi_2$

(ii) For r.h.s. $= 0$ $\Longrightarrow$ hom. differential eq. for damped harmonic oscillator!

$$\ddot{J} + 2\,\gamma\,\dot{J} + \omega_0^2\,J = 0\ ,\quad \text{with } \gamma = \frac{\chi_1}{2\,\chi_2}\ ,\ \omega_0^2 = \frac{1}{\chi_2}\ .$$

(A): $0 < (\zeta_1 - \zeta_2)^2 \Longleftrightarrow 4\,\zeta_1\,\zeta_2 < (\zeta_1 + \zeta_2)^2 \Longleftrightarrow 4\,\omega_1\,\omega_2 > (\omega_1 + \omega_2)^2$
$\Longleftrightarrow \chi_1^2 > 4\,\chi_2 \Longleftrightarrow \gamma > \omega_0 \Longrightarrow$ **overdamped!**

(B): $4\,|\omega_1|^2 > (2\,\mathrm{Im}\,\omega_1)^2 \Longleftrightarrow -\,4\,\omega_1\,\omega_2 > -(\omega_1 - \omega_1^*)^2 = -(\omega_1 + \omega_2)^2$
$\Longleftrightarrow \chi_1^2 < 4\,\chi_2 \Longleftrightarrow \gamma < \omega_0 \Longrightarrow$ **underdamped!**

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (IX)
Two Simple Poles of the Retarded Green's Function (IV)

(iii) overdamped case, scenario (A) : if $\zeta_2 \gg \zeta_1$

$$\Longrightarrow \quad \chi_1 \equiv \frac{1}{\zeta_1} + \frac{1}{\zeta_2} \simeq \frac{1}{\zeta_1} \equiv \tau_R$$

$$\Longrightarrow \quad \chi_2 \equiv \frac{1}{\zeta_1\,\zeta_2} = \tau_R^2\,\frac{\zeta_1}{\zeta_2} \ll \tau_R^2 \quad \Longrightarrow \quad \chi_2\,\partial_t^2 J \ll \chi_1\,\partial_t J \simeq \tau_R\,\partial_t J$$

$\Longrightarrow$ **neglect 2nd order time derivative** $\Longrightarrow$ **relaxation-type eq. for J !**

$$\boxed{\tau_R\,\partial_t\,J + J \simeq D_0\,F + D_1\,\partial_t\,F + D_2\,\partial_t^2\,F + \mathcal{O}\left(\partial_t^3 F\right)}$$

(iv) underdamped case, scenario (B) :

$\Longrightarrow$ **approach to asymptotic solution is always oscillatory !**

$\Longrightarrow$ **equation of motion is never of relaxation-type !**

Deriving the Equation of Motion for $\pi^{\mu\nu}$ (X)
N Poles of the Retarded Green's Function

Assumption: retarded Green's function $\tilde{G}_R(\omega, \vec{q}\,)$ has N simple poles in the vicinity of $\omega = 0$ (and possibly other singularities, e.g. branch cuts, beyond that region)

$\Longrightarrow$ $$\tilde{G}_R(Q) = \sum_{i=1}^{N} \frac{f_i(Q)}{\omega - \omega_i(\vec{q}\,)} = \frac{\Xi(Q)}{[\omega - \omega_1(\vec{q}\,)] \cdots [\omega - \omega_N(\vec{q}\,)]} \,, \quad f_i(Q)\,, \Xi(Q) \text{ analytic}$$

$\Longrightarrow$ differential eq. for $J(X)$ of order N in time!

$$\chi_N\, \partial_t^N\, J + \ldots + \chi_1\, \partial_t\, J + J = D_0\, F + \ldots + D_N\, \partial_t^N\, F + \mathcal{O}\left(\partial_t^{N+1}\, F, \vec{\nabla}\right)$$

where $$\chi_m = (-i)^m \sum_{1 \le i_1 \ldots < i_m \le N} \frac{1}{\omega_{i_1}(\vec{0}) \cdots \omega_{i_m}(\vec{0})} \,, \quad D_k = i^k \frac{(-1)^N}{k!} \frac{\partial_\omega^k\, \Xi(\omega, \vec{0})\big|_{\omega=0}}{\omega_1(\vec{0}) \cdots \omega_N(\vec{0})} \,.$$

If pole closest to origin is purely imaginary and others are sufficiently far away

$\Longrightarrow$ $$\chi_1 = \sum_{i=1}^{N} \frac{1}{i\, \omega_i} \simeq \frac{1}{i\, \omega_1} \equiv \frac{1}{\zeta_1} \equiv \tau_R \,, \quad \chi_1\, \partial_t\, J \gg \chi_m\, \partial_t^m\, J \,, \;\; m \ge 2$$

$\Longrightarrow$ relaxation eq. for J: $$\tau_R\, \partial_t\, J + J \simeq D_0\, F + \ldots + D_N\, \partial_t^N\, F + \mathcal{O}\left(\partial_t^{N+1}\, F, \vec{\nabla}\right)$$

Applications (I)
Boltzmann equation (I)

$$\boxed{K \cdot \partial\, f_k = \mathcal{C}[f]}$$

Decompose $f_k \equiv f_{0k} + \delta f_k$, with $f_{0k} = e^{\alpha_0 - \beta_0\, U \cdot K}$, where $\alpha_0 \equiv \dfrac{\mu}{T}$, $\beta_0 \equiv \dfrac{1}{T}$

$\Longrightarrow$ linearize Boltzmann equation in δf_k around background with
$\alpha_0 = const.$, $\beta_0 = const.$, $\dot{u}^\mu \equiv U \cdot \partial\, u^\mu = 0$, $\theta \equiv \partial \cdot U = 0$.

$\Longrightarrow$ in local rest frame, $U = (1, 0, 0, 0)$:

$$\boxed{\left[\partial_t + \vec{v} \cdot \vec{\nabla} - \hat{\mathcal{C}}(K)\right] \delta f_k(X) = \mathcal{S}(X, K)}$$

where $\vec{v} \equiv \vec{k}/E_k$, $E_k = \sqrt{k^2 + m^2}$, $\hat{\mathcal{C}}(K)$ collision operator,
and $\mathcal{S}(X, K) = \beta_0\, f_{0k}\, E_k^{-1}\, K^{\langle\mu} K^{\nu\rangle}\, \sigma_{\mu\nu}(X)$.

$\Longrightarrow$ solution in Fourier space: $\delta \tilde{f}_k(Q) = \left[-i\,\omega + i\,\vec{v} \cdot \vec{q} - \hat{\mathcal{C}}(K)\right]^{-1} \tilde{\mathcal{S}}(Q, K)$

Since $\pi^{\mu\nu}(X) = \displaystyle\int \mathrm{d}K\, K^{\langle\mu} K^{\nu\rangle}\, \delta f_k(X)$, $\mathrm{d}K \equiv \dfrac{\mathrm{d}^3\vec{k}}{(2\pi)^3 E_k}$

$\Longrightarrow$ $$\boxed{\tilde{\pi}^{\mu\nu}(Q) = \int \mathrm{d}K\, K^{\langle\mu} K^{\nu\rangle} \left[-i\,\omega + i\,\vec{v} \cdot \vec{q} - \hat{\mathcal{C}}(K)\right]^{-1} \beta_0\, f_{0k}\, E_k^{-1}\, K^{\langle\alpha} K^{\beta\rangle} \tilde{\sigma}_{\alpha\beta}(Q)}$$

Applications (II)
Boltzmann equation (II)

This has the form $$\boxed{\tilde{\pi}^{\mu\nu}(Q) = \tilde{G}_R^{\mu\nu\alpha\beta}(Q)\, \tilde{\sigma}_{\alpha\beta}(Q)}$$

with $\tilde{G}_R^{\mu\nu\alpha\beta}(Q) = \int \mathrm{d}K\, K^{\langle\mu}K^{\nu\rangle} \left[-i\,\omega + i\,\vec{v}\cdot\vec{q} - \hat{\mathcal{C}}(K)\right]^{-1} \beta_0\, f_{0k}\, E_k^{-1}\, K^{\langle\alpha}K^{\beta\rangle}$ **.**

For $\tau_R \equiv \tau_\pi \equiv [i\,\omega_1(\vec{0})]^{-1}$ **,** $\eta \equiv \bar{D}_0 \equiv \tilde{G}_R(0,\vec{0})$ **need only** $\tilde{G}_R(\omega,\vec{0})$

$\Longrightarrow$ $\tilde{G}_R^{\mu\nu\alpha\beta}(\omega,\vec{0}) = \int \mathrm{d}K\, K^{\langle\mu}K^{\nu\rangle}\, B^{\alpha\beta}(\omega,K)$ **,**

where $B^{\alpha\beta}(\omega,K) \equiv \left[-i\,\omega - \hat{\mathcal{C}}(K)\right]^{-1} \beta_0\, f_{0k}\, E_k^{-1}\, K^{\langle\alpha}K^{\beta\rangle}$ **.**

$B^{\alpha\beta}(\omega,K)$ **can be written in the form** $B^{\alpha\beta}(\omega,K) = f_{0k}\, K^{\langle\alpha}K^{\beta\rangle} \sum_{n=0}^{\infty} a_n(\omega) E_k^n$ **.**

$\Longrightarrow$ $\tilde{G}_R^{\mu\nu\alpha\beta}(\omega,\vec{0}) = \sum_{n=0}^{\infty} a_n(\omega) \int \mathrm{d}K\, K^{\langle\mu}K^{\nu\rangle}K^{\langle\alpha}K^{\beta\rangle}\, E_k^n\, f_{0k} \equiv 2\,\Delta^{\mu\nu\alpha\beta}\tilde{G}_R(\omega,\vec{0})$ **,**

where $\tilde{G}_R(\omega,\vec{0}) \equiv \sum_{n=0}^{\infty} I_{n+4,2}\, a_n(\omega)$ **,** **with** $I_{nq} = \frac{1}{(2q+1)!!}\int \mathrm{d}K\, f_{0k}\, E_k^{n-2q}\left(m^2 - E_k^2\right)^q$ **.**

$\Longrightarrow$ $$\boxed{\tilde{\pi}^{\mu\nu}(\omega,\vec{0}) = 2\,\tilde{G}_R(\omega,\vec{0})\, \tilde{\sigma}^{\mu\nu}(\omega,\vec{0})}$$

Applications (III)
Boltzmann equation (III)

Determine coefficients $a_n(\omega)$:

insert ansatz for $B^{\alpha\beta}(\omega, K)$ into eq. for $B^{\alpha\beta}(\omega, K)$, multiply by $E_k^m\, K^{\langle\mu}K^{\nu\rangle}$,

and integrate over $\mathrm{d}K$ $\Longrightarrow$ $\sum_{n=0}^{\infty}\left(-i\,\omega\,\mathcal{D}^{mn} + \mathcal{A}^{mn}\right) a_n(\omega) = \beta_0\, I_{m+3,2}$, where

$$\mathcal{D}^{mn} = \frac{1}{5!!}\int \mathrm{d}K\, f_{0k}\, E_k^{m+n}\left(m^2 - E_k^2\right)^2 , \quad \mathcal{A}^{mn}\,\Delta^{\mu\nu\alpha\beta} = -\frac{1}{2}\int \mathrm{d}K\, E_k^m\, K^{\langle\mu}K^{\nu\rangle}\,\hat{\mathcal{C}}(K)\, f_{0k}\, E_k^n\, K^{\langle\alpha}K^{\beta\rangle} .$$

This has the formal solution $a_m(\omega) = \beta_0 \sum_{n=0}^{\infty}\left[\left(-i\,\omega\,\mathcal{D} + \mathcal{A}\right)^{-1}\right]^{mn} I_{n+3,2}$.

$$\Longrightarrow \quad \boxed{\tilde{G}_R(\omega, \vec{0}) = \beta_0 \sum_{m,n=0}^{\infty} I_{m+4,2}\left[\left(-i\,\omega\,\mathcal{D} + \mathcal{A}\right)^{-1}\right]^{mn} I_{n+3,2}}$$

$\Longrightarrow$ poles of $\tilde{G}_R(\omega, \vec{0})$ are given by roots of $\det\left(-i\,\omega\,\mathcal{D} + \mathcal{A}\right) = 0$.

Since $\mathcal{D}$, $\mathcal{A}$ real, all poles $\omega_1(\vec{0})$, $\omega_2(\vec{0})$, ... are purely imaginary !

Applications (IV)
Boltzmann equation (IV)

Example: take only first term in expansion of $B^{\alpha\beta}(\omega, K)$.

With $\mathcal{D}_{00} \equiv I_{42}$ $\Longrightarrow$ $$\tilde{G}_R(\omega, \vec{0}) = \frac{\beta_0 \, I_{32}}{-i\,\omega + \mathcal{A}_{00}/I_{42}}$$

$\Longrightarrow$ $$\tau_\pi = \frac{1}{i\,\omega_1} \equiv \frac{\mathcal{A}_{00}}{I_{42}}$$

$\Longrightarrow$ τ_π is microscopic time scale, determined by collisions of particles through $\mathcal{A}_{00}$

$\Longrightarrow$ $$\eta = \tilde{G}_R(0, \vec{0}) = \frac{\beta_0 \, I_{42} \, I_{32}}{\mathcal{A}_{00}} \equiv \frac{\beta_0 \, I_{32}}{\tau_\pi}$$

$\Longrightarrow$ $$\frac{\tau_\pi}{\Gamma_s} = \frac{3\,(\epsilon_0 + p_0)\,\tau_\pi}{4\,\eta} = \frac{3\,(\epsilon_0 + p_0)}{4\,\beta_0\,I_{32}}$$

$\Longrightarrow$ coincides with matching kinetic theory to fluid dynamics à la Israel and Stewart

G.S. Denicol, T. Koide, DHR, PRL 105 (2010) 162501

Applications (V)
Metric Perturbations (I)

Consider metric perturbation $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$,

$h^{xy} = h^{xy}(t, z) \neq 0$, $h^{\mu\nu} = 0$ **for** $(\mu\nu) \neq (xy)$

$$\Longrightarrow \qquad \boxed{\delta T^{xy}(t,z) = \int \mathrm{d}t' \, dz' \, G_R^{xyxy}(t-t',\, z-z')\, h_{xy}(t',\, z')}$$

D.T. Son, A.O. Starinets, Ann. Rev. Nucl. Part. Sci. 57 (2007) 95

On the other hand,

$$\boxed{\delta T^{xy} = T^{xy}(\eta^{\mu\nu} + h^{\mu\nu}) - T^{xy}(\eta^{\mu\nu}) = -p_0\, h^{xy} + \delta\pi^{xy}}$$

$\delta\pi^{xy}$ **shear-stress tensor generated by metric perturbations**

$$\Longrightarrow \qquad \delta\pi^{xy} = p_0\, h^{xy} + \int \mathrm{d}t' \, \mathrm{d}z' \, G_R^{xyxy}(t-t',\, z-z')\, h_{xy}(t',\, z')$$

$$\Longleftrightarrow \qquad \boxed{\delta\tilde{\pi}^{xy}(Q) = \tilde{G}_R(Q)\, \tilde{h}_{xy}(Q)} \,, \quad \textbf{with} \quad \tilde{G}_R(Q) = -p_0 + \tilde{G}_R^{xyxy}(Q)$$

Applications (VI)

Metric Perturbations (II)

If first pole of $\tilde{G}_R(Q)$ is purely imaginary $\Longrightarrow$ relaxation eq. for $\delta\pi^{xy}$

$$\tau_\pi\,\partial_t\,\delta\pi^{xy} + \delta\pi^{xy} = D_0\,h_{xy} + D_1\,\partial_t\,h_{xy} + D_2\,\partial_t^2\,h_{xy} + \mathcal{O}\left(\partial_t^3\,h_{xy}, \partial_z^2\,h_{xy}\right)$$

with

$$\tau_\pi = \frac{1}{i\,\omega_1(\vec{0})}\,,\quad D_0 = \tilde{G}_R(0,\vec{0}) = -p_0 + \tilde{G}_R^{xyxy}(0,\vec{0}) \equiv -p_0 + p_0 = 0\,,$$

$$D_1 = i\,\partial_\omega\,\tilde{G}_R(\omega,\vec{0})\Big|_{\omega=0} + \tau_\pi\,D_0 = i\,\partial_\omega\,\tilde{G}_R(\omega,\vec{0})\Big|_{\omega=0} \equiv \eta\,,$$

$$D_2 = -\frac{1}{2}\,\partial_\omega^2\,\tilde{G}_R(\omega,\vec{0})\Big|_{\omega=0} + D_1\,\tau_\pi - D_0\,\tau_\pi^2 \equiv -\frac{1}{2}\,\partial_\omega^2\,\tilde{G}_R(\omega,\vec{0})\Big|_{\omega=0} + \eta\,\tau_\pi\,.$$

Remark:

Since $2\,\sigma^{xy} \equiv \partial_t\,h_{xy}$, $D_1 \equiv \eta$, not D_0 (which actually $\equiv 0$ in this case).

Discussion (I)

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100
proposed heuristic way to derive relaxation-type eq. from gradient expansion.
In our notation:

- **to first order in gradient expansion** $J \simeq \bar{D}_0 \, F$
- **replace** $\partial_t \, F \simeq \partial_t \, \dfrac{J}{\bar{D}_0}$ **in second term on r.h.s. of gradient expansion**
- **define** $\bar{\tau}_R \equiv -\dfrac{\bar{D}_1}{\bar{D}_0}$ **(quantity with dimension of time)**
- $\Longrightarrow$ **modified gradient expansion:**

$$J \simeq \bar{D}_0 \, F + \frac{\bar{D}_1}{\bar{D}_0} \, \partial_t \, J + \mathcal{O}(\partial_t^2 \, F) \equiv \bar{D}_0 \, F - \bar{\tau}_R \, \partial_t \, J + \mathcal{O}(\partial_t^2 \, F)$$

$$\Longleftrightarrow \quad \boxed{\bar{\tau}_R \, \partial_t \, J + J \simeq \bar{D}_0 \, F + \mathcal{O}(\partial_t^2 \, F)}$$

Discussion (II)

Remarks:

(i) Relaxation to asymptotic value of J is enforced by hand, even if true dynamics is oscillatory!

(ii) In general $\tau_R \neq \bar{\tau}_R$

(iii) If $\Xi(\omega, \vec{0}) = const.$

$$\Longrightarrow \quad 0 = D_1 = i\, \partial_\omega \tilde{G}_R(\omega, \vec{0})\Big|_{\omega=0} + D_0\, \tau_R \equiv \bar{D}_1 + \bar{D}_0\, \tau_R$$

$$\Longleftrightarrow \quad \tau_R = -\frac{\bar{D}_1}{\bar{D}_0} \equiv \bar{\tau}_R$$

(iv) Metric perturbations: if $0 = D_2 = \bar{D}_2 + \eta\, \tau_\pi$

$$\Longrightarrow \quad \eta\, \tau_\pi = -\bar{D}_2 \equiv \frac{1}{2}\, \partial^2_\omega\, \tilde{G}_R(\omega, \vec{0})\Big|_{\omega=0}$$

Note: BRSSS: $\eta\, \bar{\tau}_\pi = \frac{1}{2}\left[\partial^2_\omega\, \tilde{G}_R(\omega, \vec{q})\Big|_{\omega=\vec{q}=0} - \partial^2_{q_z}\, \tilde{G}_R(\omega, \vec{q})\Big|_{\omega=\vec{q}=0}\right]$

Extra term due to admixture of space-time curvature in definition of $\bar{\tau}_\pi$

(v) AdS/CFT theories: oscillatory approach to asymptotic solution

P.K. Kovtun, A.O. Starinets, PRD 72 (2005) 086009

Conclusions

1. Asymptotic Causality Condition: $\frac{\tau_\pi}{\Gamma_s} \geq \frac{1}{1 - c_s^2}$
   $\Longrightarrow$ τ_π/η must not be too small!

2. Gradient expansion of dissipative current J
   $\Longleftrightarrow$ Taylor expansion of associated retarded Green's function $\tilde{G}_R(Q)$

3. Equations of motion for J determined by analytic structure of $\tilde{G}_R(Q)$

4. First pole $\omega_1(\vec{0})$ purely imaginary $\Longrightarrow$ relaxation-type equation!
   $\Longrightarrow$ exponential approach to asymptotic solution given by gradient expansion
   $\Longrightarrow$ relaxation time given by $\tau_R = \left[i\, \omega_1(\vec{0}) \right]^{-1}$

5. Example: Boltzmann equation
   $\Longrightarrow$ τ_π determined by scattering processes
   $\Longrightarrow$ transient dynamics determined by slowest microscopic, not fastest macroscopic (fluid-dynamical) time scale

6. Poles off imaginary axis
   $\Longrightarrow$ oscillatory approach to asymptotic solution given by gradient expansion

7. Constructing relaxation-type equations by hand from gradient expansion in general fails to give correct τ_R, may miss true (oscillatory) dynamics