1(b) $0 = \Delta_{\alpha\nu}\partial_\mu T^{\mu\nu} = \Delta_{\alpha\nu}\partial_\mu(\epsilon u^\mu u^\nu + W^\mu u^\nu + W^\nu u^\mu - P\Delta^{\mu\nu} + \pi^{\mu\nu})$

$$= \epsilon \underbrace{\Delta_{\alpha\nu}\dot{u}^\nu}_{\equiv \dot{u}_\alpha} + W^\mu \underline{\Delta_{\alpha\nu}\partial_\mu u^\nu} + W_\alpha\Theta + \Delta_{\alpha\nu}\dot{W}^\nu$$

$$- \Delta^\mu_\alpha \partial_\mu P - P\Delta_{\alpha\nu}\underbrace{\partial_\mu \Delta^{\mu\nu}}_{-\partial_\mu(u^\mu u^\nu) = -\dot{u}^\nu} + \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}$$

$$= (\epsilon+P)\dot{u}_\alpha + W^\mu \underbrace{\partial_\mu u_\alpha}_{\equiv \nabla_\mu u_\alpha} + W_\alpha\Theta + \Delta_{\alpha\mu}\dot{W}^\mu$$

$$- \nabla_\alpha P + \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}$$

or $\boxed{(\epsilon+P)\dot{u}_\alpha = \nabla_\alpha P - \Delta_{\alpha\mu}\dot{W}^\mu - W_\alpha\Theta - W^\mu \underbrace{\nabla_\mu u_\alpha}_{\frac{1}{3}\Delta_{\mu\alpha}\Theta + \sigma_{\mu\alpha} + \omega_{\mu\alpha}} - \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}}$

$$\boxed{= \nabla_\alpha P - \Delta_{\alpha\mu}\dot{W}^\mu - \frac{4}{3}\Theta W_\alpha - W^\mu\sigma_{\mu\alpha} - W^\mu\omega_{\mu\alpha} - \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}}$$

acceleration equation

Eckart frame: particle no. conservation $0 = \dot{n} + n\Theta$

energy conservation (as above)

acceleration eq. (as above)

Landau frame: particle no. conservation (as above)

energy conservation: $0 = \dot{\epsilon} + (\epsilon+P)\Theta - \pi^{\mu\nu}\sigma_{\mu\nu}$

acceleration eq. : $(\epsilon+P)\dot{u}_\alpha = \nabla_\alpha P - \Delta_{\alpha\nu}\partial_\mu\pi^{\mu\nu}$

# II. The Boltzmann equation

The Boltzmann equation describes the time evolution of the single-particle distribution function $f_{\vec{k}}(X)$ in phase-space ~~i.e., the probability~~

Assumptions are:

(i) Binary collisions ~~Thus~~ $1+2 \to 1'+2'$.
(Note this can be easily generalized to $1+2 \to 1'$, $1+2 \to 1'+2'+3'$, etc.)

(ii) Hypothesis of molecular chaos. Statistical assumption about the number of binary collisions $\sim f_{\vec{k}_1} f_{\vec{k}_2} \underbrace{W_{12\to 1'2'}}_{\text{transition rate}}$
(probability for collision process)

(iii) $\cancel{|\partial_\mu f| \ll}$ $f_{\vec{k}}$ changes slowly over the characteristic space-time scale of the binary collision process $\Rightarrow$ ~~collision is point-like~~ collision happens at single space-time point $X$
"localized collisions"

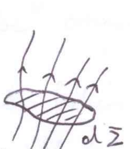
~~A. Collisionless~~ case

Note that $N^\mu(X) \equiv g \overbrace{\int \frac{d^3\vec{k}}{(2\pi)^3 k_0}}^{\equiv \int_k} k^\mu f_{\vec{k}}(X) \Rightarrow N^0 \equiv g \int \frac{d^3\vec{k}}{(2\pi)^3} f_{\vec{k}}$ particle density
$\equiv \int_k f_{\vec{k}}$

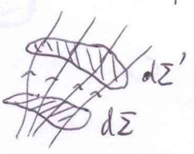
This is simply the relationship between single-particle distribution fct. $f_{\vec{k}}$ and particle density

Now count the no. of particle world lines crossing ~~the~~ hypersurface element $d\Sigma$

$$dN = \int_{d\Sigma} d\Sigma_\mu N^\mu = \int_{d\Sigma} d\Sigma_\mu \int_k k^\mu f_{\vec{k}}$$



Now ~~count the no. of~~ the same particle world lines cross ~~crossing~~ a different hypersurface element $d\Sigma'$,



$$dN' = \int_{d\Sigma'} d\Sigma'_\mu \int_k k^\mu f_{\vec{k}}$$

Obviously, the no. of world lines crossing $d\Sigma$ and $d\Sigma'$ are the same, so

$$0 = dN' - dN = \int_{d\Sigma'} d\Sigma'_\mu \int_k k^\mu f_{\vec{k}} - \int_{d\Sigma} d\Sigma_\mu \int_k k^\mu f_{\vec{k}}$$

## 1. Collisionless case

Note that the relationship between single-particle distribution fct. $f_{\vec{k}}(X)$ and particle density $N^0(X)$ is

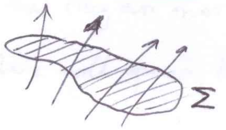
$$N^0(X) = g\int \frac{d^3\vec{k}}{(2\pi)^3} f_{\vec{k}}(X) \equiv g\int \frac{d^3\vec{k}}{(2\pi)^3 k_0} k_0 f_{\vec{k}}(X)$$

The particle 4-current is then given by

$$\boxed{N^\mu(X) = g\int \frac{d^3\vec{k}}{(2\pi)^3 k_0} k^\mu f_{\vec{k}}(X)}$$

Now counts the number of ~~par~~ world lines crossing a hypersurface $\Sigma$ with momenta between $\vec{k}$ and $\vec{k}+d^3\vec{k}$. These are

$$\boxed{k^0 \frac{dN}{d^3\vec{k}} \equiv \int_\Sigma d\Sigma_\mu\, k_0 \frac{dN^\mu}{d^3\vec{k}} = \cancel{\int_\Sigma d\Sigma_\mu k^\mu g f_{\vec{k}}(x)}\ \frac{g}{(2\pi)^3}\int_\Sigma d\Sigma_\mu\, k^\mu f_{\vec{k}}(x)}\ .$$



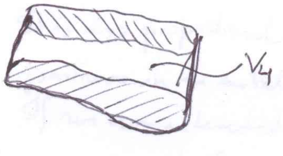
Cooper-Frye formula

Now follow the world lines to a different hypersurface $\Sigma'$. In the collisionless case, the momenta of the particles along the world lines remain the same, and if $\Sigma'$ encloses all world lines that crossed $\Sigma$, we have

$$0 = \frac{k^0 dN'}{d^3\vec{k}} - \frac{k_0 dN}{d^3\vec{k}}$$

$$\Leftrightarrow 0 = \int_{\Sigma'} d\Sigma'_\mu\, k^\mu f_{\vec{k}}(X) - \int_\Sigma d\Sigma_\mu\, k^\mu f_{\vec{k}}(X)$$

Now take the 4-volume $V_4$ bounded by $\Sigma$, $\Sigma'$, and the "tube" connecting them:



Since no world lines cross the tube (there are no collisions that kick particles out of $V_4$ before their world lines reach $\Sigma'$), we have

$$0 = \oint_{\partial V_4} d\Sigma_\mu\, k^\mu f_{\vec{k}}(X) \underset{\substack{\uparrow \\ \text{Gauß' theorem}}}{=} \int_{V_4} d^4X\, \partial_\mu\left(k^\mu f_{\vec{k}}(X)\right)$$

$$= \int_{V_4} d^4X\, k^\mu \partial_\mu f_{\vec{k}}(X).$$

But since $V_4$ was arbitrary, we have

$$\boxed{k^\mu \partial_\mu f_{\vec{k}}(x) = 0}$$

This is the Boltzmann eq. in the collisionless case.

2. Collisions

The ~~collision~~ case with collisions is easily obtained as a generalization of the previous case. Obviously, the number of world lines in $V_4$ will change on account of collisions, since a collision will change the momentum of the particles such that they are no longer inside the momentum-space interval $d^3\vec{k}$. This gives rise to a loss term. On the other hand, collisions may change the momenta of particles ~~in such a wa~~ previously not in $d^3\vec{k}$ in such a way that they are inside $d^3\vec{k}$ after the collision. This gives rise to a gain term. We will specify these terms in the following.

Consider a binary collision where (at least) one of the particles in the initial state had momentum $\vec{k}$ in $d^3\vec{k}$. Its collision partner had momentum $\vec{k}_1$. After the collision, they have momenta $\vec{k}'$ and $\vec{k}_1'$.

The (relativistically invariant) average number of collisions of this type is proportional to

$$\frac{1}{2} \frac{g\,d^3\vec{k}}{(2\pi)^3 k_0} \frac{g\,d^3\vec{k}_1}{(2\pi)^3 k_{1,0}} \frac{g\,d^3\vec{k}'}{(2\pi)^3 k_0'} \frac{g\,d^3\vec{k}_1'}{(2\pi)^3 k_{1,0}'} f_{\vec{k}}(x)\, f_{\vec{k}_1}(x)\, W_{\vec{k}\vec{k}_1 \to \vec{k}'\vec{k}_1'}$$

The factor $\frac{1}{2}$ takes into accounts that the particles ~~of~~ are indistinguishable, i.e., ~~does~~ the state $\vec{k}\vec{k}_1 \to \vec{k}'\vec{k}_1'$ is indistinguishable from $\vec{k}\vec{k}_1 \to \vec{k}_1'\vec{k}'$.
The number of collisions is proportional to the invariants momentum-space intervals of the collision partners, multiplied by internal degeneracy factors.
It is also proportional to the single-particle distribution fcts. of the collision partners in the initial state.
If we are interested in the loss of particles with momentum $\vec{k}$ in $d^3\vec{k}$, per $\frac{d^3\vec{k}}{k_0}$, irrespective of the momentum of the collision partner, and of the momenta of the final-state particles, we have to integrate over $\vec{k}_1$, $\vec{k}'$, and $\vec{k}_1'$.
The loss term in the Boltzmann eq. finally becomes

$$C_{loss}[f] = \frac{1}{2} \int_{\vec{k}_1, \vec{k}', \vec{k}_1'} f_{\vec{k}}(x) f_{\vec{k}_1}(x) W_{\vec{k}\vec{k}_1 \to \vec{k}'\vec{k}_1'}$$

Similarly, the gain term is

$$C_{gain}[f] = \frac{1}{2}\int_{k_1 k' k_1'} f_{\vec{k}'}(x) f_{\vec{k}_1'}(x) W_{\vec{k}'\vec{k}_1' \to \vec{k}\vec{k}_1}$$

where ~~now~~ the initial momenta are $\vec{k}', \vec{k}_1'$ and the final momenta are $\vec{k}, \vec{k}_1$.
The net change of particles with momenta $\vec{k}$ in $d^3\vec{k}$, in terms of the single-particle distribution fct. $f_{\vec{k}}(x)$ is

$$\boxed{C[f] = \frac{1}{2}\int_{k_1 k' k_1'} \left[ f_{\vec{k}'}(x) f_{\vec{k}_1'}(x) W_{\vec{k}'\vec{k}_1' \to \vec{k}\vec{k}_1} - f_{\vec{k}}(x) f_{\vec{k}_1}(x) W_{\vec{k}\vec{k}_1 \to \vec{k}'\vec{k}_1'} \right]}$$

This is the collision term ~~on the r.~~ on the r.h.s. of the Boltzmann eq.,

$$\boxed{k^\mu \partial_\mu f_{\vec{k}}(x) = C[f]}$$

It can be further simplified under the assumption that

$$W_{\vec{k}'\vec{k}_1' \to \vec{k}\vec{k}_1} = W_{\vec{k}\vec{k}_1 \to \vec{k}'\vec{k}_1'} \left(= W_{\vec{k}\vec{k}_1 \to \vec{k}_1'\vec{k}'}\right):$$

$$C[f] = \frac{1}{2}\int_{k_1 k' k_1'} \left[ f_{\vec{k}'}(x) f_{\vec{k}_1'}(x) - f_{\vec{k}}(x) f_{\vec{k}_1}(x) \right] W_{\vec{k}\vec{k}_1 \to \vec{k}'\vec{k}_1'}$$

## 3. Fields

We can include external fields under the assumption that the fields are ~~so weak~~ sufficiently weak so that they do not influence the dynamics of the collision process. Then, we may simply consider the collisionless case.

~~The previous consideration goes through up to the point when we count the number of world lines going through $\Sigma'$~~

Now something interesting happens. ~~The momenta~~ We may still count the no. of worldlines going through $\Sigma$, and having momenta between $\vec{k}$ and $\vec{k} + d^3\vec{k}$.
But when we follow the world lines further until $\Sigma'$, the momenta will have changed on account of the external field.
~~The~~ The relativistic version of Newton's law tells us that

$$F^\mu = \frac{dk^\mu}{d\tau},$$

where $\tau$ is the proper time in the rest frame of the particle.

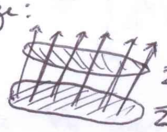
Therefore, the momentum change of a particle during a proper time interval $d\tau$ along the world line due to an external field is

$$dk^\mu = F^\mu \, d\tau \quad , \quad k^\mu \to k^\mu + F^\mu d\tau$$

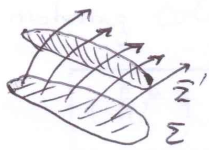
Note that since $m^2 = k^\mu k_\mu = \text{const.}$, $0 = d(k^\mu k_\mu) = 2k^\mu dk_\mu$, $= 2k^\mu F_\mu d\tau$

the momentum change is 4-orthogonal to $k^\mu$.

Now consider world lines going through $\Sigma$ and $\Sigma'$. In the case without fields we had the situation that the slope of the worldlines did not change:



Now, on account of the field, all world lines will be "bent" on account of the field:



This means that we may lose some worldlines, since they are "bent out" of $\Sigma'$, but we may also gain some world lines, as they are "bent into" $\Sigma'$

~~This effect~~

This effect has to be compensated by a "tilt" and a "shift" of $\Sigma' \to \Sigma''$ such that we do not lose or gain world lines. One can convince oneself that this tilt should be such that $d\Sigma''_\mu F^\mu \equiv 0$. Then,

$$0 = \int_{\Sigma''} d\Sigma''_\mu \, (k^\mu + F^\mu d\tau) \, f_{\vec{k}+\vec{F}d\tau}(x) - \int_{\Sigma} d\Sigma_\mu \, k^\mu f_{\vec{k}}(x)$$

$$\underset{d\Sigma''_\mu F^\mu = 0}{=} \int_{\Sigma''} d\Sigma''_\mu \, k^\mu \left( f_{\vec{k}}(x) + \frac{\partial f_{\vec{k}}(x)}{\partial k^\nu} F^\nu d\tau \right) - \int_{\Sigma} d\Sigma_\mu \, k^\mu f_{\vec{k}}(x)$$

~~Rename $\Sigma'' \to \Sigma'$, $d\Sigma''_\mu \to d\Sigma'_\mu$~~

Now add again the tube connecting $\Sigma$ and $\Sigma''$. Since no worldlines are added or lost due to the shift $\Sigma' \to \Sigma''$, we obtain

$$0 = \oint_{\partial V_4} d\Sigma_\mu\, k^\mu\, f_{k}(X) + \int_{\Sigma''} d\Sigma''_\mu\, d\tau\, k^\mu \frac{\partial f_k(X)}{\partial k^\nu} F^\nu$$

But $\frac{k^\mu}{m} \equiv v^\mu$, the 4-velocity of the particles, and $d\Sigma''_\mu\, d\tau\, v^\mu \equiv d^4X$, is just the 4-volume "swept" by the world lines when going from $\Sigma$ to $\Sigma''$

$$\Rightarrow \quad 0 = \int_{V_4} d^4X \left[\partial_\mu (k^\mu f_k(X)) + m F^\nu \frac{\partial f_k(X)}{\partial k^\nu}\right]$$

$$= \int_{V_4} d^4X \left[k^\mu \partial_\mu + m F^\mu \frac{\partial}{\partial k^\mu}\right] f_k(X)$$

Since $V_4$ was arbitrary,

$$\left(k^\mu \partial_\mu + m F^\mu \frac{\partial}{\partial k^\mu}\right) f_k(X) = 0,$$

or, restoring collisions,

$$\boxed{\left(k^\mu \partial_\mu + m F^\mu \frac{\partial}{\partial k^\mu}\right) f_k(X) = C[f]}$$

This is the Boltzmann - Vlasov equation. In the collisionless case, one often also calls this Vlasov equation.

4. Mixtures

Let us consider the system of Boltzmann eqs. for $N$ different species,

$$k_i^\mu \partial_\mu f^{(i)}_{k_i}(X) = \sum_{j=1}^{N} C_{ij}[\vec f] \qquad \vec f \equiv (f^{(1)}, f^{(2)}, \ldots, f^{(N)})$$

Here, the collision term is (taking particles to be distinguishable)

$$C_{ij}[\vec f] = \sum_{k,l=1}^{N} \int_{k_j k_k k_l} \left( f^{(k)}_{k_k} f^{(l)}_{k_l} W_{k_k k_l \to k_i k_j} - f^{(i)}_{k_i} f^{(j)}_{k_j} W_{k_i k_j \to k_k k_l}\right)$$

In order to simplify the notation, we will suppress the "$k$"s

$$\boxed{k_i^\mu \partial_\mu f_i(X) = \sum_{j=1}^{N} C_{ij}[\vec{f}]}$$

$$C_{ij}[\vec{f}] = \sum_{k,l=1}^{N} \int_{jkl} \left( f_k f_l W_{kl\to ij} - f_i f_j W_{ij\to kl} \right)$$

We shall now show that

$$\boxed{\sum_{i,j=1}^{N} \int_i \psi_i(X)\, C_{ij}[\vec{f}] = 0}$$

for a function $\psi_i(X) = a_i(X) + b_\mu(X)\, k_i^\mu$, i.e., which is at most linear in the momentum $k_i^\mu$. The functions $a_i(X)$ must be such that, for collision partners $ij$, $kl$:

$$a_i(X) + a_j(X) = a_k(X) + a_l(X),$$

i.e., the functions must be additively conserved.

Now we compute

$$\sum_{i,j=1}^{N} \int_i \psi_i(x)\, C_{ij}[\vec{f}] = \sum_{i,j,k,l}^{N} \int_{ijkl} \psi_i \left( f_k f_l W_{kl\to ij} - \underbrace{f_i f_j W_{ij\to kl}}_{(ij\leftrightarrow kl)} \right)$$

Changing integration variables and summation indices in the last term:

$$= \sum_{i,j,k,l} \int_{ijkl} (\psi_i - \psi_k)\, f_k f_l W_{kl\to ij}$$

But $W_{kl\to ij} = \cancel{W_{ij\to kl}}\ W_{lk\to ji}$

$$= \frac{1}{2} \sum_{i,j,k,l} \int_{ijkl} \underbrace{(\psi_i - \psi_k + \psi_j - \psi_l)}\, f_k f_l W_{kl\to ij}$$

$$= \underbrace{a_i + a_j - a_k - a_l}_{=0} + b_\mu \underbrace{(k_i^\mu + k_j^\mu - k_k^\mu - k_l^\mu)}_{=0}$$

by above constraints — by energy-momentum conservation in binary collisions

5. Conservation laws

(a) Integrate the Boltzmann eq. over $\frac{g\,d^3k_i}{(2\pi)^3 k_{0,i}}$, sum over $i$, and use the previous proof with $\psi_i(X) \equiv 1$:

$$\sum_{i=1}^{N} \int_i k_i^\mu \partial_\mu f_i(X) = \sum_{i,j=1}^{N} \int_i C_{ij}[f] \equiv 0$$

$$\Leftrightarrow \quad 0 = \partial_\mu \underbrace{\sum_{i=1}^{N} \underbrace{\int_i k_i^\mu f_i(X)}_{\equiv N_i^\mu \text{ 4-current of particle species } i}}_{N^\mu \text{ total 4-current of all particle species}}$$

$$\Rightarrow \quad 0 = \partial_\mu N^\mu \qquad \text{particle no. conservation}$$

(b) Multiply the Boltzmann eq. with $k_i^\nu$, integrate over $\frac{g\,d^3k_i}{(2\pi)^3 k_{0i}}$, sum over $i$, and use previous proof with $\psi_i(X) \equiv k_i^\mu$:

$$\sum_{i=1}^{N} \int_i k_i^\nu k_i^\mu \partial_\mu f_i(X) = \sum_{i,j=1}^{N} \int_i k_i^\nu C_{ij}[f] \equiv 0$$

$$\Leftrightarrow \quad 0 = \partial_\mu \underbrace{\sum_{i=1}^{N} \underbrace{\int_i k_i^\mu k_i^\nu f_i(X)}_{\equiv T_i^{\mu\nu} \text{ energy-momentum tensor of particle species } i}}_{\equiv T^{\mu\nu} \text{ total energy-momentum tensor}}$$

$$\Rightarrow 0 = \partial_\mu T^{\mu\nu} \qquad \text{energy-momentum conservation}$$