Some approaches to tackle ill-posed inversion problem

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Outline

- Introduction & Motivation
- Methods
- Mock data tests
- Real lattice results

Difficulties in the inversion

Relation between correlators and spectral functions:

$$G(\tau,\vec{p},T)=\int_0^\infty \frac{d\omega}{2\pi}\rho(\omega,\vec{p},T)\,K(\tau,\omega,T)$$

Discretized ~O(10) (τ)

Continuous ~O(1000) (ω)

$$K(\tau,\omega,T)=\frac{\cosh(\omega(\tau-1/2T))}{\sinh(\omega/2T)}$$

Spectral functions are crucial to understand in-medium hadron properties and transport properties of QGP:

$$G(\tau,\boldsymbol{p})=\int d\boldsymbol{x}\, e^{-i\cdot\boldsymbol{p}\cdot\boldsymbol{x}}\langle J_H(0,0)J_H{}^{+}(\tau,\boldsymbol{x})\rangle$$

$J_H(\tau,\vec{x})=\overline{q}(\tau,\vec{x})\Gamma_H q(\tau,\vec{x})$

Heavy quark diffusion coefficient :

$$D=\frac{\pi}{6\chi_{00}}\lim_{\omega\to 0}\frac{\rho_{ii}(\omega,\vec{p}=0,T)}{\omega}$$

$J_H(\tau,\vec{x})=T^{\mu\nu}(\tau,\vec{x})$

Shear viscosity:

$$\eta=\pi\lim_{\omega\to 0}\lim_{\boldsymbol{p}\to 0}\frac{\rho^{12,12}(\omega,\boldsymbol{p},T)}{\omega}$$

Maximum Entropy Method(MEM)

- A method based on Bayesian theorem:

$$\langle\rho\rangle = \int d\alpha P[\alpha|\bar{G}] \int \boldsymbol{\mathcal{D}\rho\, P[\rho|\bar{G},\alpha]\rho}$$

$$\approx \int d\alpha\, P[\alpha|\bar{G}]\hat{\boldsymbol{\rho}}_{\boldsymbol{\alpha}} \text{ (sharp-peak assumption)}$$

Goal: obtain the most probable solution

- $P[\alpha|\bar{G}] = \frac{P[\alpha]}{P[\bar{G}]} \int \mathcal{D}\rho P[\bar{G}|\rho,\alpha] \cdot P[\rho|\alpha]$

  $P[\bar{G}|\rho,\alpha] \sim \exp(-\chi^2[\rho]/2)$: likelihood function

  $P[\rho|\alpha] \sim \exp(\alpha S[\rho])$: prior probability

sharp-peak assumption

minimize $F = \frac{\chi^2}{2} - \alpha S$

- Shanon-Jaynes entropy:

$$S[\rho] = \int d\omega \left[\rho(\omega) - D(\omega) - \rho(\omega) \ln \frac{\rho(\omega)}{D(\omega)}\right]$$

- Default Model $D(\omega)$: parameter needed as input

[M. Jarrell, J. E. Gubernatis, Phy. Rep.269,133(1996)]
[M. Asakawa et al., Prog.Part.Nucl.Phys. 46(2001) 459-508]

Stochastic Analytic Inference(SAI)

- A stochastic method based on Bayesian theorem :
  $\langle n \rangle = \int d\alpha \, \langle n \rangle_\alpha \, P[\alpha|\bar{G}]$
- $n(x) = \frac{\rho(\omega)}{D(\omega)}, \qquad x \equiv \phi(\omega) = \int_0^\omega D(\nu) d\nu$ [H. Ohno, PoS(LATTICE 2015)175]

Goal: find the distribution of $P[\alpha|\bar{G}]$

- Field treatment of $n(x)$ gives:
  $\langle n \rangle_\alpha = \int \mathcal{D}n \, n \, \boldsymbol{P[n|\alpha, \bar{G}]} = \int \mathcal{D}n \, n \frac{\mathbf{1}}{\boldsymbol{Z(\alpha)}} \boldsymbol{e^{-\chi^2/2\alpha}}$
- Posterior probability: $P[n|\alpha, \bar{G}] = \frac{1}{P[\bar{G}|\alpha]} P[\bar{G}|\alpha, n] P[n|\alpha]$

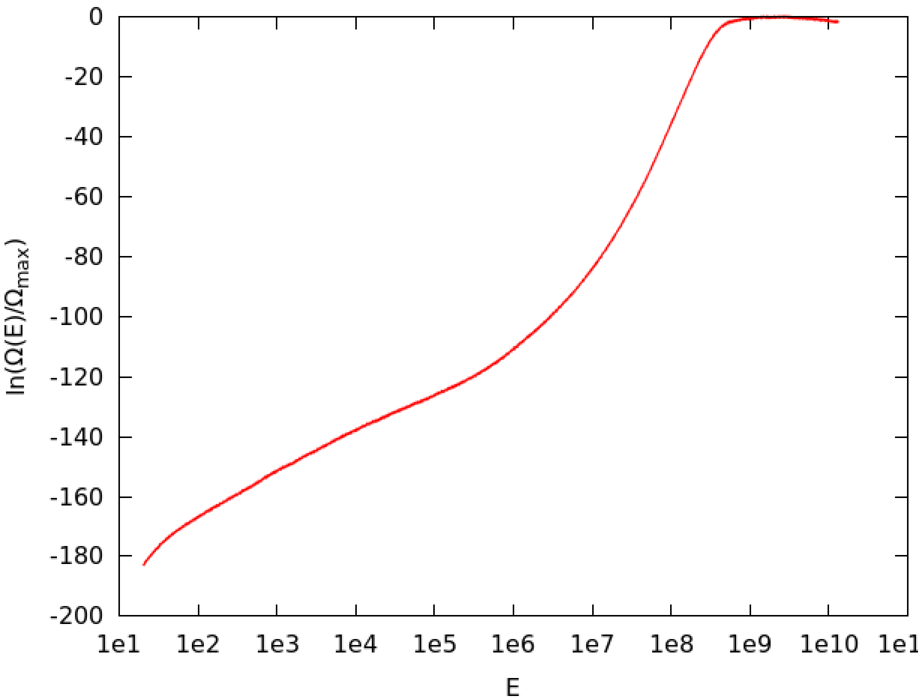
$$\Longrightarrow \begin{cases} P[\bar{G}|\alpha, n] = \frac{1}{Z'} e^{-\chi^2[n]/2\alpha}, \text{likelihood function} \\ P[n|\alpha] = \Theta(n)\delta\left(\int_0^{x_{max}} dx \, n(x) - 1\right), \text{prior probability} \\ P[\bar{G}|\alpha] = Z/Z', \qquad \text{normalization factor} \end{cases}$$

- $P[\alpha|\bar{G}] = \frac{P[\alpha]}{P[\bar{G}]} \int \mathcal{D}n \; P[\bar{G}|\alpha, n] P[n|\alpha] \sim P[\alpha] \alpha^{-\frac{N}{2}} Z(\alpha)$

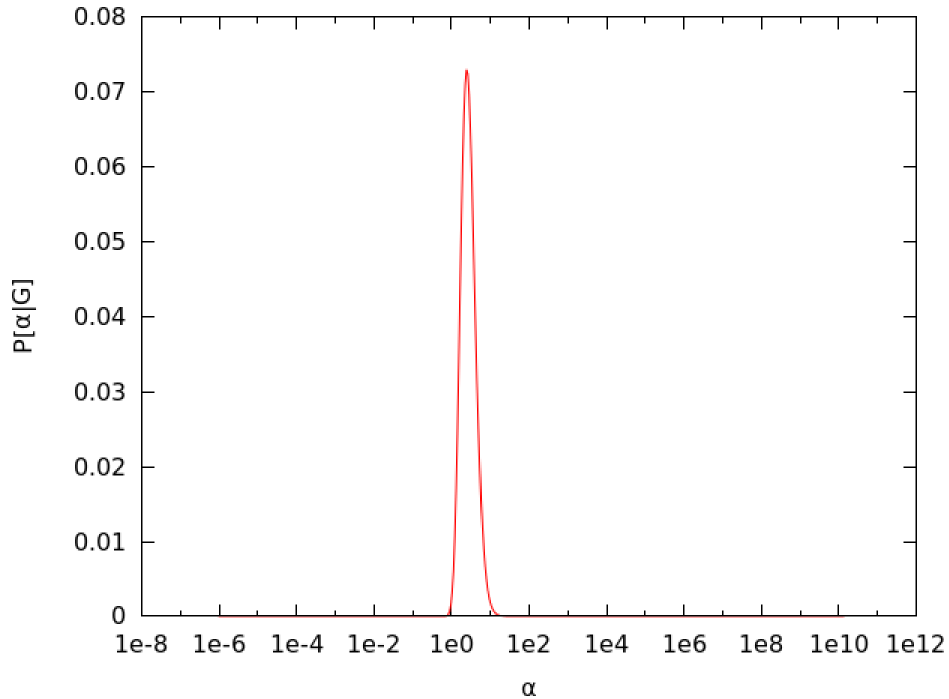
Stochastic Analytic Inference

- Density of States(DoS): $\Omega(E) = \int \mathcal{D}n\, \delta(\chi^2[n]/2 - E)$
- $P[\alpha|\bar{G}] = P[\alpha]\alpha^{-\frac{N}{2}} \int dE\Omega(E)e^{-E/\alpha}$

F.-G. Wang, D.P. Landau arXiv:cond-mat/0107006



Density of States.



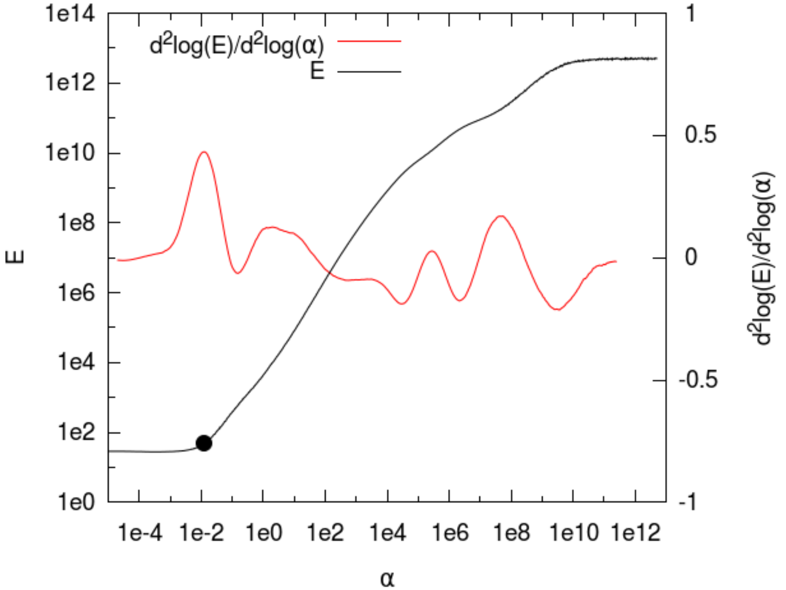
$P[\alpha|\bar{G}]$.

Stochastic Optimization Method(SOM)

[H.-T. Shu, H.-T. Ding, O. Kaczmarek, S. Mukherjee, H. Ohno, PoS(LATTICE 2015)180]

- Based on Central Limit Theorem.
- **No prior information** is needed. All information comes from correlators: $E = \chi^2[\rho]/2$ (Fictitious energy)
- Field treatment of ρ. Evolves with fictitious temperature α.
- Possible solution obtained when **phase transition** occurs.

[K.S.D. Beach arXiv:cond-mat/0403055]



Phase transition occurs at the *kink(black spot)*.

SAI to SOM

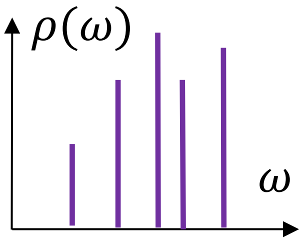
- SAI reduces to SOM when using **constant default model**!
  $x = \omega, n(x) = \rho(\omega)$

Basis of Stochastic approaches

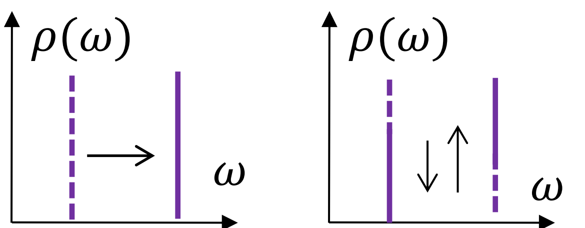
SAI

- Ingredients: δ functions

$$\rho(\omega) = \sum_{t}^{K} r_i \delta(\omega - a_i),$$



- Elementary updates



1.Shift 2. Change residues

- Normalization

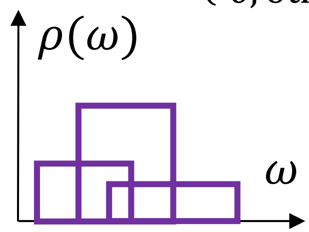
$$\sum r_i = \bar{G}(\tau_0)$$

SOM

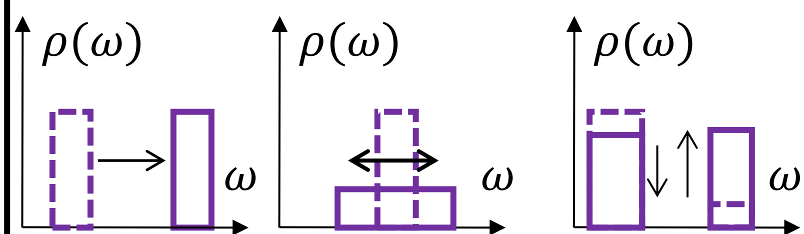
- Ingredients: boxes

$$\rho(\omega) = \sum_{t}^{K} \eta\,(P_t)(\omega),$$

$$\eta(P_t)(\omega) = \begin{cases} h_t, \omega \in [c_t - w_t/2, c_t + w_t/2,] \\ 0, \text{otherwise} \end{cases}$$



- Elementary updates

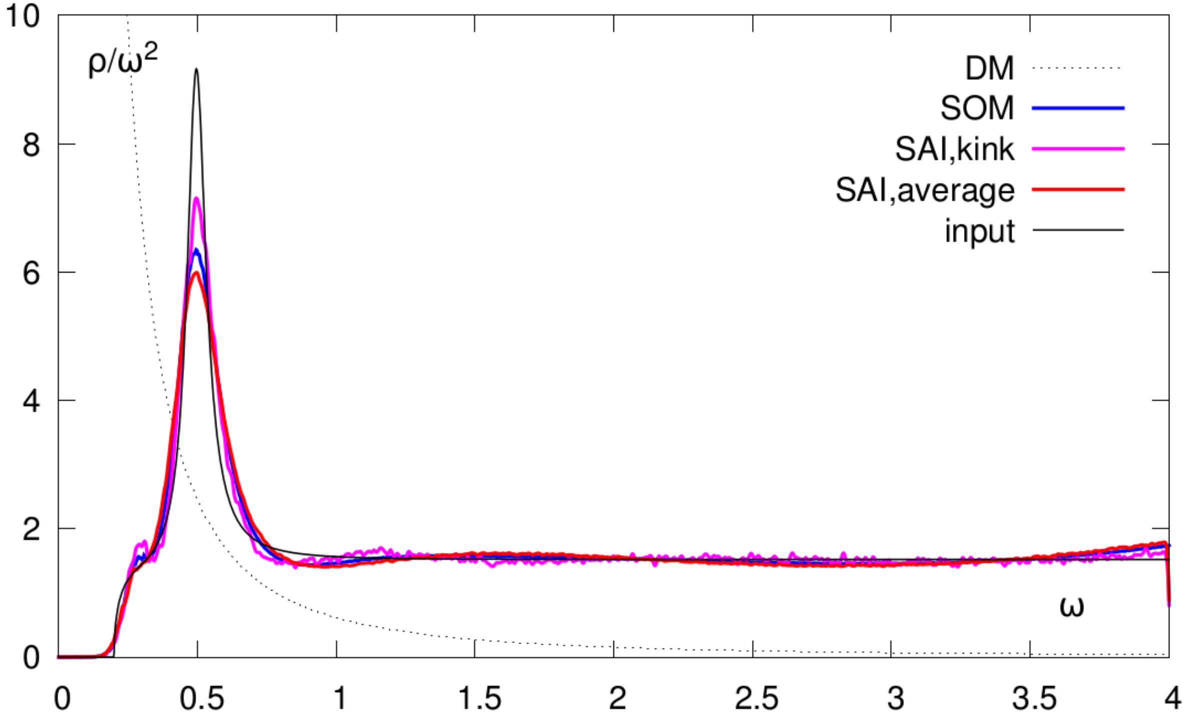


1.Shift 2. Change width 3. Change heights

- Normalization

$$\sum h_i w_i = \bar{G}(\tau_0)$$

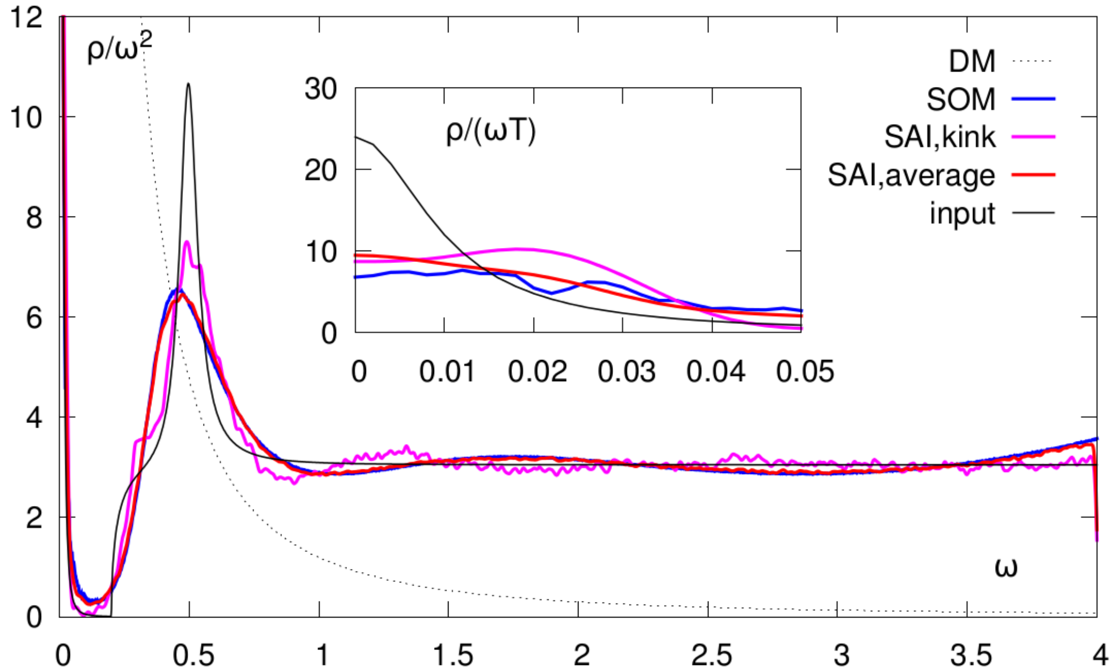
Mock data test:Different spectral functions



$\rho(\omega) = \rho_{res} + \rho_{cont}$,
corresponding to $T < T_c$
Error in mock data:
$\sigma(\tau) = \epsilon G(\tau)\tau$
$N_\tau = 48, \epsilon = 10^{-4}$

1. SOM gives similar results to SAI with constant DM.
2. SAI&SOM reconstruct the input well.

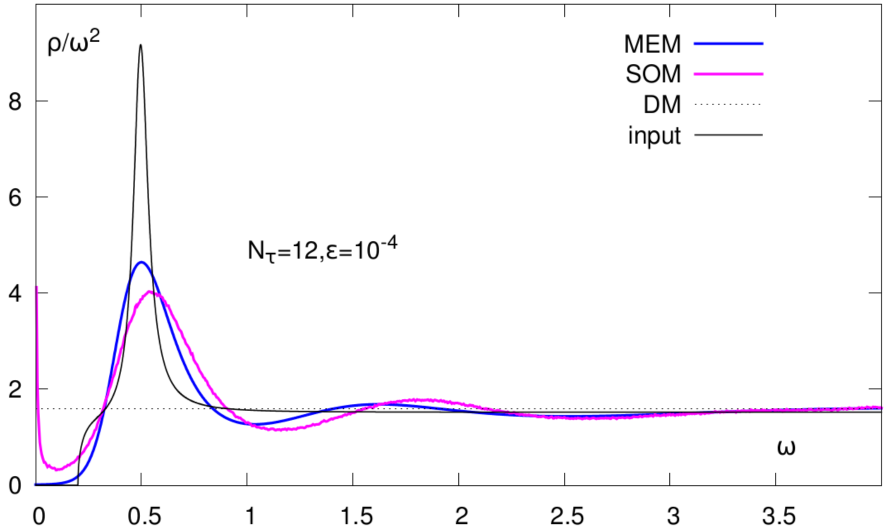
Mock data test: Different spectral functions

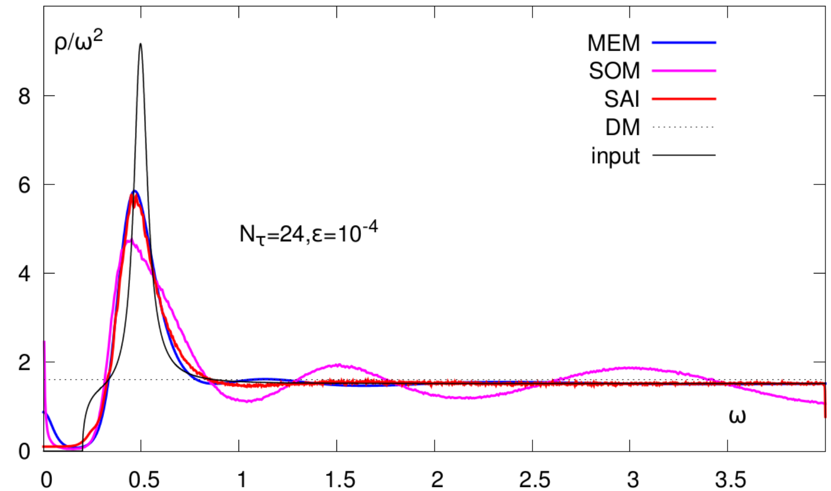


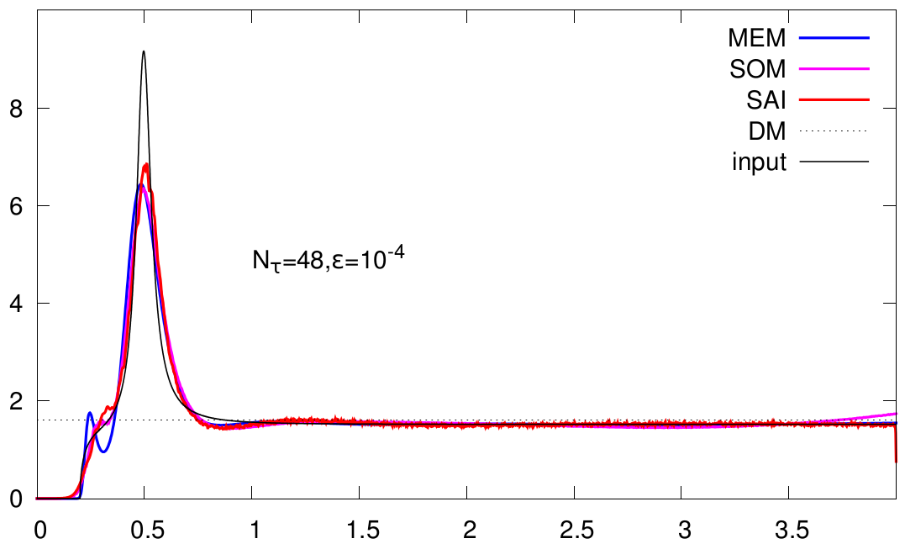
$\rho(\omega) = \rho_{trans} + \rho_{res} + \rho_{cont}$,
corresponding to $T > T_c$
$N_\tau = 48, \epsilon = 10^{-4}$

1. MEM&SAI with constant DM and SOM can not reconstruct the transport peak precisely.
2. MEM&SAI&SOM reconstruct the resonance peak, but the width and peak-location differ from the input.
3. SAI&SOM can reconstruct the continuum part well.

Mock data test:Dependence on N_τ

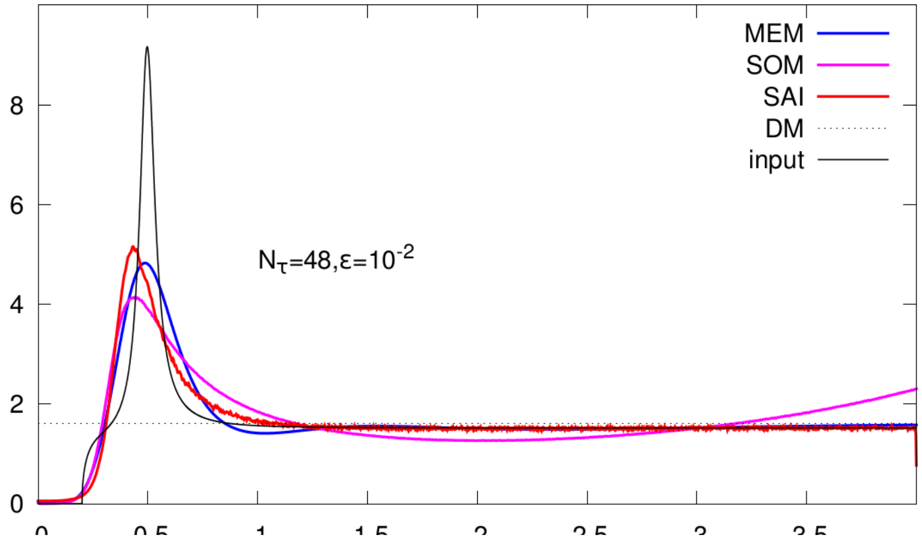


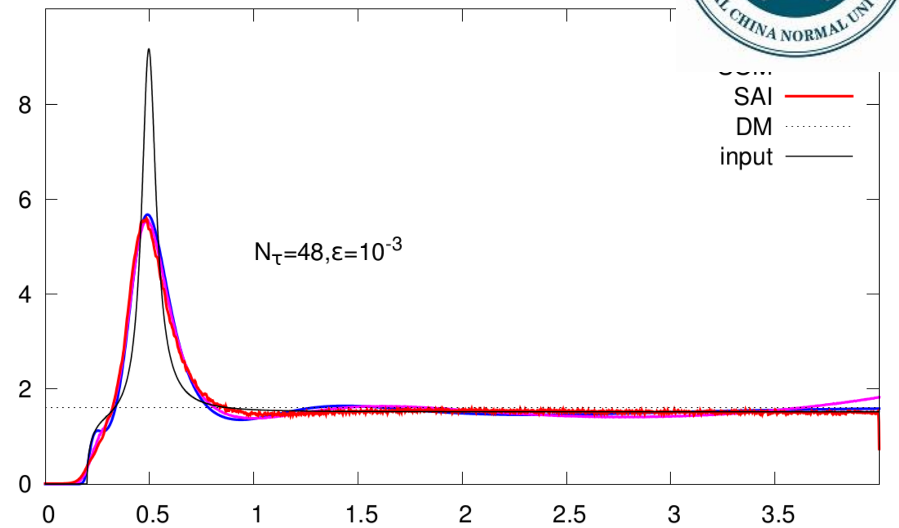


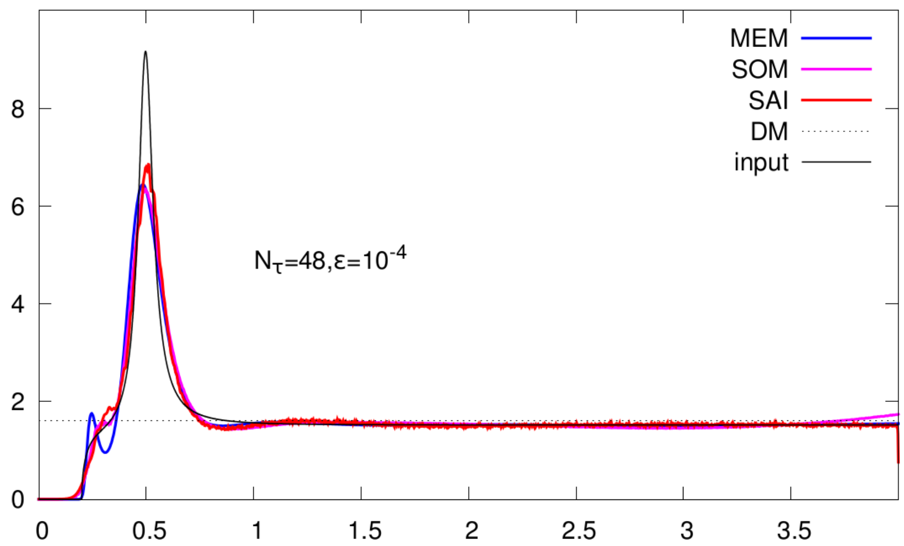


1. MEM reconstructs the resonance peak well even at small N_τ.
2. SAI&SOM give fake transport peak at small N_τ .
3. SAI&SOM reconstruct the continuum part well.
4. MEM gives fake resonance peaks.

Mock data test: Dependence on noise level



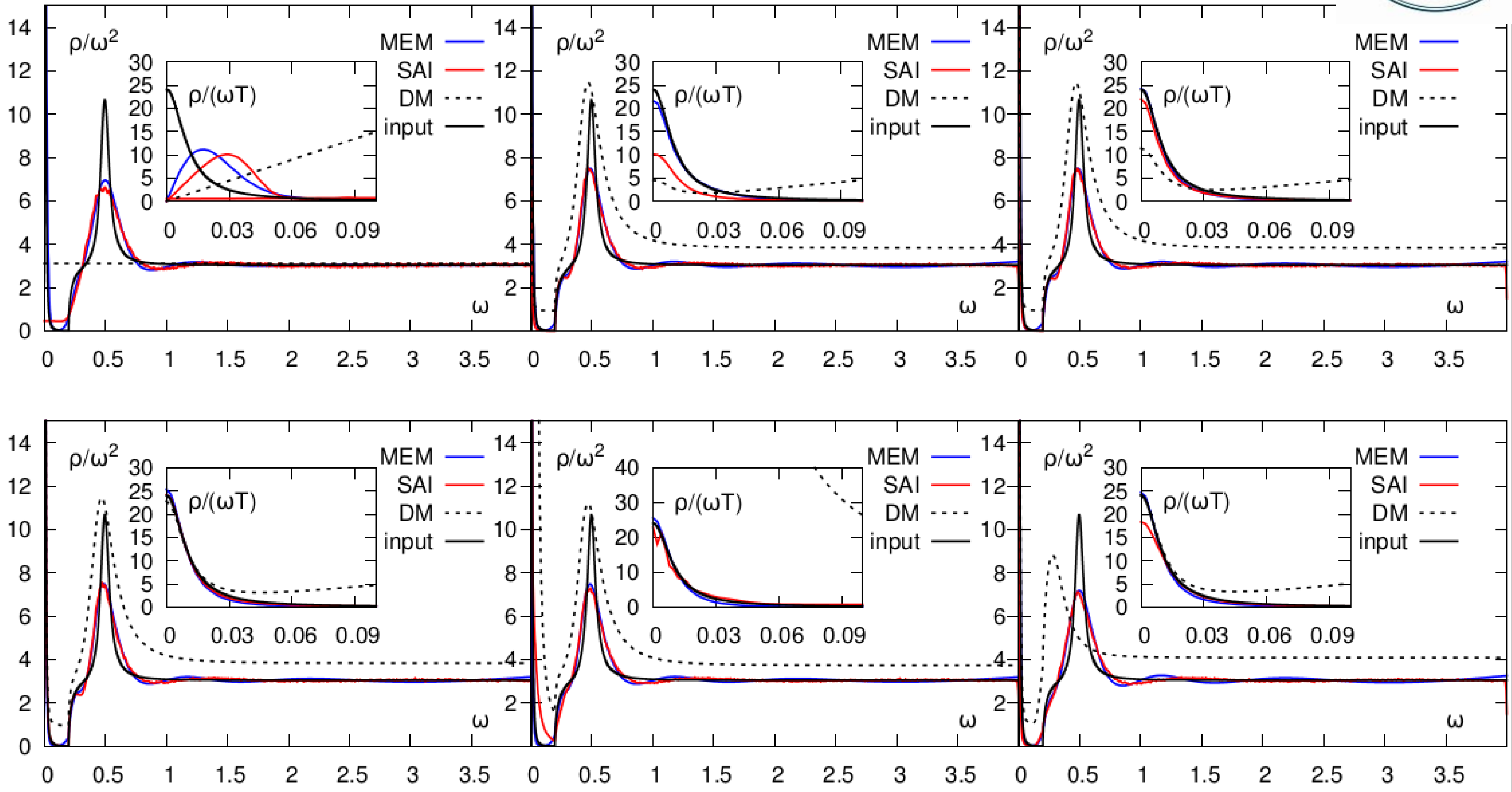




1. MEM gives bad output with noisy data.
2. SAI&SOM give a rough resonance peak with noisy data.
3. SAI&SOM reconstruct the continuum part well.
4. As noise becomes weak, the output approaches to the input for all methods.

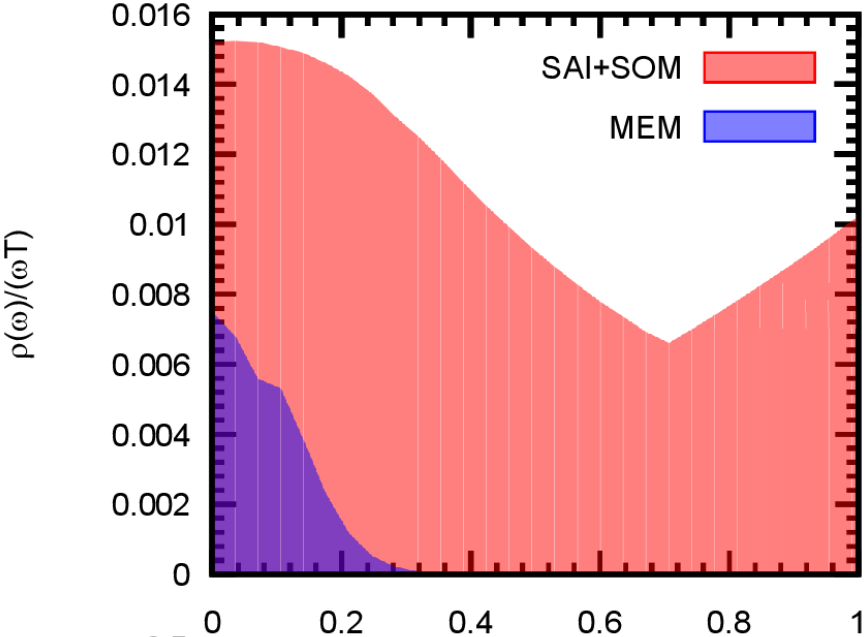
Mock data test: Dependence on DM

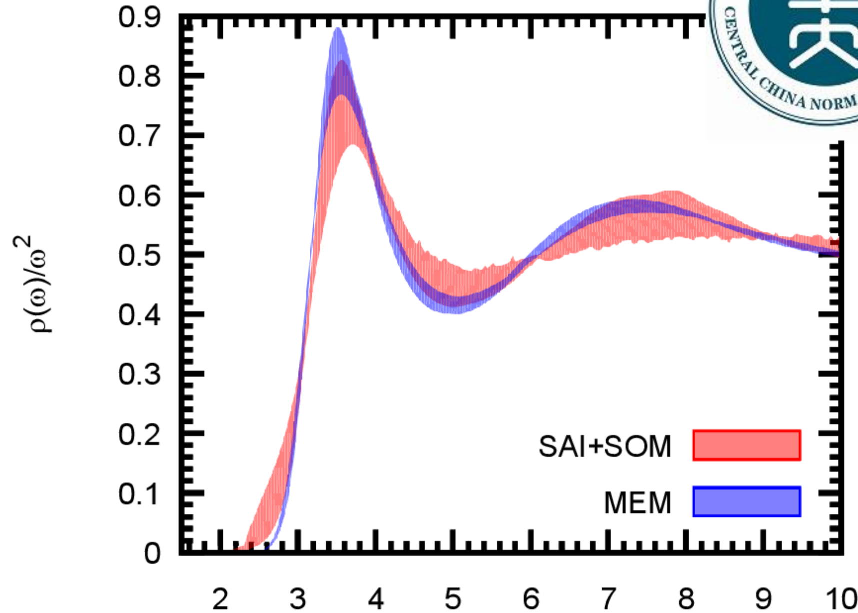
$N_\tau = 48, \epsilon = 10^{-4}$ for all.

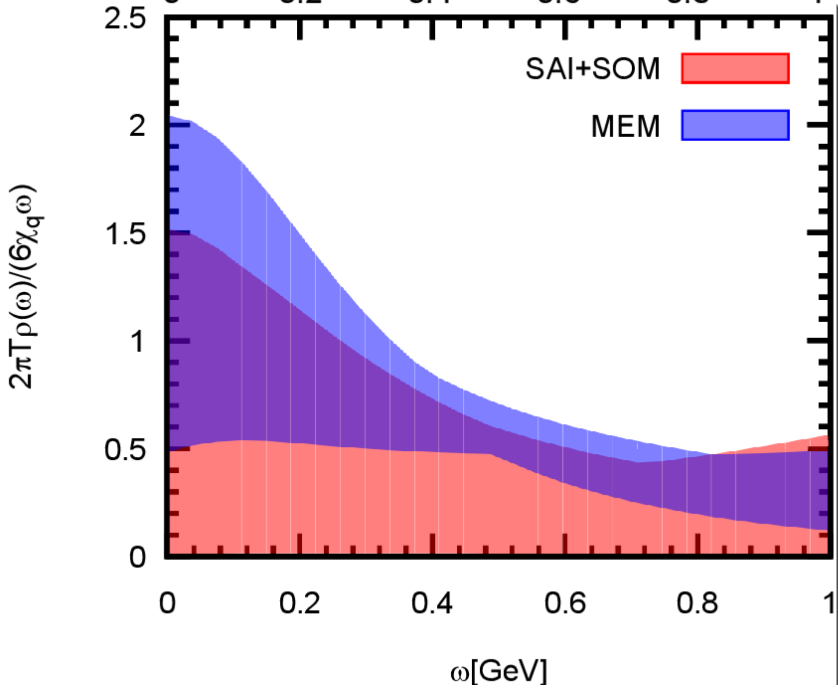


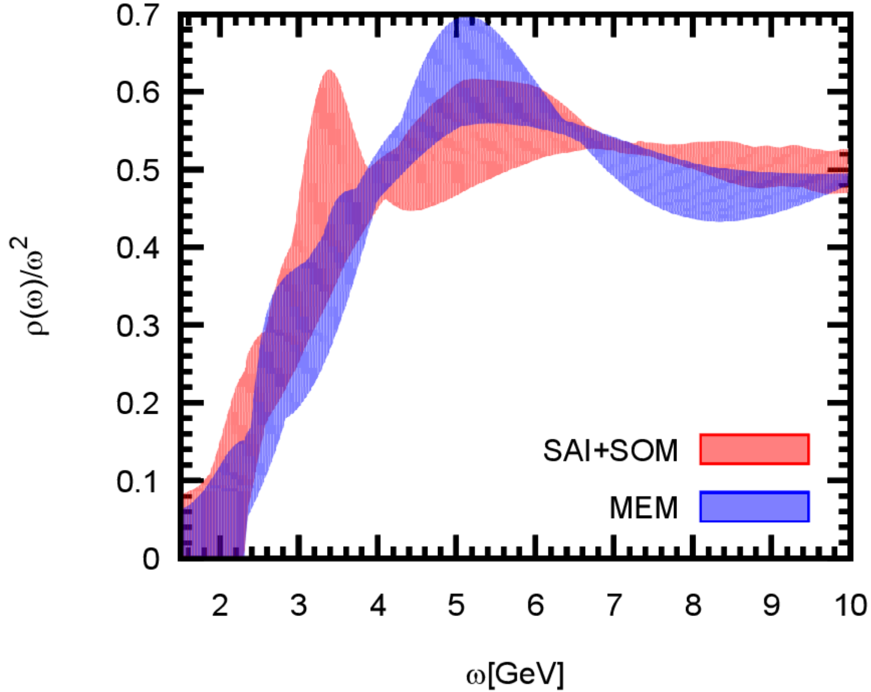
1. Dependence is weak for resonance peak and continuum.
2. There is dependence for transport peak. But upper bound exists in this case.

Real Lattice Results at 0.75Tc & 1.5Tc









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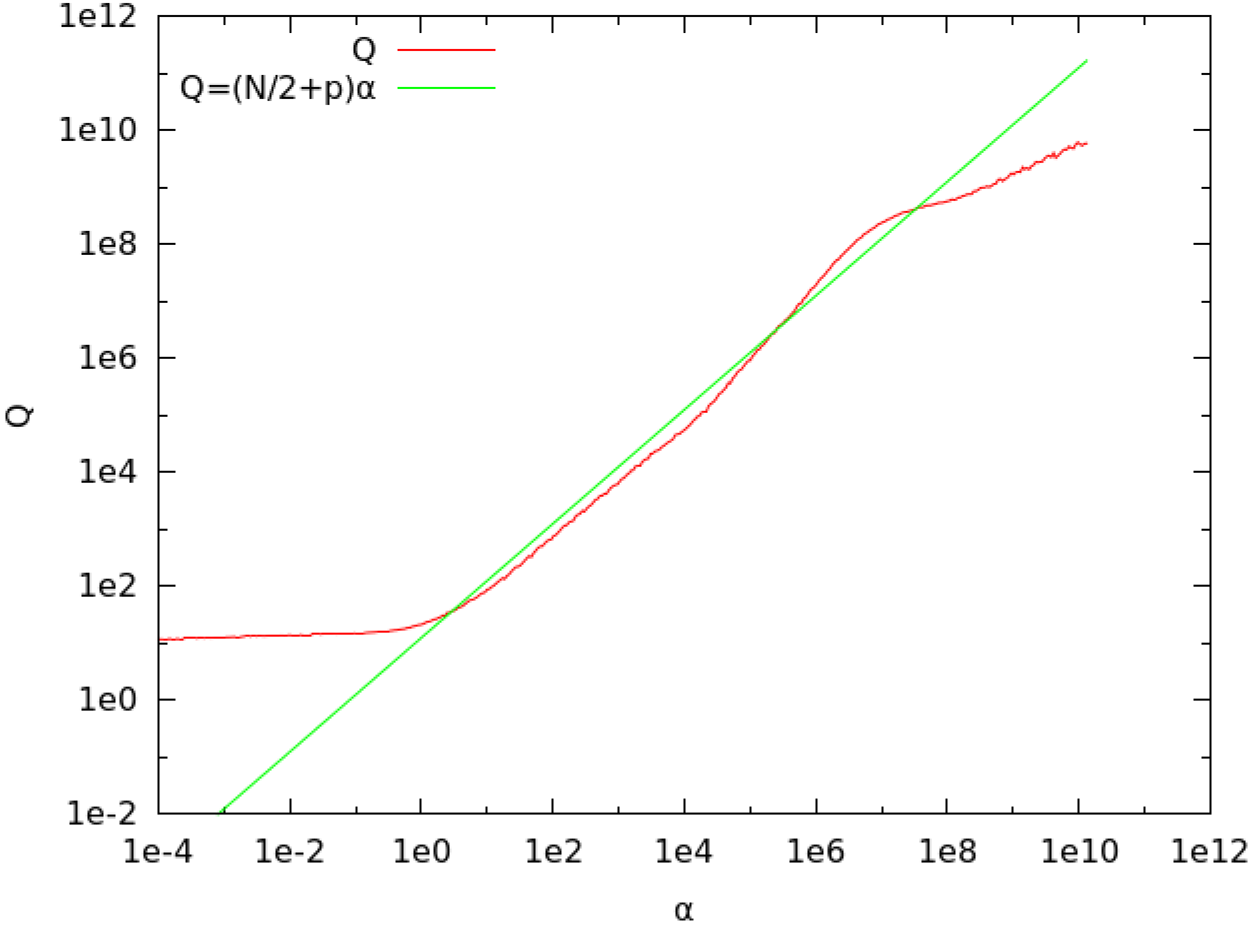

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Mock SPFs

- Resonance peak : $\rho_{res} = c_{res} \frac{\Gamma(\omega,\omega_0,\gamma_0)M}{(\omega^2-M^2)^2+M^2\Gamma^2(\omega,\omega_0,\gamma_0)} \frac{\omega^2}{\pi}$

  Where $\Gamma(\omega,\omega_0,\gamma_0) = \theta(\omega-\omega_0)\gamma_0(1-\frac{\omega^2{}_0}{\omega^2})^5$
- Transport peak:

  $\rho_{trans} = c_{trans} \frac{\eta\omega}{(\omega^2-\eta^2)^2}$
- Free continuum:

$$\rho_{cont} = c_{cont} \frac{N_c}{8\pi^2}\Theta(\omega^2-4m^2)\omega^2 \tanh\left(\frac{\omega}{4T}\right)$$

$$\times \sqrt{1-\left(\frac{2m}{\omega}\right)^2} \left[a^1 + a^2\left(\frac{2m}{\omega}\right)^2\right]$$

- Free Wilson:

$$\rho_{Wilson} = c_{Wilson} \frac{N}{L^3}\sum_k \sinh\left(\frac{\omega}{2T}\right)\left[b^1 - b^2 \frac{(\sum_{i=1}^{3}\sin^2 k_i)}{\sinh^2 E_k(m)}\right]$$

Where $\cosh E_k(m) = 1 + \frac{K_k^2+M_k^2(m)}{2(1+M_k(m))}$, $K_k = \sum_{i=1}^{3}\gamma_i \sinh k_i$

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Mock SPFs parameters

Elements in SPF	Parameters
ρ_{res}^1	$c_{res} = 1, \omega_0 = 0.2, \gamma_0 = 0.20, M = 0.5$
ρ_{res}^2	$c_{res} = 4, \omega_0 = 0.2, \gamma_0 = 0.25, M = 1.2$
ρ_{res}^3	$c_{res} = 6, \omega_0 = 0.2, \gamma_0 = 0.20, M = 2.5$
ρ_{trans}	$\mathrm{c}_{trans} = 0.2, \eta = 0.01$
ρ_{cont}	$c_{cont} = 20, a^1 = 2, a^2 = 1, m = 0.1$
ρ_{Wilson}	$c_{Wilson} = 1, b^1 = 3, b^2 = 1, m = 0.112$

$\mathrm{N}_\tau = 48, a = 1, \tau_{min} = 1, \omega \in [0,4]$.
Error in mock data: $\sigma(\tau) = \epsilon G(\tau)\tau$