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The Bc spectrum at N^3LO

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Based on the work in collaboration with A. Pineda and M. Stahlhofen:
arxiv:1511.08210, arxiv:1706.03971, arxiv:17xx.xxxxx

Outline

1. The effective field theory for bound states
2. The potential for different masses
3. The B_c spectrum
4. Final remarks

The effective field theory for bound states

EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: $m_r \gg |\mathbf{p}| \gg E$

when bounded by QCD, we need to take into account the relation of the scales to Λ_{QCD}

- **NR limit:** $m_r \gg \Lambda_{\mathrm{QCD}}$

  Strong coupling regime: $|\mathbf{p}| \sim \Lambda_{\mathrm{QCD}}$

  Weak coupling regime: $|\mathbf{p}| \gg \Lambda_{\mathrm{QCD}}$

EFTs for bound states: NR systems

Non-relativistic systems fulfil the relation: $m_r \gg |\mathbf{p}| \gg E$

When bounded by QCD in the weak coupling regime $\alpha_s \sim v \sim 0.3 - 0.5$

Scales in bound state		Coulomb interaction
Hard scale: m_r	$\longrightarrow$	m_r
Soft scale: $|\mathbf{p}|$	$\longrightarrow$	$m_r\alpha_s$
Ultrasoft scale: E	$\longrightarrow$	$m_r\alpha_s^2$

Scales are well separated

We can integrate out the hard and soft scales to obtain **pNRQCD**

It describes systems such as: J/ψ, Υ, $\bar{t}t$ near threshold, B_c, etc.

The physics of heavy quarks

- Bound states of heavy quarks are naturally NR systems
- We focus in the situation $m_1 \sim m_2 \sim m_r$

Strict weak coupling regime $mv^2 \gg \Lambda_{\mathbf{QCD}}$

$$\left.\begin{array}{l}\left(i\partial_0 - \frac{\mathbf{p}^2}{2m_r} - V^{(0)}(r)\right)\phi(\mathbf{r}) = 0 \\ + \text{ corrections to the potential} \\ + \text{ interaction with other low-energy degrees of freedom}\end{array}\right\} \text{pNRQCD.}$$

- The potentials depend on the matching procedure: on-shell, off-shell in Coulomb and Feynman gauges, with Wilson loops, ...

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- The singlet potential: $h_s(\mathbf{r}, \mathbf{p}, \mathbf{P_R}, \mathbf{S}_1, \mathbf{S}_2) = \frac{\mathbf{p}^2}{2\,m_r} + \frac{\mathbf{P_R^2}}{2\,M} + V_s(\mathbf{r}, \mathbf{p}, \mathbf{P_R}, \mathbf{S}_1, \mathbf{S}_2)$

$$V_s = V^{(0)} + \frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2} + \cdots$$

The potential for different masses

The potential in position space

Basis for the potential ($\mathbf{P_R} = 0$):

- static and $1/m$ potentials ($a+b=0,1$): $V^{(a,b)} = V^{(a,b)}(r)$
- the $1/m^2$ potential ($a+b=2$): $V^{(a,b)} = V^{(a,b)}_{SD} + V^{(a,b)}_{SI}$

$$V^{(a,b)}_{SI} = \frac{1}{2}\left\{\mathbf{p^2}, V^{(a,b)}_{\mathbf{p^2}}(r)\right\} + V^{(a,b)}_{\mathbf{L^2}}(r)\frac{\mathbf{L}^2}{r^2} + V^{(a,b)}_{r}(r)$$

Poincaré invariance, invariant under $m_1 \leftrightarrow m_2$, $\psi \leftrightarrow \chi$

Many contributions over the years

Pantaleone, Yndurain, Penin, Steinhauser, Brambilla, Pineda, Soto, Vairo ...

The potential in momentum space

The potential in momentum space: $\tilde{V}_s \equiv \langle \mathbf{p}' | V_s | \mathbf{p} \rangle$

- static and $1/m$ potentials: $\tilde{V}^{(0)} = \tilde{V}^{(0)}(\mathbf{k})$
- the $1/m^2$ potential:

$$\tilde{V}_{SI}^{(2,0)} = \frac{\mathbf{p}^2 + \mathbf{p}'^2}{2\mathbf{k}^2} \tilde{D}_{\mathbf{p}^2}^{(2,0)}(k) + \tilde{D}_r^{(2,0)}(k) + \frac{(\mathbf{p}'^2 - \mathbf{p}^2)^2}{\mathbf{k}^4} \tilde{D}_{\mathrm{off}}^{(2,0)}(k)$$

The $\tilde{D}$-coefficients have $D = d + 1 = 4 + 2\epsilon$ dimensions

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- Only the $\tilde{D}_{\mathrm{off}}$-coefficients are **scheme-dependent**
- The Fourier transform of the off-shell potential contributes to the **three** position space structures

pNRQCD: Field redefinition & scheme dependence

Matching schemes are related by a field redefinition

$$h_s = h_0 + \frac{\delta V_1(r)}{m_r} + \cdots \quad \Longrightarrow \quad h_s' = h_0 + \frac{1}{2m_r^2}\delta V_{\rm FR} + \cdots$$

Unitary transformation

$$\delta \tilde{V}_{\rm FR} = \langle \mathbf{p}'|\delta V_{\rm FR}|\mathbf{p}\rangle = \frac{(\mathbf{p}'^2 - \mathbf{p}^2)^2}{\mathbf{k}^4}\tilde{g}(k)\,,$$

where $\tilde{g}(k) \sim \tilde{g}(k, V_0, \delta V_1)$

We can **exchange $\mathbf{1/m_r}$** by **$\mathbf{1/m_r^2}$ off-shell** potential terms

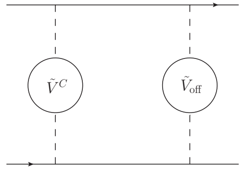
Unitary transformation acts as translation between schemes

- On-shell Green functions (S-matrix)
- Off-shell Green functions: Coulomb & Feynman gauge
- Wilson loops

Matching with on-shell Green functions

- on-shell Green functions $\equiv$ S-matrix elements
- asymptotic quarks fulfilling the free EOM order by order

Imperfect cancellation between NRQCD and pNRQCD potential loops

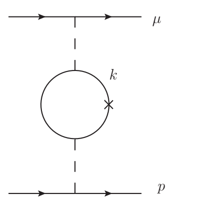


$\Rightarrow$Nontrivial mass dependence in the $1/m$ potential: $\sim \frac{1}{m_1+m_2} = \frac{m_r}{m_1 m_2}$

- As expected: $\tilde{D}^{(a,b)}_{\text{off}}(k) = 0$

Matching with off-shell Green functions

- Gauge dependent: Coulomb and Feynman gauges
- Freedom treating energy dependence:
  different choices affect the $1/m$ and $1/m^2$ potentials



Our choice: the one that exhibits the divergence structure of the on-shell potential most "naturally"

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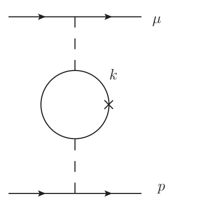

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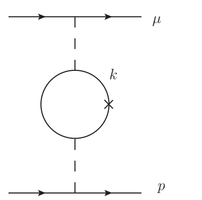

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Matching with Wilson loops

$$W_\Box \equiv \mathrm{P}\exp\left\{-ig\oint_{r\times T_W} dz^\mu A_\mu(z)\right\}$$

- Green functions in **position space** and setting the **time** of the quark and anti-quark **equal**
- **Gauge independent**: comparison with lattice is possible

The "quasi-static" energy E_s

$$\frac{\mathbf{p}^2}{2\,m_\mathrm{r}} + \frac{\mathbf{P}_\mathbf{R}^2}{2\,M} + E^{(0)} + \frac{E^{(1,0)}}{m_1} + \frac{E^{(0,1)}}{m_2} + \frac{E^{(2,0)}}{m_1^2} + \frac{E^{(0,2)}}{m_2^2} + \frac{E^{(1,1)}}{m_1 m_2} + \cdots$$

Example

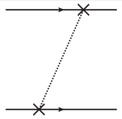
$$E^{(1,1)}_{\mathbf{L}^2}(r) = \frac{i}{d-1}\left(\delta^{ij} - d\,\frac{r^i r^j}{r^2}\right)\lim_{T\to\infty}\int_0^T dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0)\rangle\!\rangle_c$$

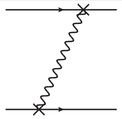
where $\langle\!\langle \ldots \rangle\!\rangle \equiv \langle \ldots W_\Box\rangle / \langle W_\Box\rangle$

The potential with Wilson loops

- We obtain: $V_{s,W}(r) = E_s(r)|_{\mathrm{soft}}$
- Feynman rules for chromoelectric insertions

Example:





$$V^{(1,1)}_{\mathbf{L}^2,W}(r) = \frac{i g_B^2 C_F}{(d-1)} \left(\delta^{ij} - d\,\frac{r^i r^j}{r^2}\right) \lim_{T\to\infty} \int_0^T dt\; t^2 \int \frac{d^D k}{(2\pi)^D}\, e^{i(k_0 t - \mathbf{k}\mathbf{r})}\, \frac{i k_0^2}{k^2 + i0}\, P_{ij}(\mathbf{k})$$

Only need the **off-shell potential** to extract the N^{3}LO potential

The B_c spectrum

The quarkonium spectrum

Pineda, Yndurain (1998)	$\mathcal{O}(m\alpha_s^4)$	$m_1 = m_2$
Brambilla et al. (2000)	$\mathcal{O}(m\alpha_s^4)$	$n = 0, m_1 \neq m_2$
Brambila et al. (1999)	$\mathcal{O}(m\alpha_s^5 \ln(\alpha_s))$	
Pineda (2001)	$\mathcal{O}(m\alpha_s^{4+m} \ln^m(\alpha_s))$	
Penin, Steinhauser (2002)	$\mathcal{O}(m\alpha_s^5)$	$n = 0, m_1 = m_2$
Beneke et al., Penin et al. (2005)	$\mathcal{O}(m\alpha_s^5)$	S-wave, $m_1 = m_2$
Kiyo, Sumino (2014)	$\mathcal{O}(m\alpha_s^5)$	$m_1 = m_2$
CP et al. (2015)	$\mathcal{O}(m\alpha_s^5)$	$m_1 \neq m_2$

The renormalized potential

The (singlet) heavy quarkonium self-energy

$$\Sigma_B(1-\text{loop}) = -g_B^2 C_F V_A^2 (1+\epsilon)\frac{\Gamma(2+\epsilon)\Gamma(-3-2\epsilon)}{\pi^{2+\epsilon}}\mathbf{r}\,(h_s - E + \Delta V)^{3+2\epsilon}\mathbf{r}$$

where $\Delta V \equiv V_o^{(0)} - V^{(0)}$

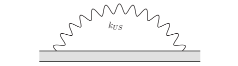
- Its UV divergences (δV_s) **cancel** the US divergences of the soft potential:

$$V_s^{\overline{\text{MS}}} + \delta V_s = V_s$$

- $V_s^{\overline{\text{MS}}}$ produces finite physical results
- δV_s is ambiguous: we choose it so that the 4-dimensional potentials are finite

Computation of the spectrum

US energy correction



Energy correction associated to the static potential

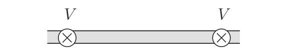
$$\delta E(n,l,s,j)\Big|_{V^{(0)}} = E_n^C \left(1 + \frac{\alpha_s}{\pi} P_1(L_\nu) + \left(\frac{\alpha_s}{\pi}\right)^2 P_2^c(L_\nu) + \left(\frac{\alpha_s}{\pi}\right)^3 P_3^c(L_\nu)\right),$$

(Kiyo, Sumino)

Energy correction associated to the relativistic potentials

$$\delta E(n,l,s,j) = E_n^C \left[\left(\frac{\alpha_s}{\pi}\right)^2 c_2^{\rm nc} + \left(\frac{\alpha_s}{\pi}\right)^3 c_3^{\rm nc}\right]$$

- $c_3^{\rm nc}$ involves the use of quantum-mechanical perturbation theory



$$\delta E_{nlj}^{V\times V} = \langle \psi_{nlj} | V \frac{1}{(E_n^C - h)'} V | \psi_{nlj} \rangle$$

$$= \int {\rm d}\mathbf{r}_2 {\rm d}\mathbf{r}_1 \psi^*_{nlj}(\mathbf{r}_2) V(\mathbf{r}_2) G'_{nl}(\mathbf{r}_1,\mathbf{r}_2) V(\mathbf{r}_1) \psi_{nlj}(\mathbf{r}_1)$$

The renormalon subtraction

- we take the masses to the renormalon subtracted scheme

RS: $m_{\rm RS}(\nu_f) = m_{OS} - \delta m_{\rm RS}$

$$\delta m_{\rm RS} = N_m \pi \nu_f \sum_{N=0}^{\infty} \left(\frac{\beta_0}{2}\right)^N \left(\frac{\alpha(n_l, \nu_f)}{\pi}\right)^{N+1} \sum_{n=0}^{\infty} \tilde{\chi}_n \frac{\Gamma(b+N+1-n)}{\Gamma(b+1-n)}$$

Pineda

RS': $m_{\rm RS'}(\nu_f) = m_{OS} - \delta m_{\rm RS'}$

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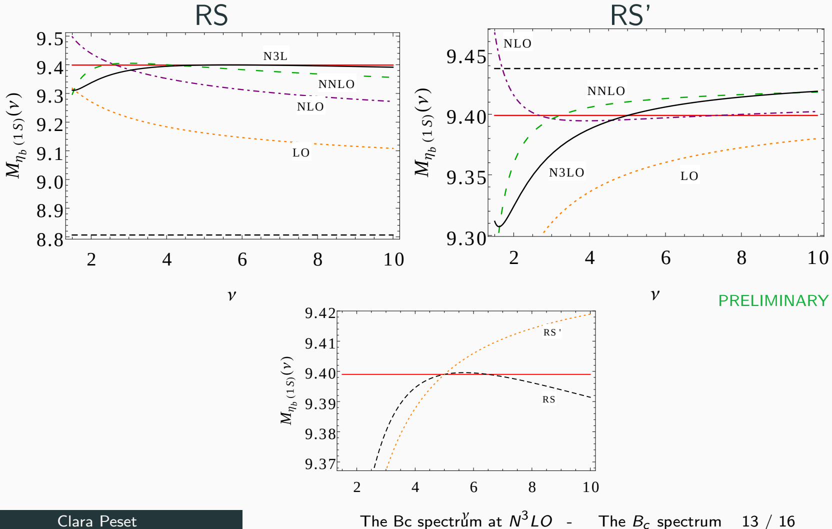
Pineda

- The mass dependence of δE is non-trivial for the B_c

The quarkonium spectrum

The η_b

- $\nu_f = 2$



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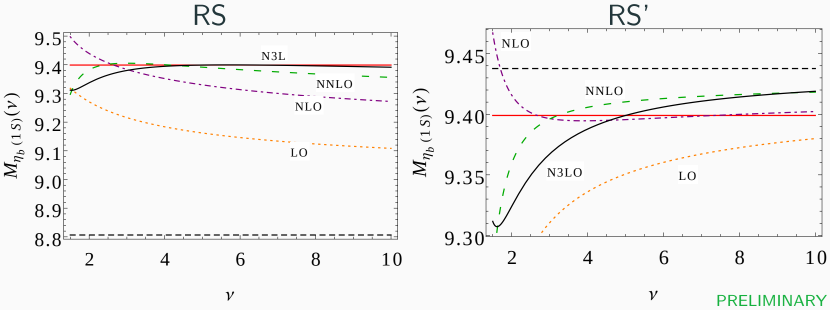


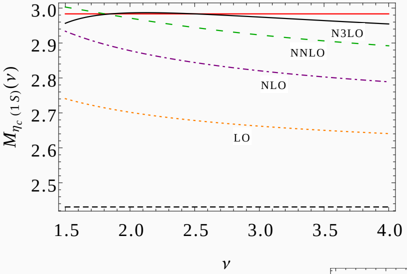
RS& RS':

$$\bar{m}_b(\bar{m}_b) \simeq 4.19(4) \text{ GeV}$$ (PRELIMINARY)

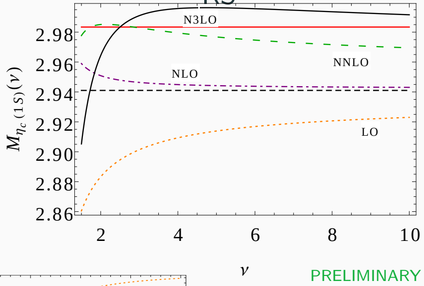
The η_c

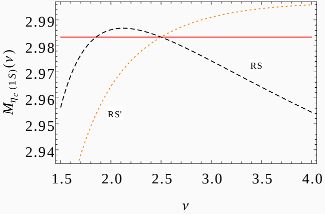
- $\nu_f = 1$

RS



RS'





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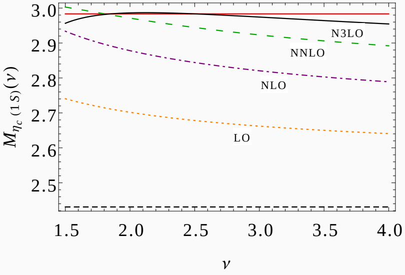

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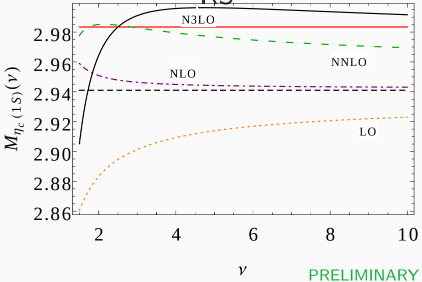


PRELIMINARY

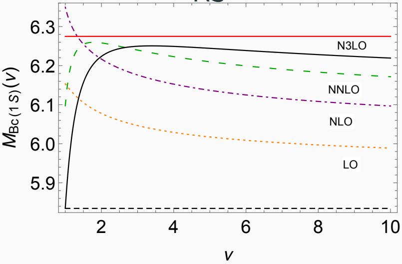
RS& RS':

$$\bar{m}_c(\bar{m}_c) \simeq 1.23(3) \text{ GeV}$$ (PRELIMINARY)

The $B_c(1S)$

- $\nu_f = 1$

RS



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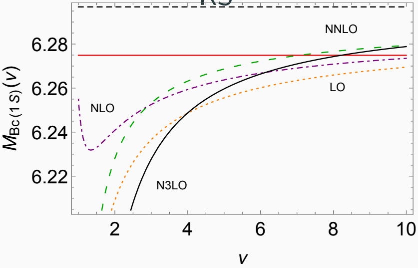
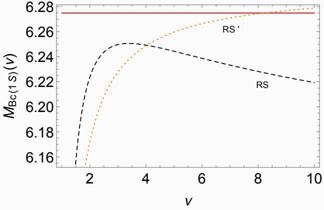


PRELIMINARY



The $B_c(1S)$

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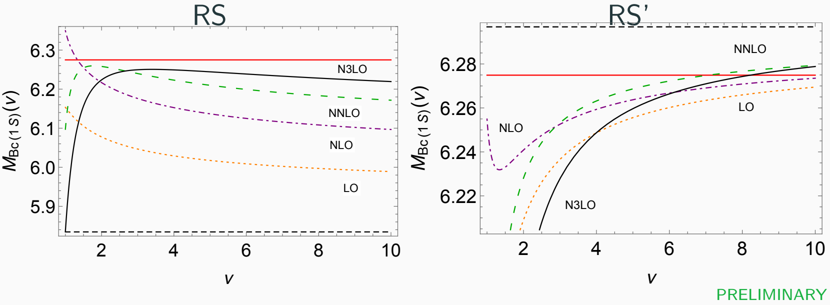


- Fit m_c from $M_{B_c} - M_{\eta_b}/2$ by fixing m_b

RS':

$$\bar{m}_c(\bar{m}_c) \simeq 1.25(X) \text{ GeV}$$ (PRELIMINARY)

Final remarks

Summary of results and final remarks

- We develop the **N^3LO potential** in pNRQCD for **different masses**
- The potentials obtained are valid for $\mathbf{mv} \gg \mathbf{\Lambda}_{\text{QCD}}$
- **The $\mathcal{O}(\alpha_s/m^2, \alpha_s^3/m)$ potential in different matching schemes**
  - all schemes are feasible
  - they are related by a field redefinition
  - $1/m$ potential in Wilson loop scheme
- The US contribution is valid for $\mathbf{mv^2} \gg \mathbf{\Lambda}_{\text{QCD}}$
- We computed the full N^3LO spectrum for different masses
- Study the B_c spectrum and decays
  - obtain the charm mass
  - numerical improvement CP, Pineda, Segovia
- Compute higher order contributions: $\mathcal{O}(m_r \alpha_s^6 \ln(\alpha_s))$

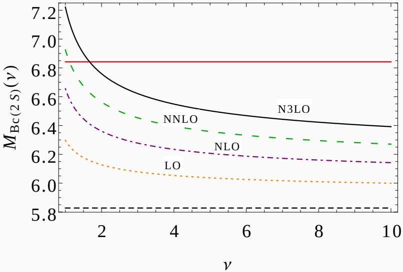
Thank you!

Back-up

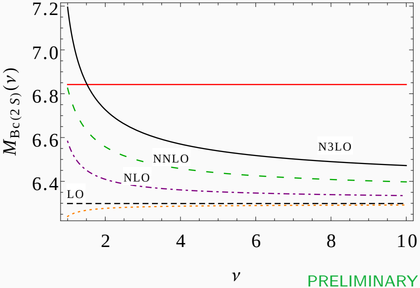
The $B_c(2S)$

- $\nu_f = 1$

RS



RS'



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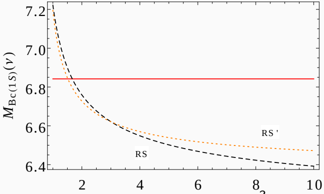
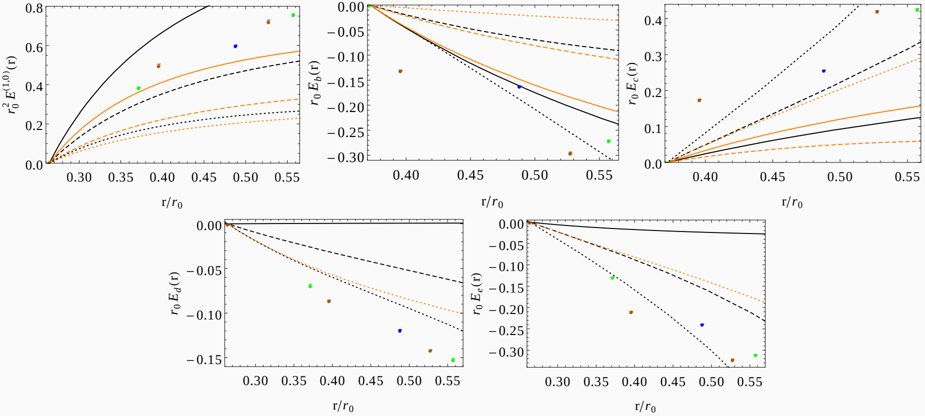


Quasi-static energies: comparison with lattice

- Quasi-static energies $E(\nu_s, \nu_{us}) = V + E_{\mathrm{US}}$ computed up to arbitrary constant in the lattice
- $E_{b,c,d,e}$ are linear combinations of the $1/m^2$ potentials, $r_0 \sim \mathrm{GeV}^{-1}$



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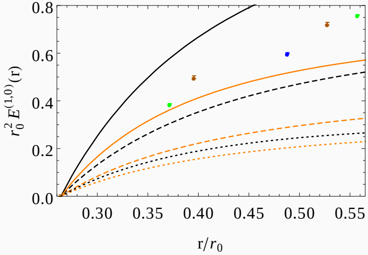

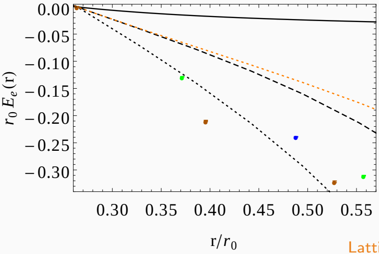

Lattice data

Koma&Koma

solid: $\nu_s = 2r_{min}/r_0^2$

dashed: $\nu_s = 3/r$

dotted: $\nu_s = 3/r, \nu_{us} = 2C_A\alpha_s(\nu_s)/r$

solid: $\nu = 2r_{min}/r_0^2$

dashed: $\nu = 1/r$

dotted: $\nu_s = 1/r, \nu_{us} = C_A\alpha_s(\nu_s)/r$

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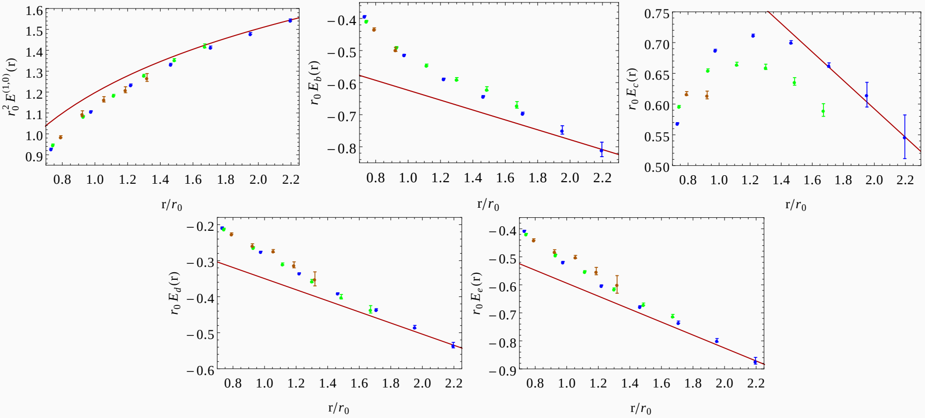

The spin-dependent and the static potentials

The static potential

- gauge independent order-by-order
- already computed to $\mathcal{O}(\alpha_s^4)$

The spin-dependent potential

- renormalized result for different masses computed in 1986 by Pantaleone et al.
- bare potential needs prescription for the D-dimensional Levi-Civita symbol
- the renormalized potential is enough to compute the spectrum

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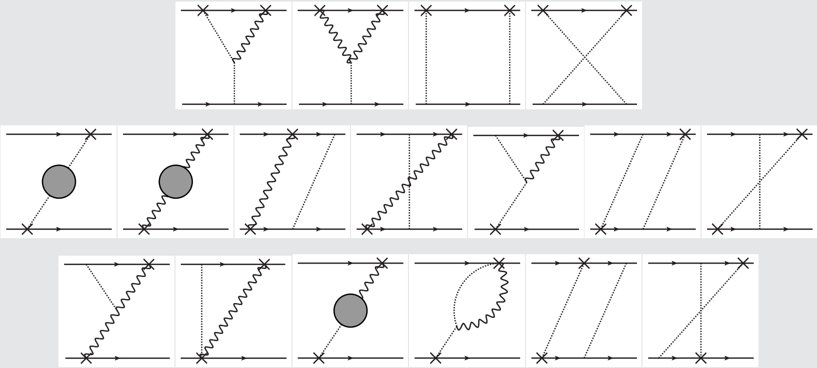
The potential with Wilson loops

- We obtain: $V_{s,W}(r) = E_s(r)|_{\text{soft}}$

Only need compute off-shell potential



Example of potential result

The $1/m_1^2$ renormalized potential with Wilson loops :

$$V_{r,W}^{(2,0),\overline{\rm MS}}(r) = \frac{C_F\alpha_s}{8}\left(c_D^{(1)} + \frac{\alpha_s}{\pi}\left\{-\frac{5}{9}\left(c_D^{(1)} + c_1^{hl\,(1)}\right)T_F n_f + \left(\frac{13}{36}c_F^{(1)\,2} + \frac{8}{3}\right)C_A \right.\right.$$
$$\left.\left. + \left(\left(\frac{4}{3} + \frac{5}{6}c_F^{(1)\,2}\right)C_A - \frac{2}{3}\left(c_D^{(1)} + c_1^{hl\,(1)}\right)T_F n_f\right)\ln(\nu)\right\}\right)4\pi\delta^{(3)}(\mathbf{r})$$
$$+ \frac{C_F\alpha_s^2}{8\pi}\left\{\left(\frac{4}{3} + \frac{5}{6}c_F^{(1)\,2}\right)C_A - \frac{2}{3}\left(c_D^{(1)} + c_1^{hl\,(1)}\right)T_F n_f\right\}\text{reg}\frac{1}{r^3},$$

$$V_{\mathbf{L}^2,W}^{(2,0),\overline{\rm MS}}(r) = \frac{C_A C_F\alpha_s^2}{4\pi r}\left(\frac{11}{3} - \frac{8}{3}\ln\left(r\nu e^{\gamma_E}\right)\right),$$

$$V_{\mathbf{p}^2,W}^{(2,0),\overline{\rm MS}}(r) = -\frac{C_A C_F\alpha_s^2}{\pi r}\left(\frac{2}{3} + \frac{1}{3}\ln\left(r\nu e^{\gamma_E}\right)\right)$$

The spectrum functions

$$\xi^{\rm SD}_{\rm FFF} = \frac{2}{3n}\frac{m_r^2}{m_1 m_2}\left\{\frac{-3(1-\delta_{l0})}{l(l+1)(2l+1)}\left(D_S + X_{\rm LS} + \frac{m_1}{m_2}X_{{\rm LS}_2} + \frac{m_2}{m_1}X_{{\rm LS}_1}\right)\right.$$
$$\left. - 4\mathcal{S}_{12}\delta_{l0}\left[2 + 3\frac{m_1 m_2}{m_2^2 - m_1^2}\ln\left(\frac{m_1^2}{m_2^2}\right)\right]\right\},$$
$$\xi^{\rm SD}_{\rm FFnf} = \frac{2m_r^2}{9n^2 m_1 m_2}\left\{\frac{1-\delta_{l0}}{l(l+1)(2l+1)}\left[2n(4\mathcal{S}_{12} - D_S)\right.\right.$$
$$+ 6\left(D_S + \frac{m_2}{m_1}X_{{\rm LS}_1} + \frac{m_1}{m_2}X_{{\rm LS}_2} + 2X_{\rm LS}\right)\left(\frac{3n}{2l+1} + \frac{n}{2l(l+1)(2l+1)} + l + \frac{1}{2}\right.$$
$$\left.\left. + 2n\left\{S_1(l+n) + S_1(2l-1) - 2S_1(2l+1) - l(\Sigma_1^{(k)} + \Sigma_1^{(m)}) + n\Sigma_b - \Sigma_1^{(m)} + \frac{1}{6}\right\}\right)\right]$$
$$\left. + 8\delta_{l0}\mathcal{S}_{12}\left[1 + 4n\left(\frac{11}{12} - \frac{1}{n} - S_1(n-1) - S_1(n) + nS_2(n)\right)\right]\right\},$$

Logarithmic functions

$E_n^C = -\frac{C_F^2 \alpha_s^2 m_r}{2n^2}$

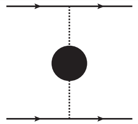
$L_\nu = \ln\left(\frac{n\nu}{2m_r C_F \alpha_s}\right) + S_1(n+l)$ $L_{US} = \ln\left(\frac{C_F \alpha_s n}{2}\right) + S_1(n+l)$

$L_H = \ln\left(\frac{n}{C_F \alpha_s}\right) + S_1(n+l)$

The Bethe logarithm is defined as

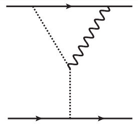
$$L_n^E = \frac{1}{(C_F\alpha_s)^2 E_n^C} \int_0^\infty \frac{d^3k}{(2\pi)^3} |\langle \mathbf{r} \rangle_{\mathbf{k}n}|^2 \left(E_n^C - \frac{k^2}{2m_r}\right)^3 \ln \frac{E_1^C}{E_n^C - \frac{k^2}{2m_r}}$$

Energy dependence in the Coulomb gauge



$$= -\frac{i g_B^4}{3} \frac{k_0^2 \mathbf{k}^{2\epsilon-4} \csc(\pi \epsilon)}{2^{4\epsilon+4} \pi^{\epsilon+\frac{1}{2}} \Gamma\left(\epsilon+\frac{5}{2}\right)} \Big[3 C_F T_F n_f \epsilon(1+\epsilon)$$

$$- \frac{C_A C_F}{4} \left((\epsilon+1)(\epsilon(56\epsilon+121)+60)\right) - \frac{5\Gamma\left(\epsilon+\frac{3}{2}\right)^2}{\sqrt{\pi}\Gamma\left(2\epsilon+\frac{5}{2}\right)} 4^{\epsilon+1}(\epsilon+1)(2\epsilon+3)(4\epsilon+3) \Big]$$



$$= -\frac{i g_B^4}{3 m_i} C_A C_F \frac{\mathbf{k}^{2\epsilon-4}(\epsilon+1)(2\epsilon+1)\Gamma(1-\epsilon)\Gamma(2\epsilon)}{4^{2\epsilon+1} \pi^{\epsilon+\frac{3}{2}} \Gamma\left(2\epsilon+\frac{3}{2}\right)}$$

$$\left[E_i \left(\mathbf{k}^2 - (\mathbf{p}'^2 - \mathbf{p}^2)^2 \right) + E_i' \left(\mathbf{k}^2 + (\mathbf{p}'^2 - \mathbf{p}^2)^2 \right) \right]$$