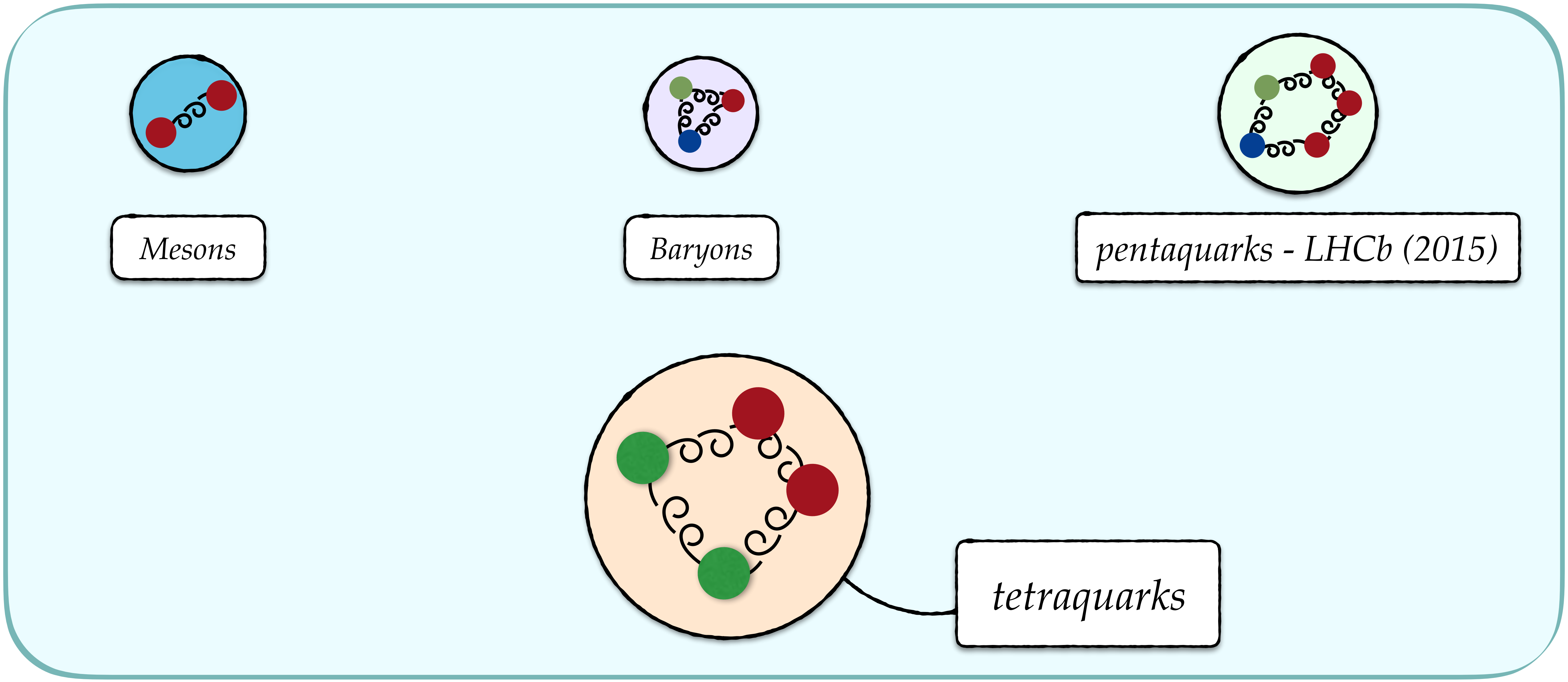
Searching for Beauty-fully Bound Tetraquarks Using Lattice NRQCD

Ciaran Hughes, Estia Eichten, Christine Davies



QWG, 2017

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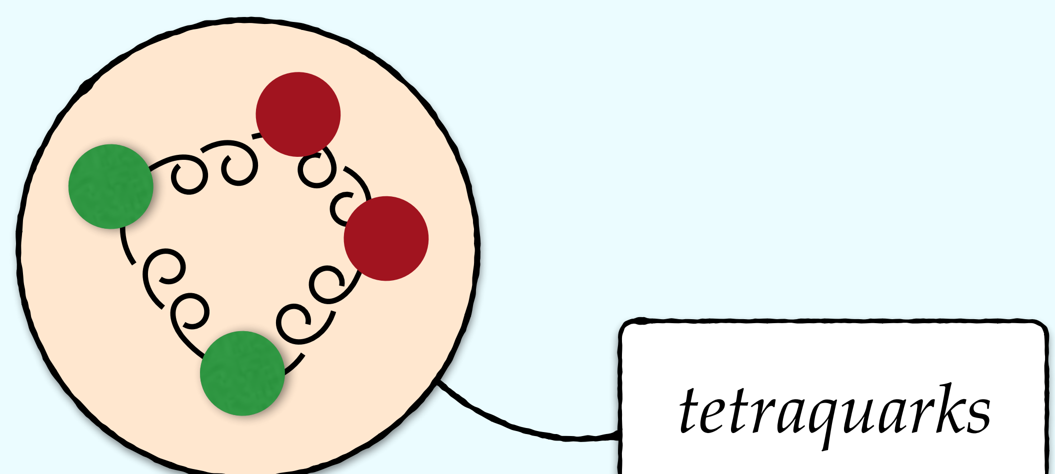
- This talk will be a bigger picture sketch of results from arxiv: 1710.03236
- For more details, please contact me (hyperlink)!



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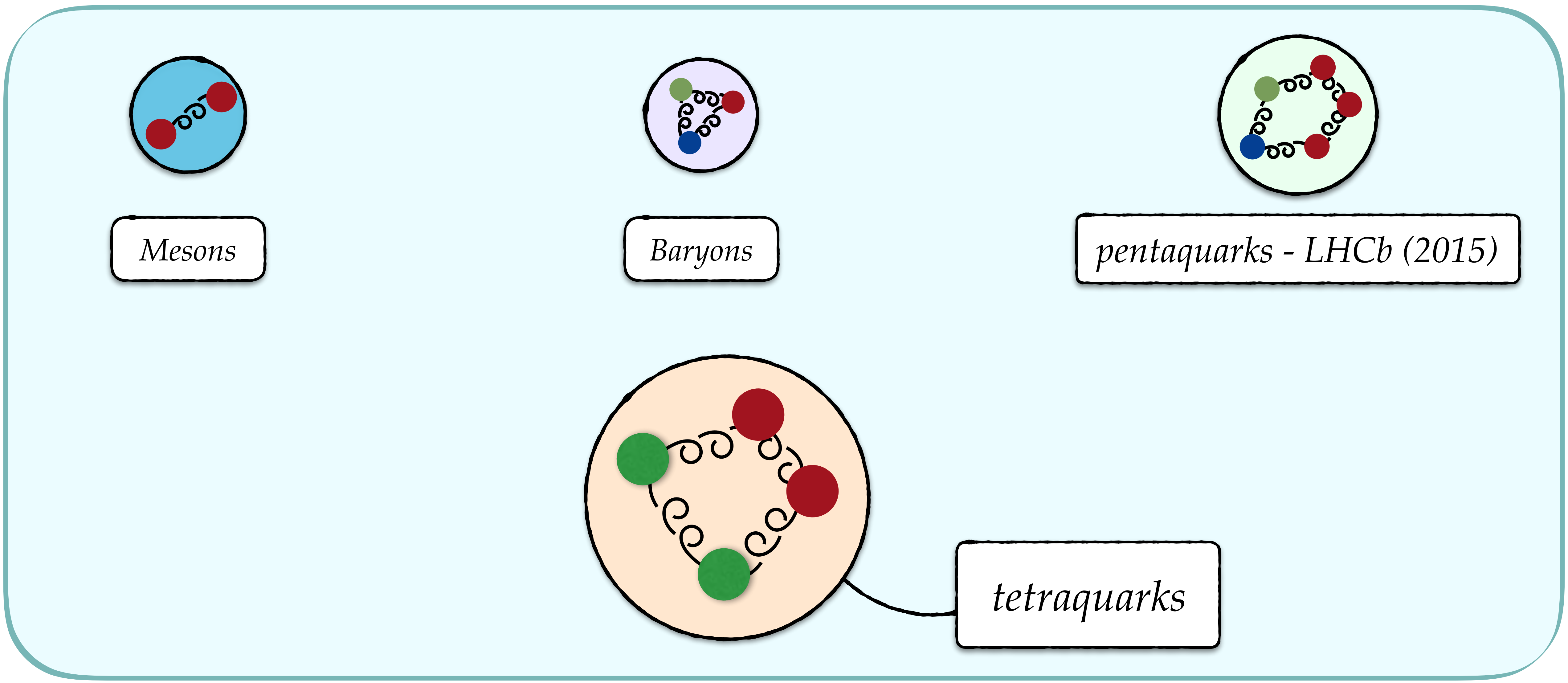

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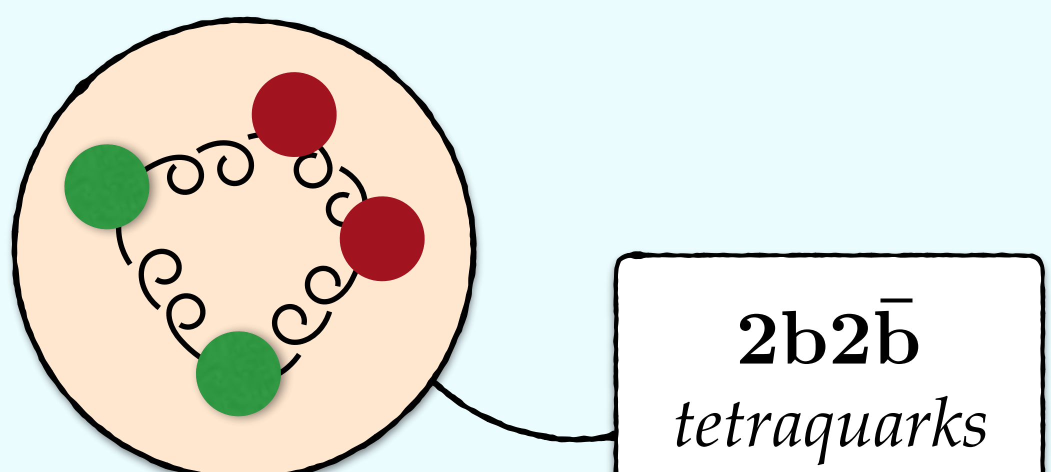

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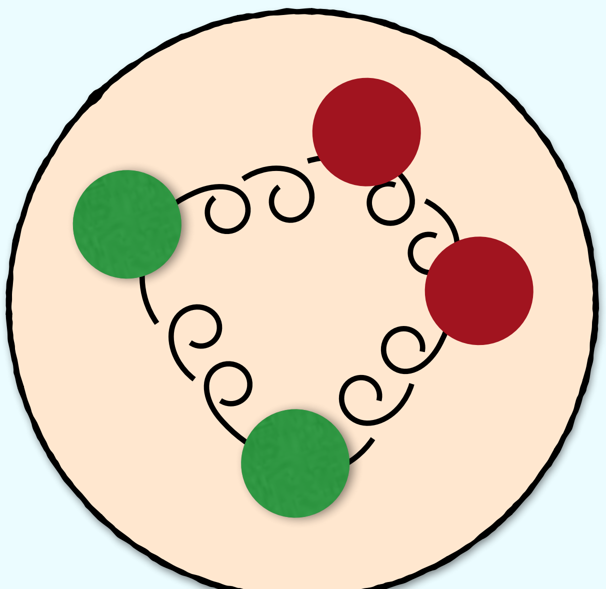
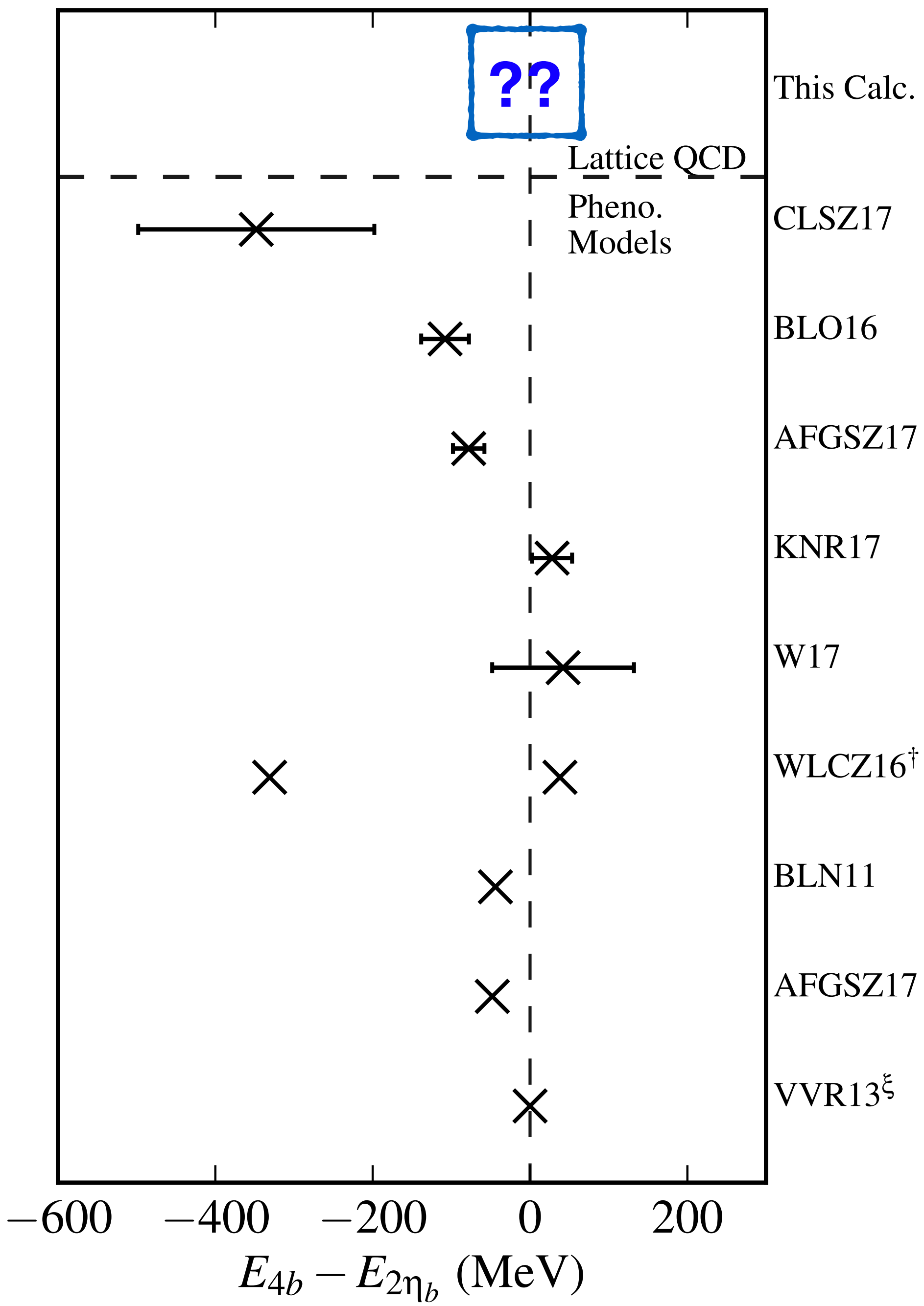


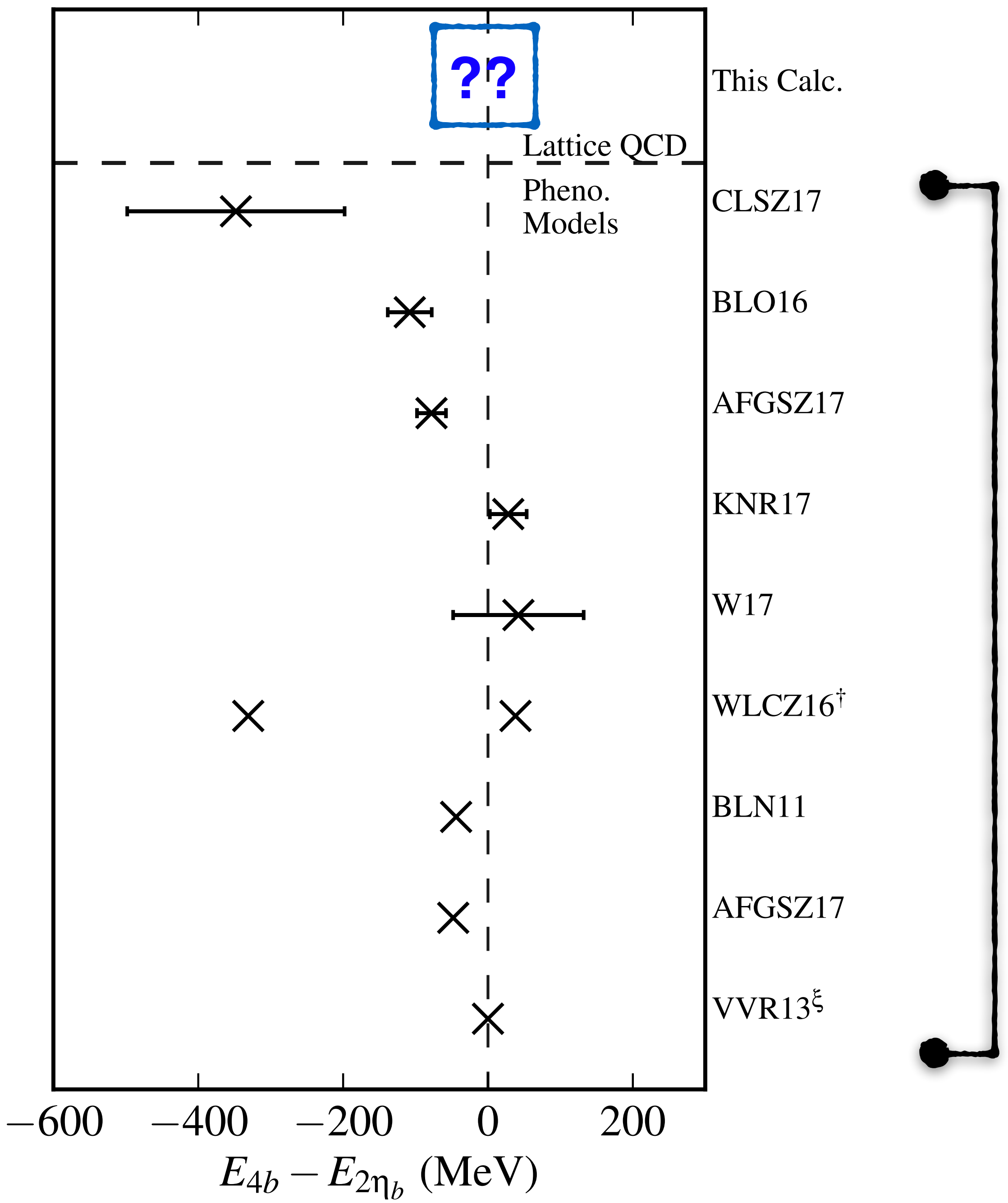
tetraquarks

But is it strong enough to bind $2\mathrm{b}2\bar{\mathrm{b}}$ into a stable state below the $2\eta_{\mathrm{b}}$ threshold.

Model Predictions for 0^{++} $\mathbf{2b2\bar{b}}$ tetraquark



Model Predictions for 0^{++} $\mathbf{2b2\bar{b}}$ tetraquark



- Results Very Model Dependent!!
  - Not from first-principles
  - Inconclusive whether tetraquark bound or not?

Quantum Chromodynamics

"The fundamental theory of the strong nuclear force"

- QCD is non-perturbative
  - Solution: 'solve' numerically via LQCD

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- Lattice QCD results are from first-principles

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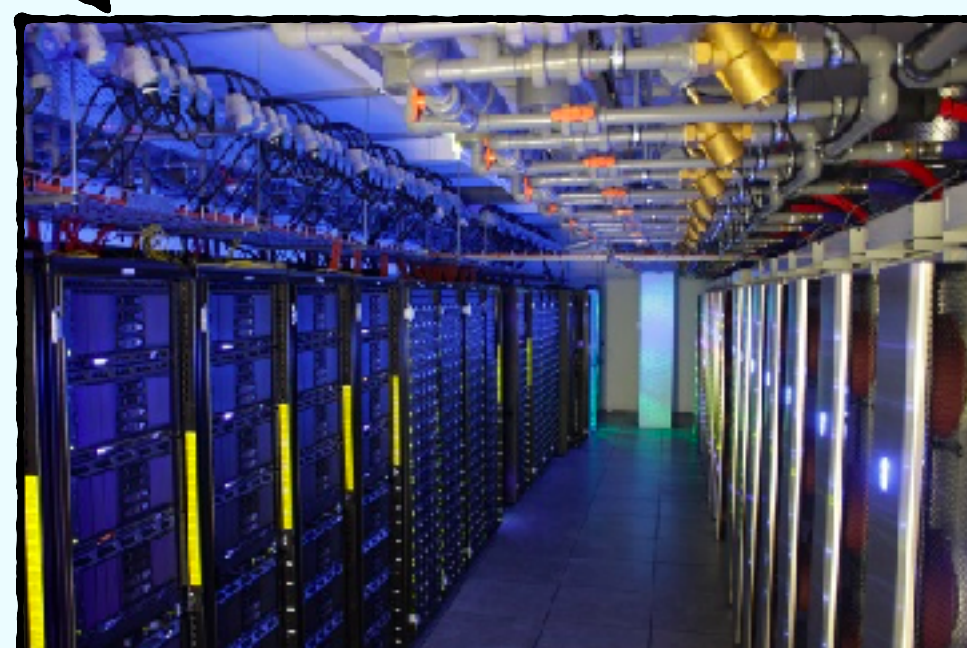
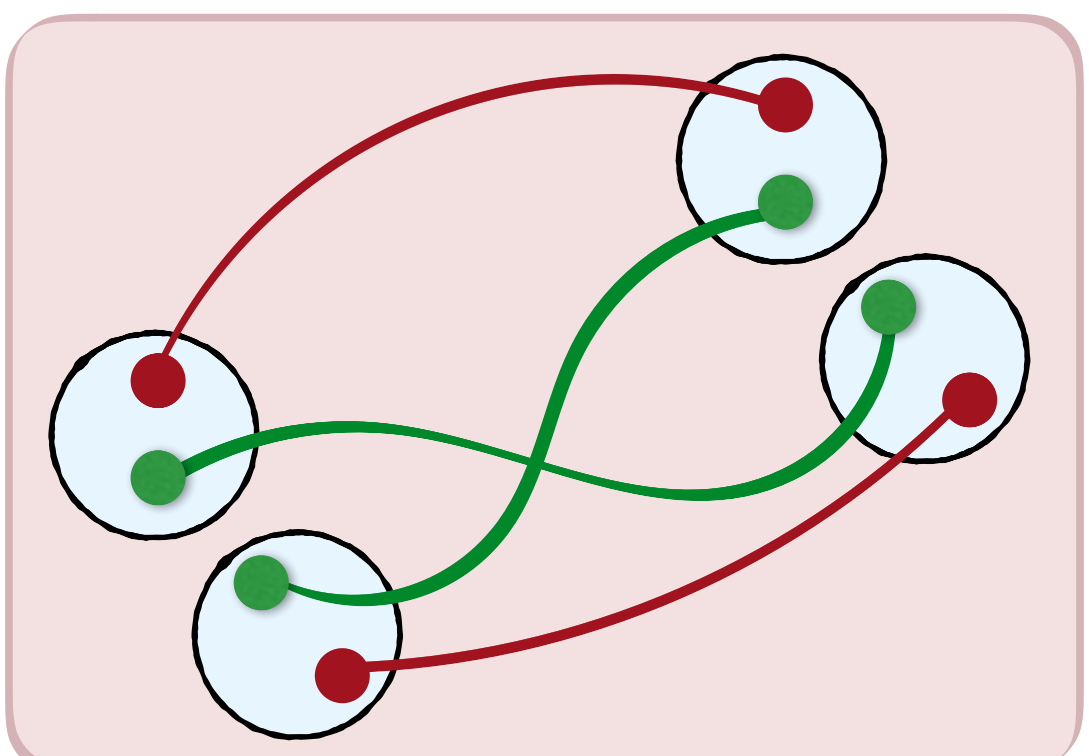
- Lattice QCD results are from first-principles

$$\mathcal{L}_{\mathbf{QCD}} = \bar{\psi}_{\mathbf{f}} \left(\mathbf{i} \not{D} - \mathrm{m_f}\right) \psi_{\mathbf{f}} - \frac{1}{4}\mathrm{tr}\left(\mathrm{GG}\right)$$





A (Bigger Picture) Lattice Spectrum Calculation: The Two-Point Correlator $C_{ab}^{2\mathrm{pt.}}(t) = \langle 0|\mathcal{O}_b(y_4)\mathcal{O}_a^\dagger(x_4)|0\rangle$

$$C_{ab}^{2\mathrm{pt.}}(x_4 - y_4) = \Big\langle \qquad \Big\rangle$$

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- **Feynman Path Integral** Approach to QFT: (Numerically) Integrate over all field configurations

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  - Create some (superposition of) state with $\mathcal{O}_a^\dagger(x_4)$

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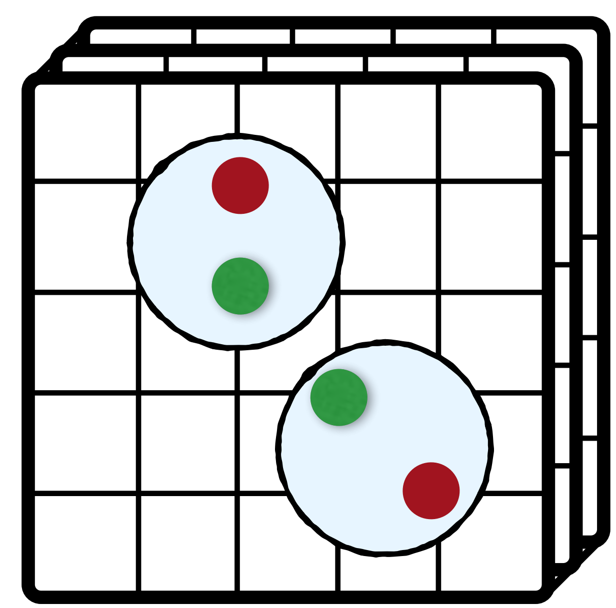

$t = x_4$

A (Bigger Picture) Lattice Spectrum Calculation: The Two-Point Correlator $C_{ab}^{2\text{pt.}}(t) = \langle 0|\mathcal{O}_b(y_4)\mathcal{O}_a^\dagger(x_4)|0\rangle$

- **Feynman Path Integral** Approach to QFT: (Numerically) Integrate over all field configurations
  - Create some (superposition of) state with $\mathcal{O}_a^\dagger(x_4)$
  - Propagate elementary fields for time t

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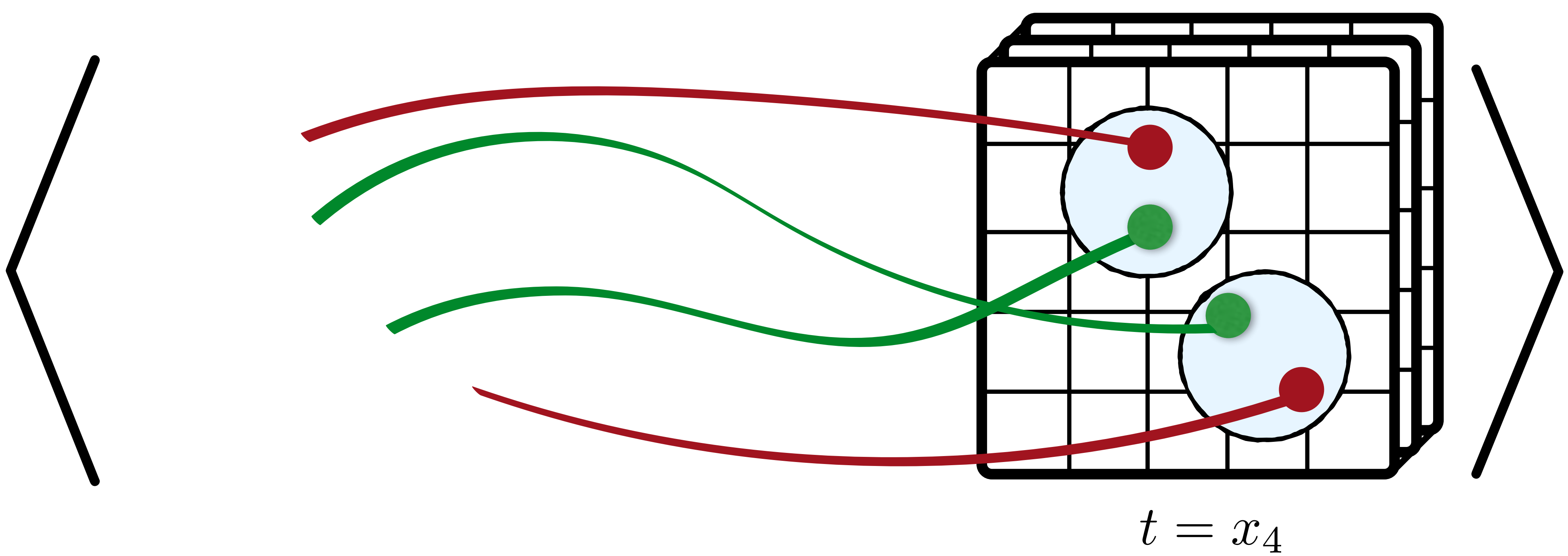

A (Bigger Picture) Lattice Spectrum Calculation: The Two-Point Correlator $C_{ab}^{2\text{pt.}}(t) = \langle 0|\mathcal{O}_b(y_4)\mathcal{O}_a^\dagger(x_4)|0\rangle$

- **Feynman Path Integral** Approach to QFT: (Numerically) Integrate over all field configurations
  - Create some (superposition of) state with $\mathcal{O}_a^\dagger(x_4)$
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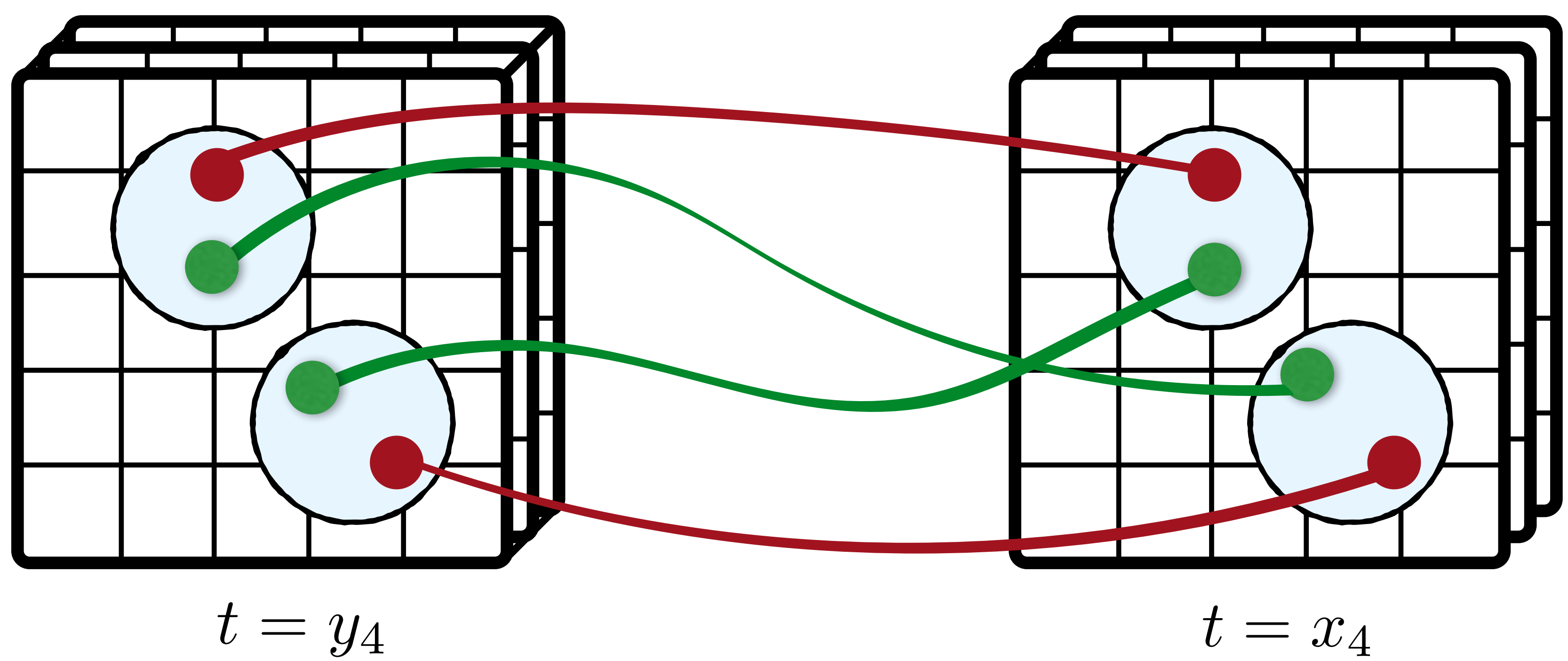
  - Destroy some (superposition of) state with $\mathcal{O}_b(y_4)$

$$C_{ab}^{2\text{pt.}}(x_4 - y_4) = \left\langle \vphantom{\int} \right.$$  $$\left. \vphantom{\int} \right\rangle$$

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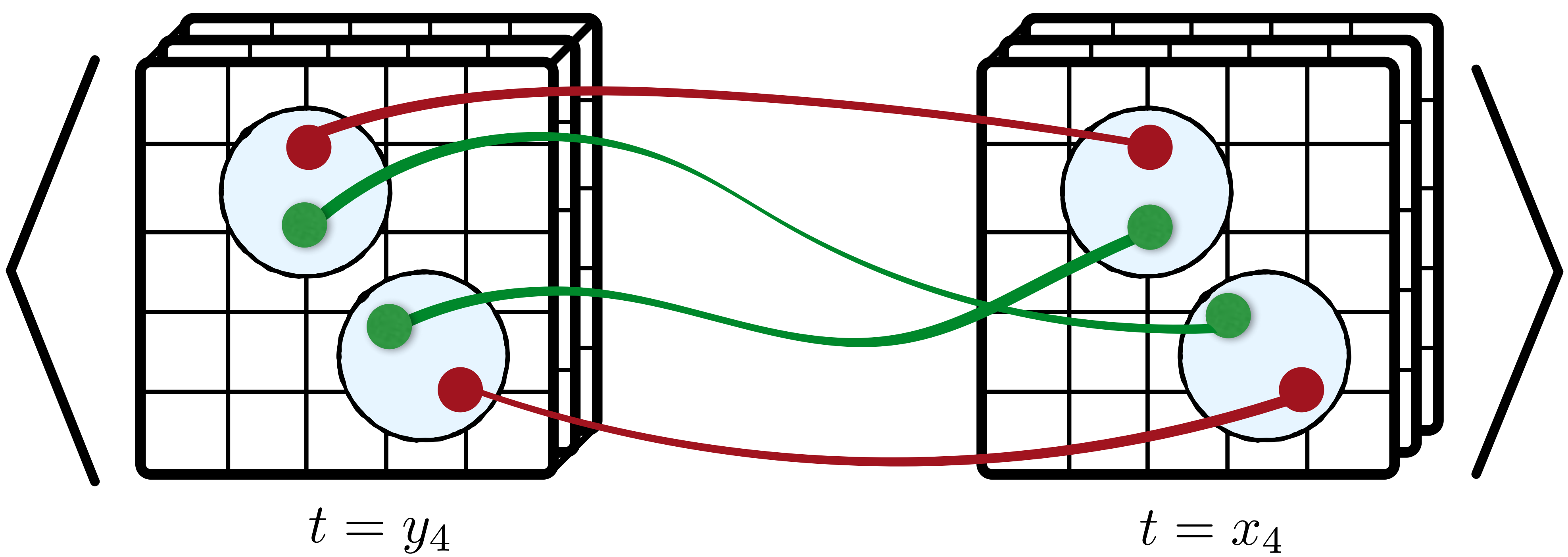
- **Hilbert Space Formalism**:
  - Insert a complete set of QCD eigenstates

$$C_{ab}^{\text{2pt.}}(x_4 - y_4) =$$



A (Bigger Picture) Lattice Spectrum Calculation: The Two-Point Correlator $C_{ab}^{\text{2pt.}}(t) = \langle 0|\mathcal{O}_b(y_4)\mathcal{O}_a^\dagger(x_4)|0\rangle$

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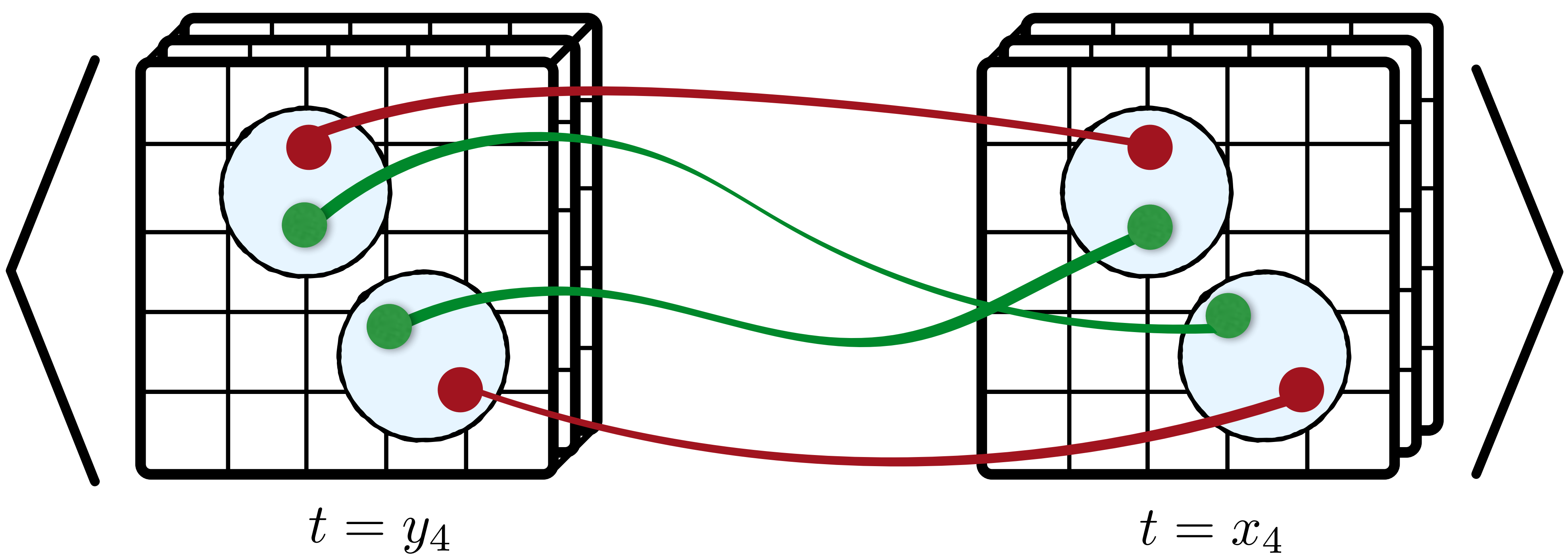
$$C_{ab}^{2pt.}(t,\mathbf{P}) \equiv \langle 0|\mathcal{O}_b(t,\mathbf{P})\mathcal{O}_a^\dagger(0,\mathbf{P})|0\rangle = \sum_n Z_{b,n} Z_{a,n}^\dagger e^{-E_n t}$$

$$C_{ab}^{\text{2pt.}}(x_4 - y_4) =$$



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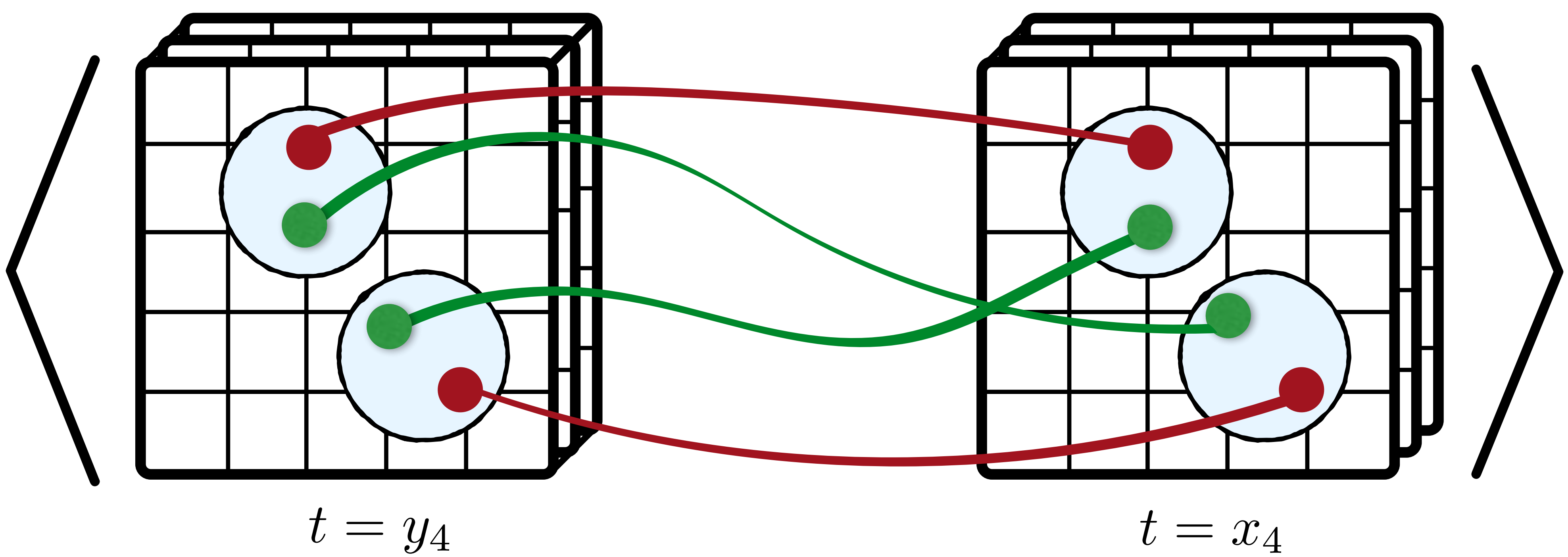
Extract QCD Energy Eigenstates

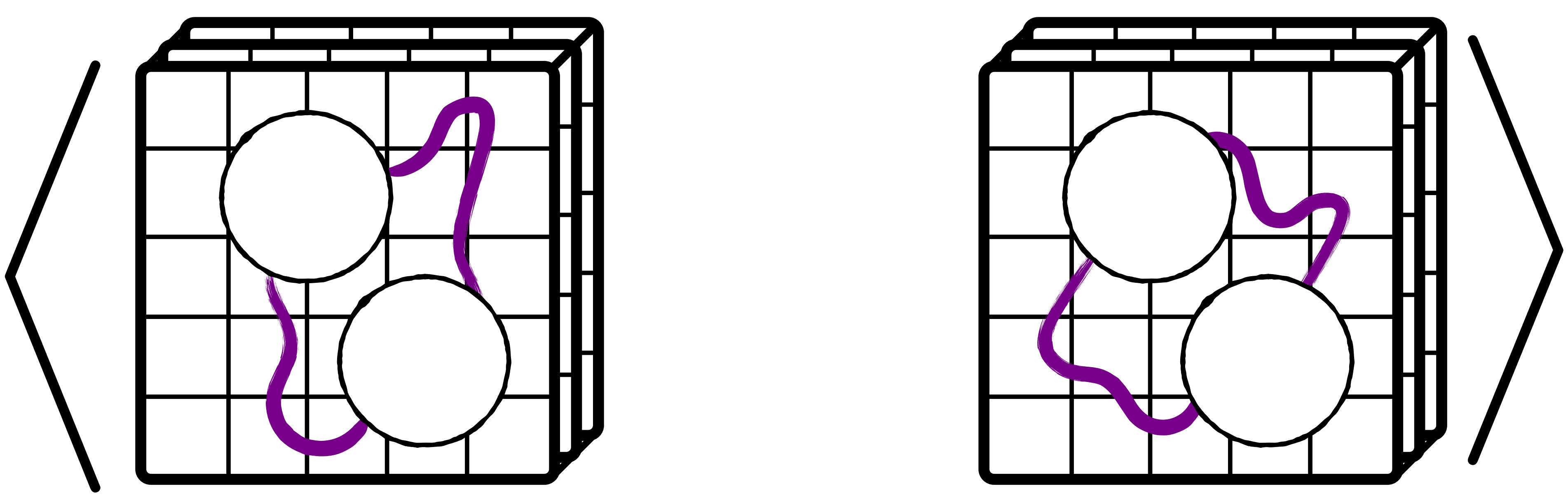
$$C_{ab}^{\mathrm{2pt.}}(x_4 - y_4) =$$



Operators Used for 0^{++} $\mathbf{2b2\bar{b}}$ State

0^{++}	
source	sink
$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$	

$$C^{\text{2pt.}}_{ab}(x_4 - y_4) =$$



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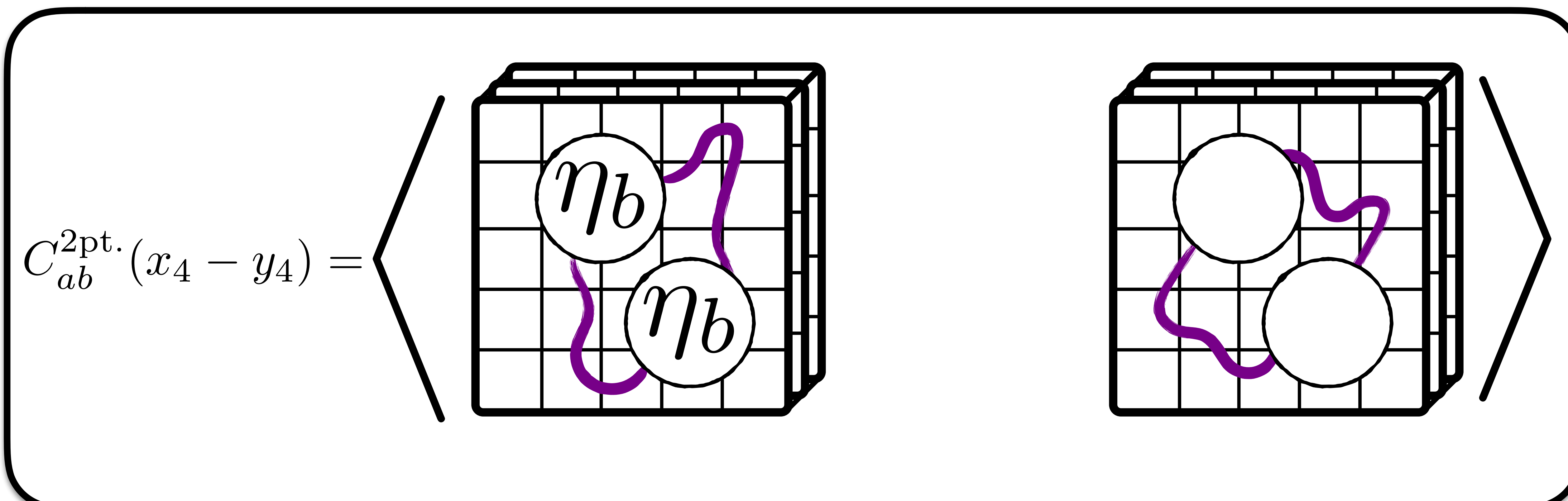

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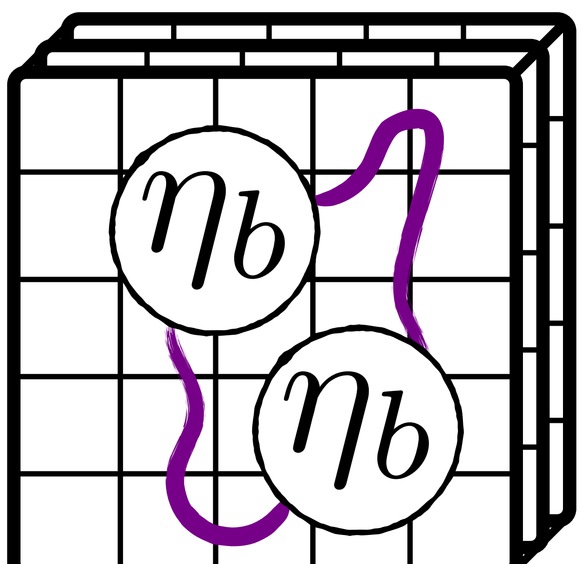
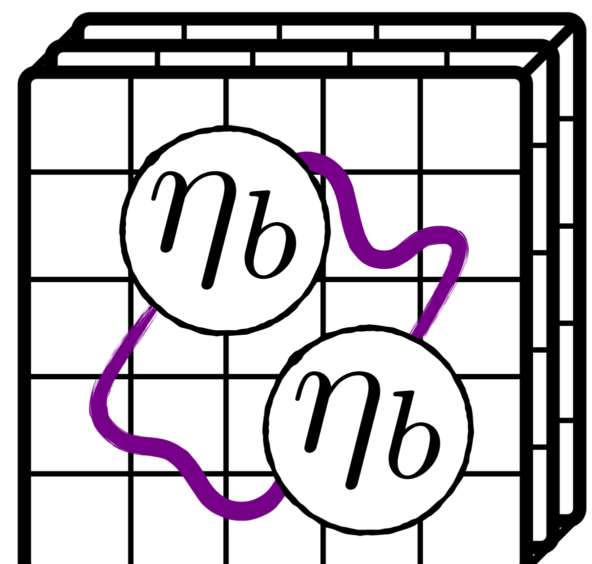
$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$	$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$

$$C^{2\text{pt.}}_{ab}(x_4 - y_4) = \Big\langle$$





$$\Big\rangle$$

Operators Used for 0^{++} $\mathbf{2b2\bar{b}}$ State

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source	sink
$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$	$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$
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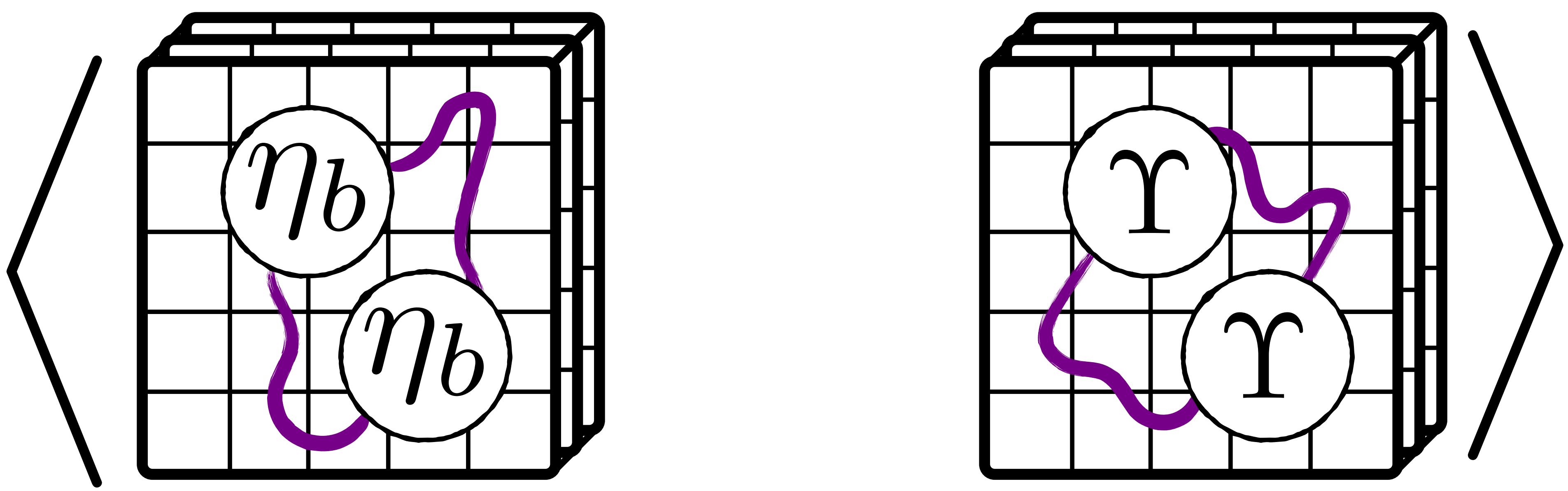
$$C_{ab}^{2\text{pt.}}(x_4 - y_4) =$$



Operators Used for 0^{++} $\mathbf{2b2\bar{b}}$ State

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source	sink
$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$	$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$
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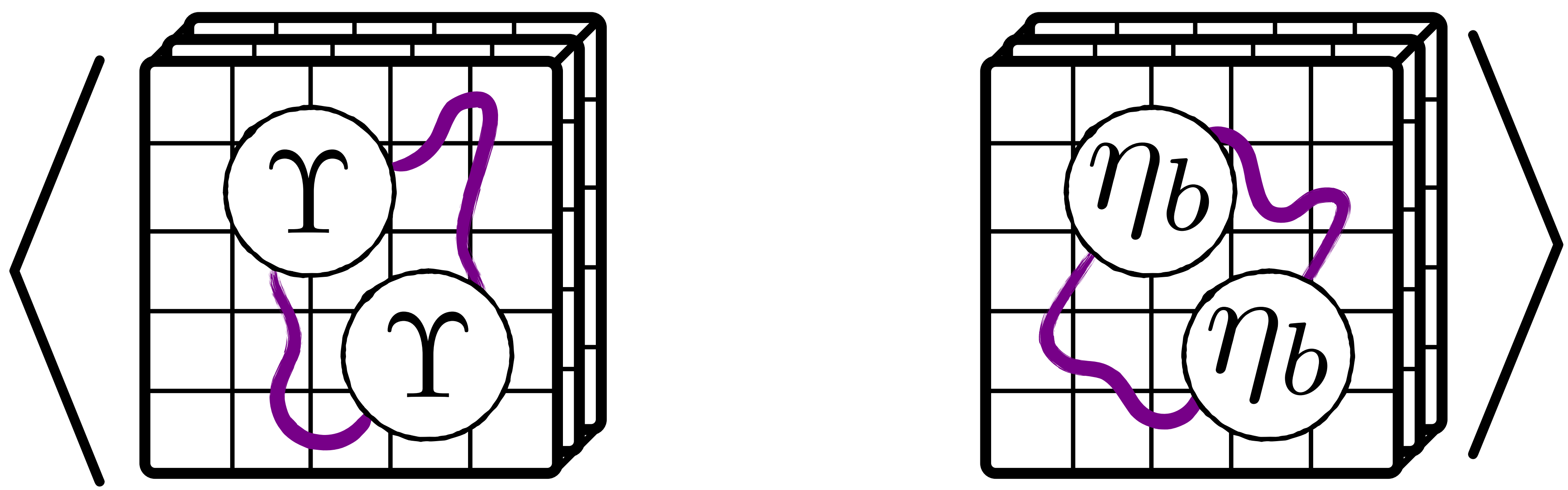

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$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$	$\mathcal{O}^{A_1}_{(\eta_b,\eta_b)}$
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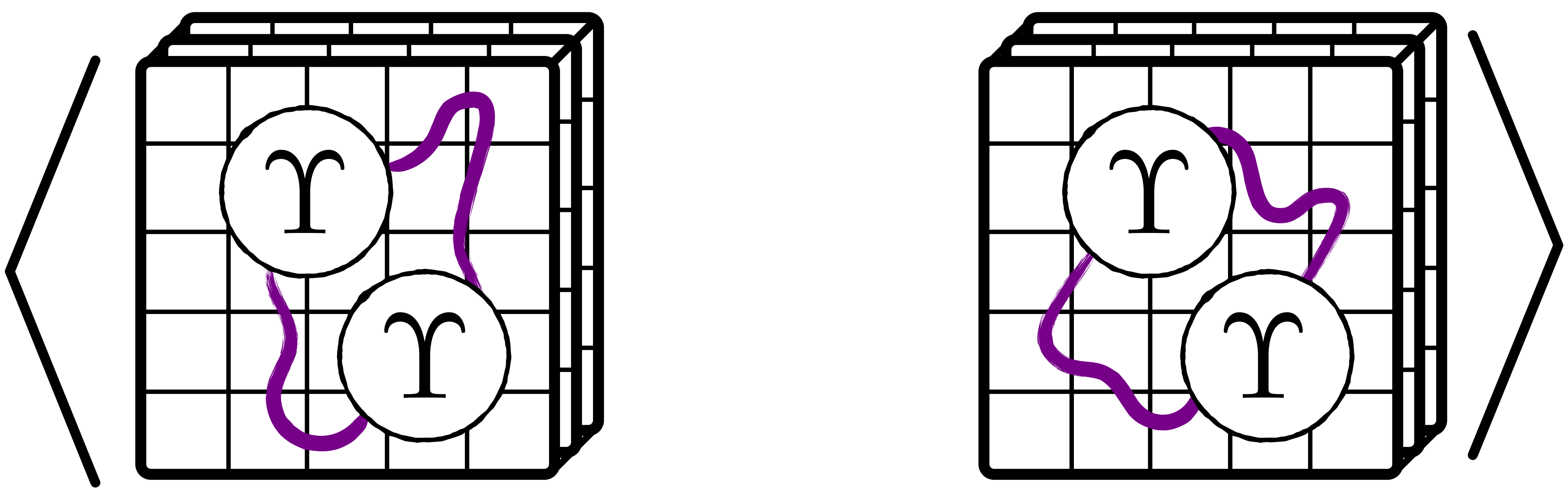

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$\mathcal{O}^{A_1}_{(\Upsilon,\Upsilon)}$	$\mathcal{O}^{A_1}_{(\Upsilon,\Upsilon)}$
$\mathcal{O}^{A_1}_{(D_{3_c},A_{3_c})}$	$\mathcal{O}^{A_1}_{(D_{3_c},A_{3_c})}$

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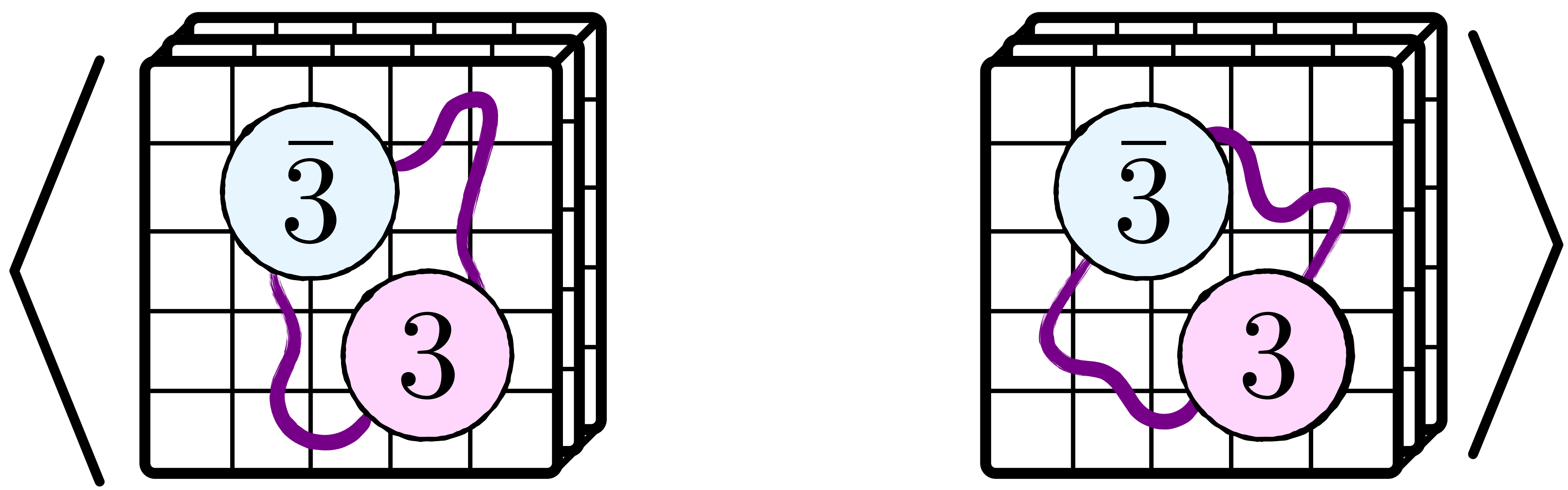

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$\mathcal{O}^{A_1}_{(D_{\bar{3}_c},A_{3_c})}$	$\mathcal{O}^{A_1}_{(D_{\bar{3}_c},A_{3_c})}$
$\mathcal{O}^{A_1}_{(D_{6_c},A_{\bar{6}_c})}$	$\mathcal{O}^{A_1}_{(D_{6_c},A_{\bar{6}_c})}$

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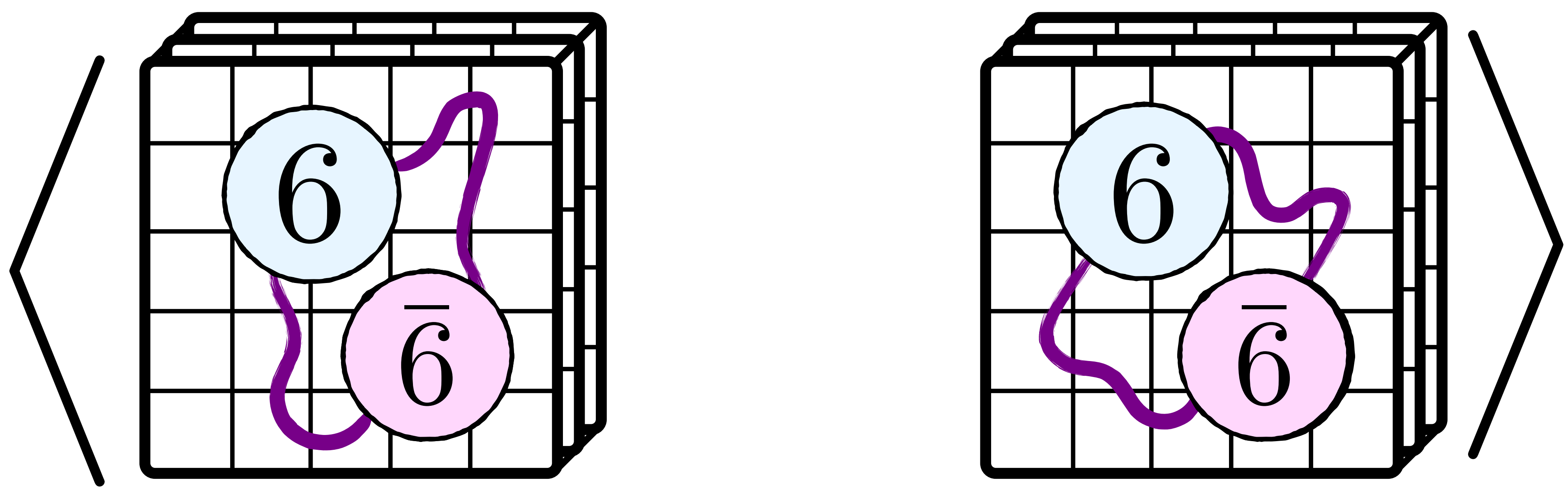

We perform a Bayesian fit to all the data within a certain channel

- We perform a Bayesian fit to all the data within a certain channel
- But you want to see the actual data! What can we easily show?

The Lattice Effective Mass

$$aE^{\text{eff}} = \log\left(\frac{C(t)}{C(t+1)}\right)$$

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$$= aE_0 + \frac{Z_1^2}{Z_0^2} e^{-(E_1-E_0)t}(1 - e^{-(E_1-E_0)}) + \ldots$$

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Excited State Contributions Exponentially decay away with time

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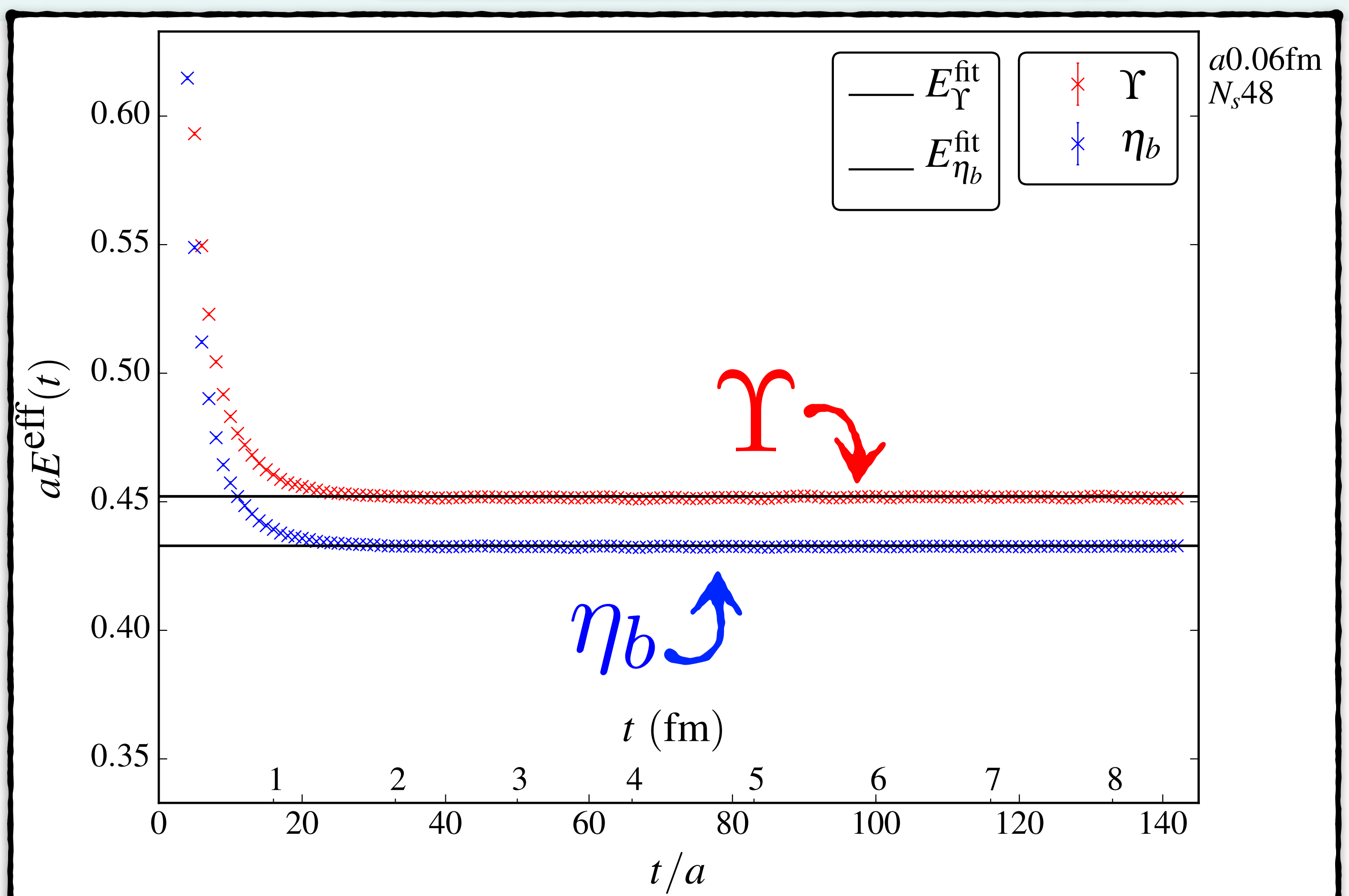
Excited State Contributions Exponentially decay away with time



Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

"What might you expect to see?"

Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

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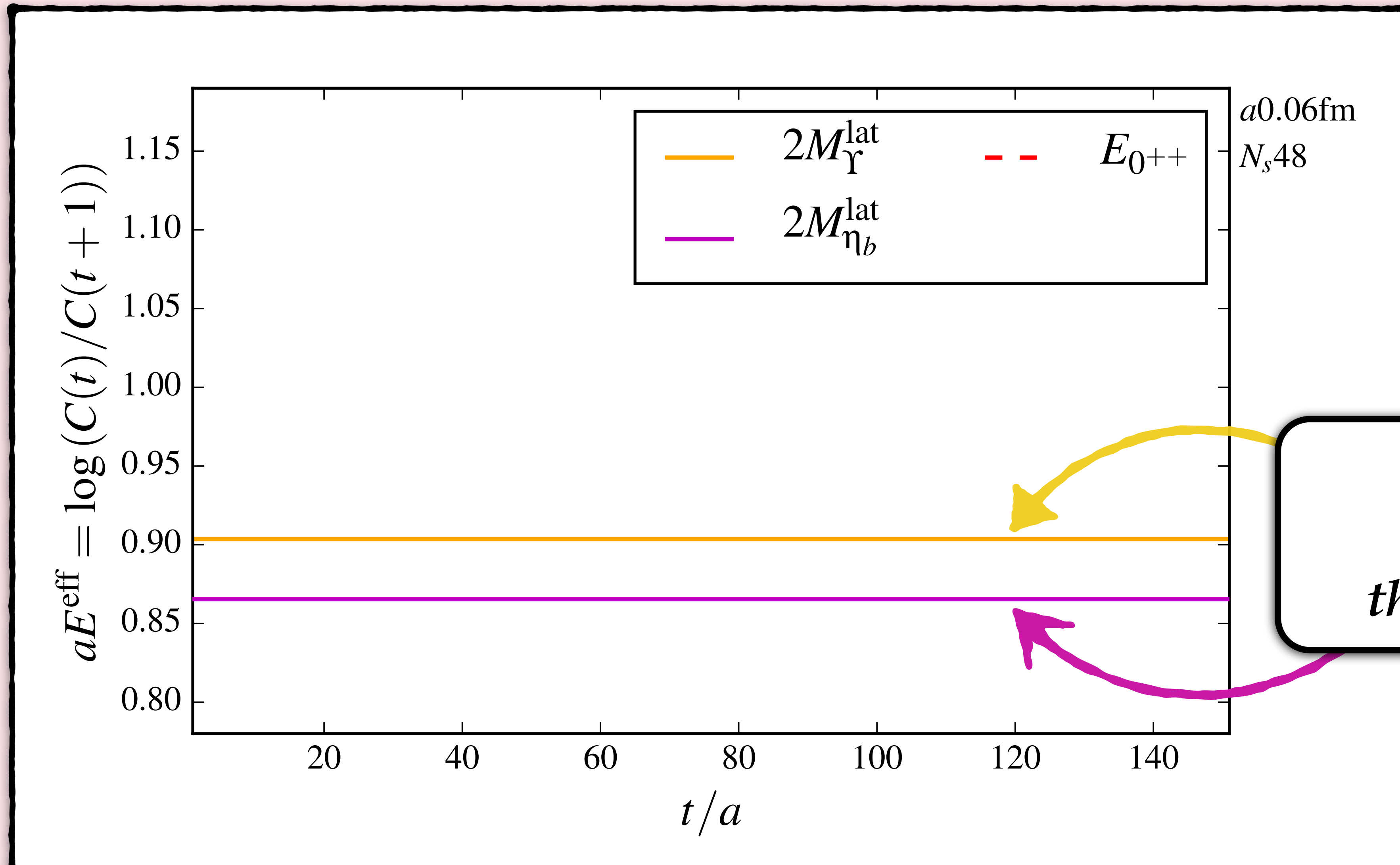
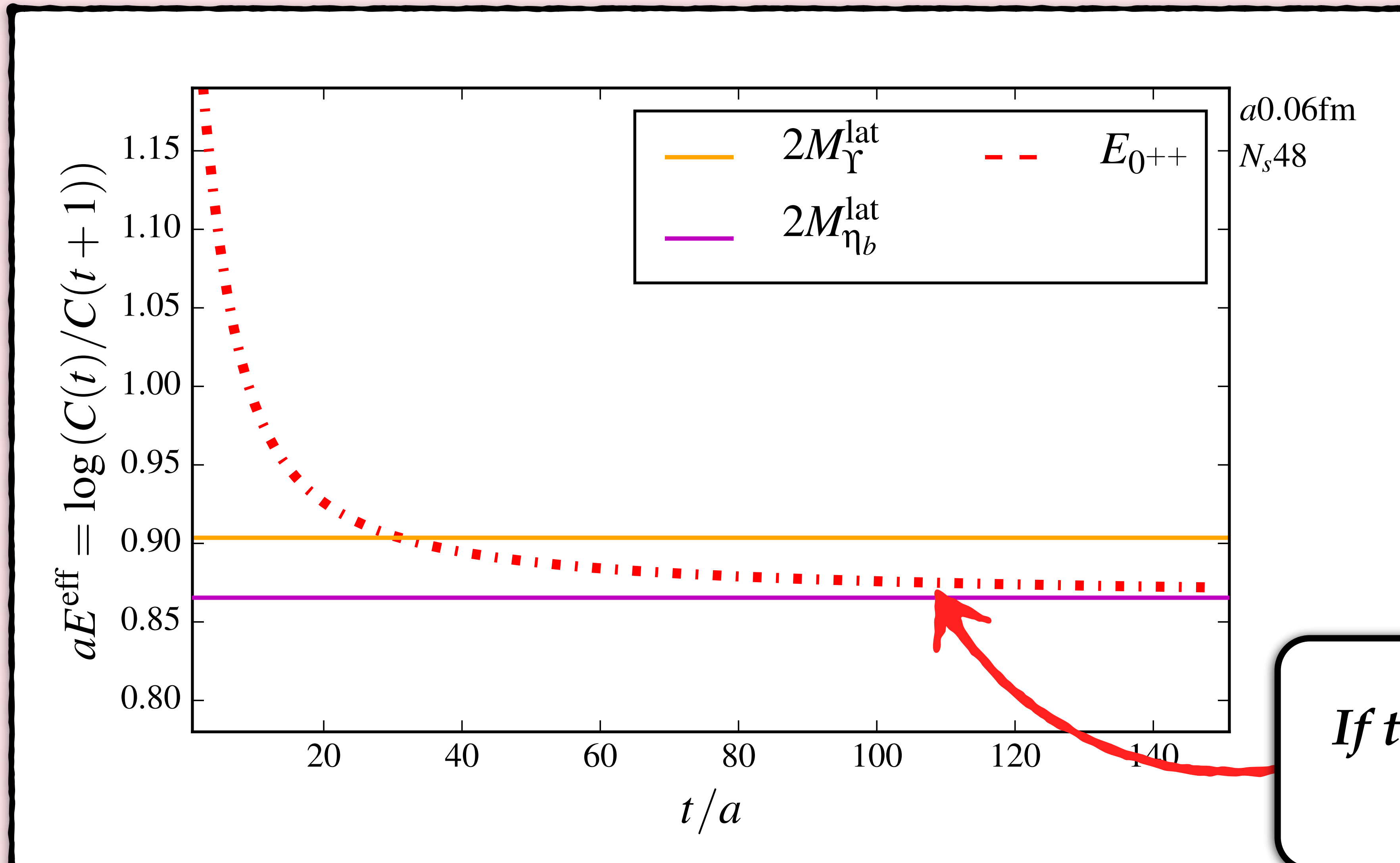


The non-interacting $2\eta_b$ and 2Υ thresholds for reference

Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

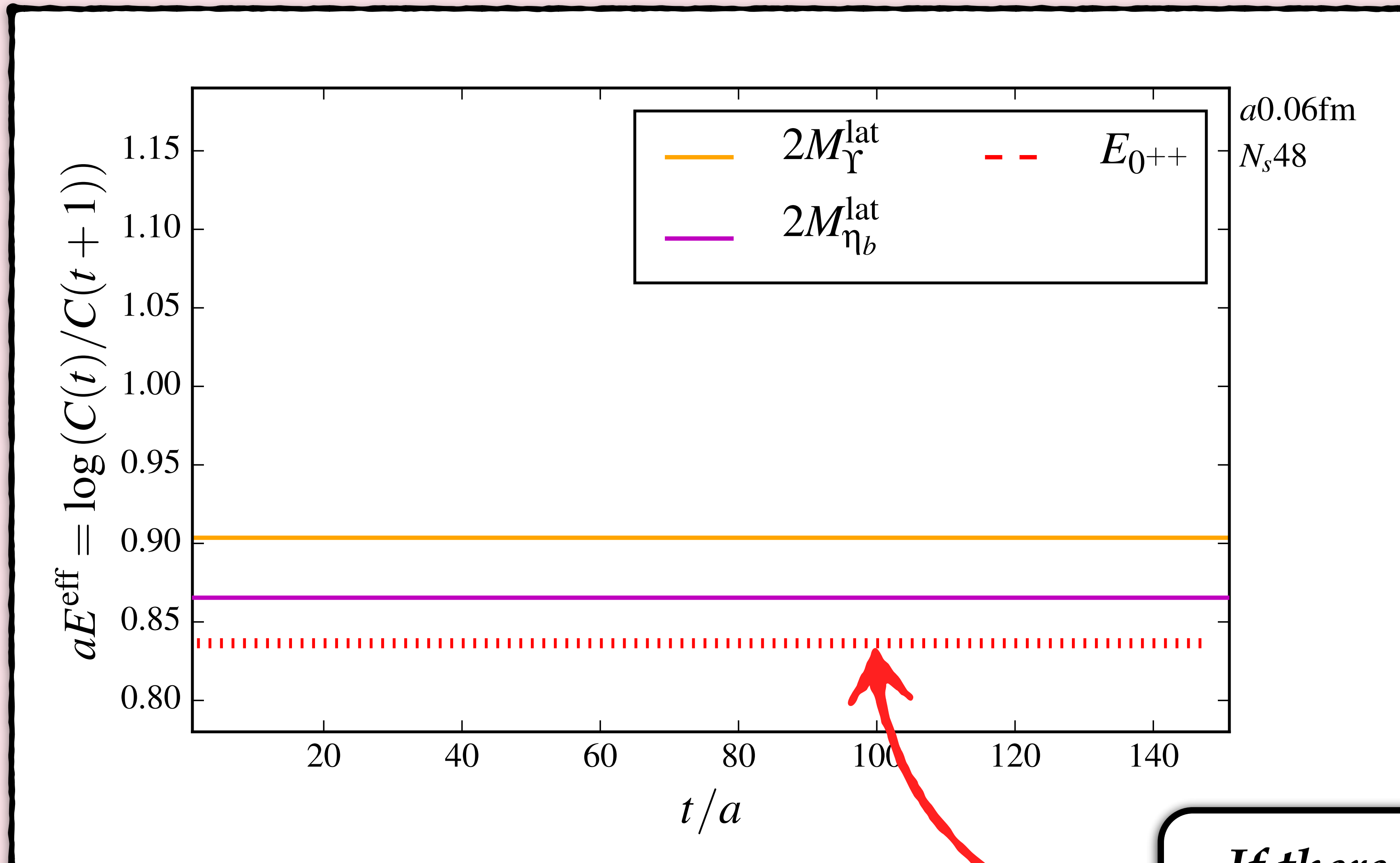
"What might you expect to see?"



If there was no new stable tetraquark

Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

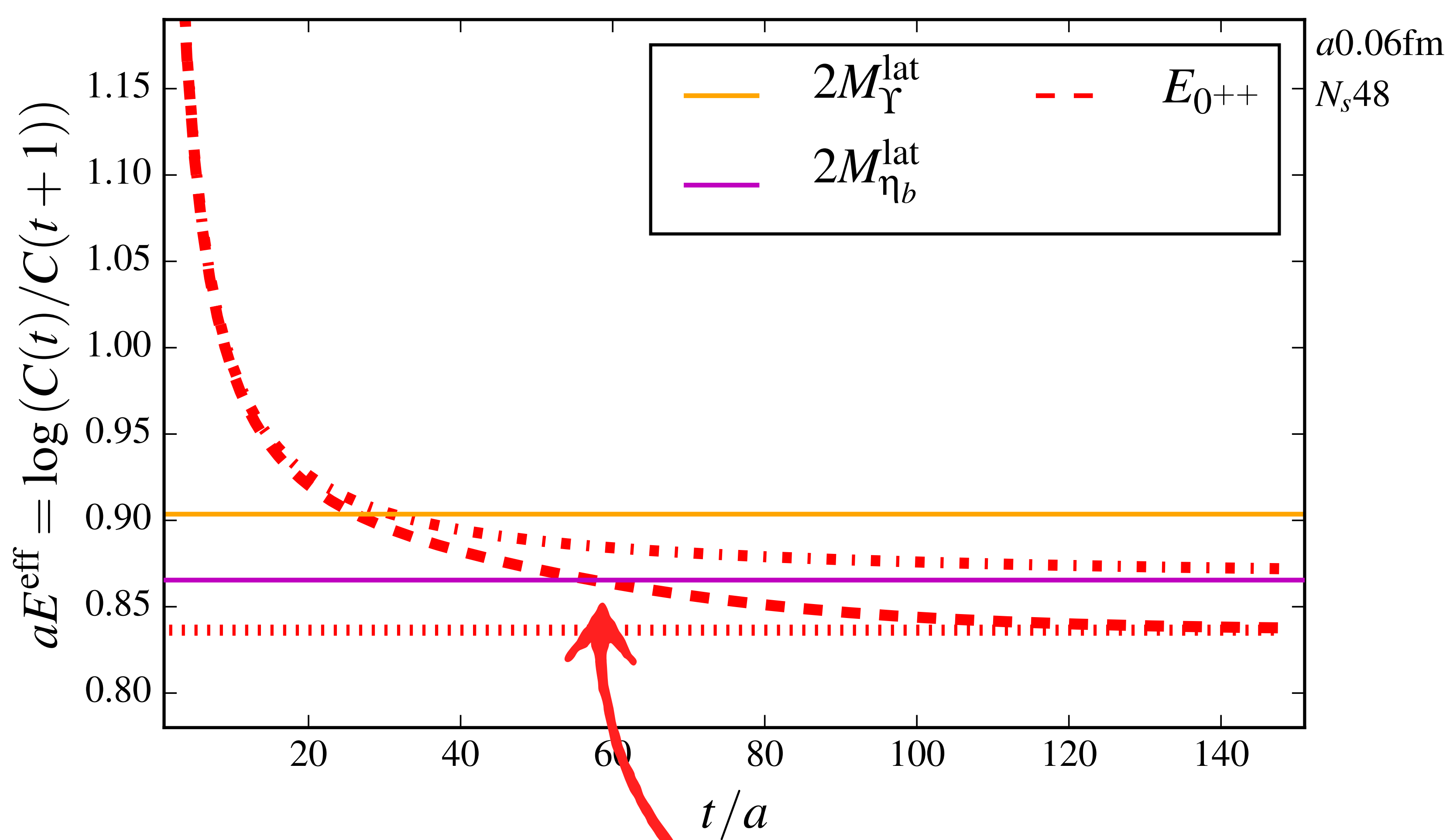
"What might you expect to see?"



If there was ONLY a new state 100 MeV below threshold

Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

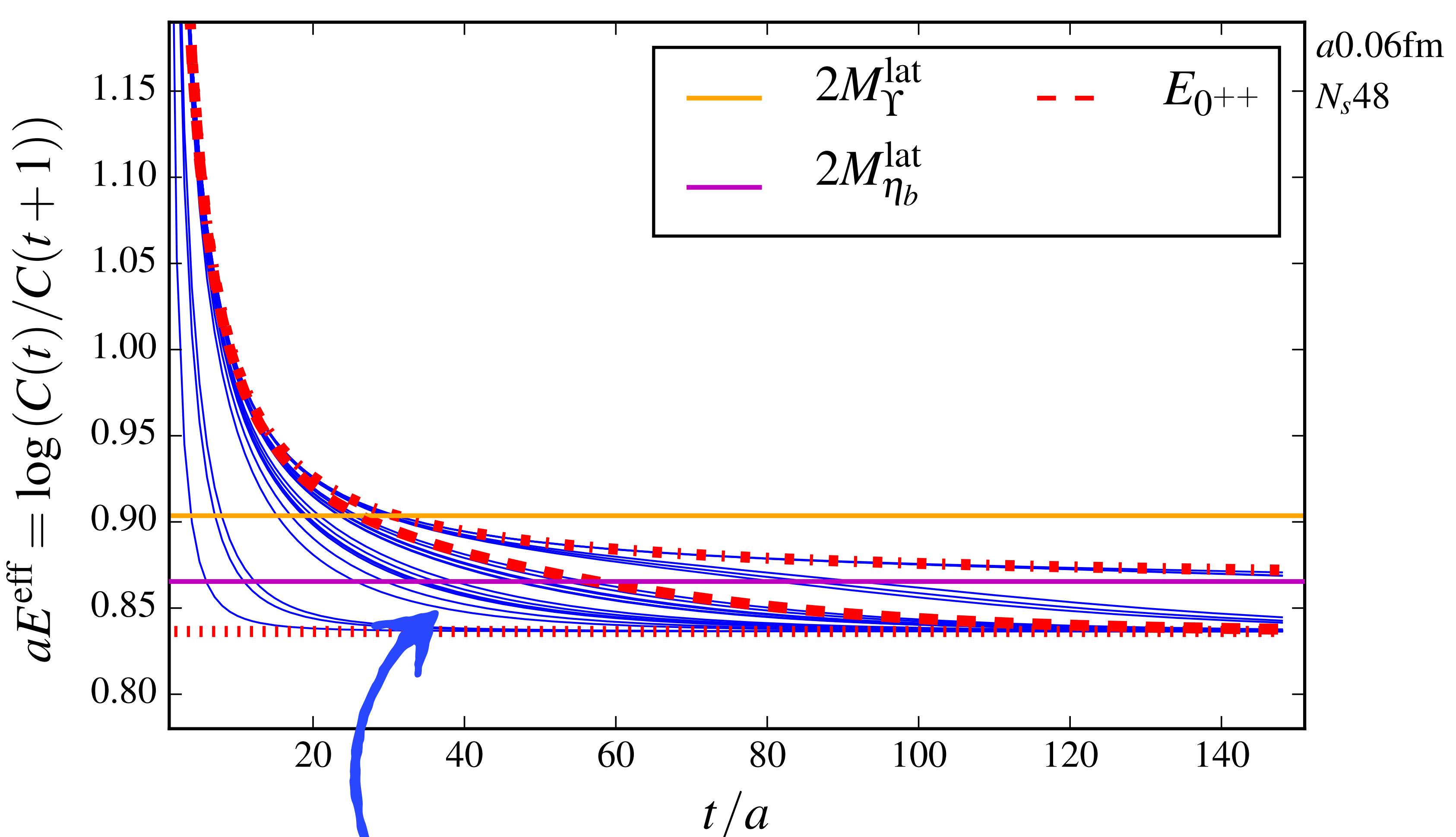
"What might you expect to see?"



If the new state was 100 MeV below threshold and had $Z_0 = Z_1$

Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

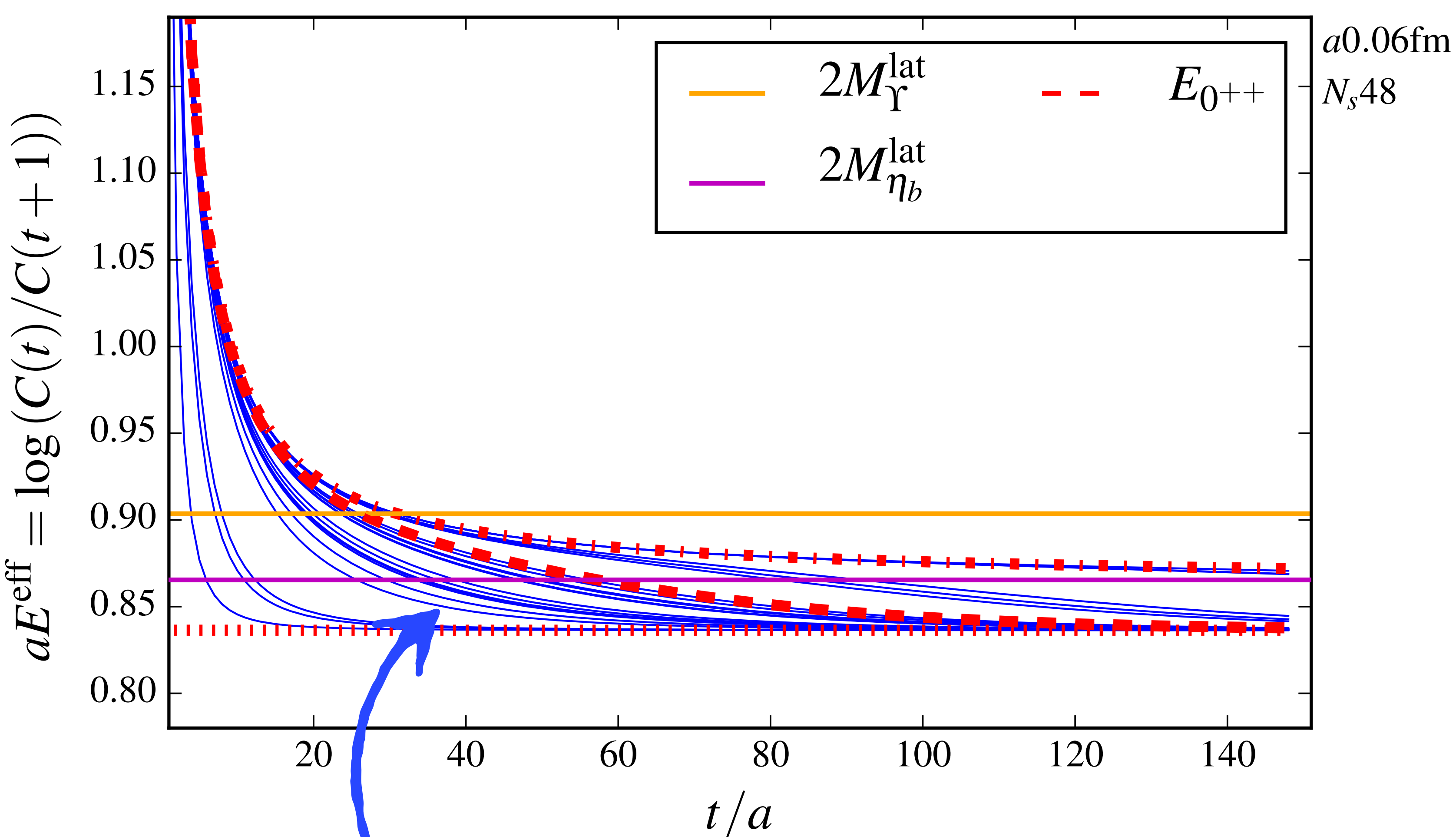
"What might you expect to see?"



If the new state was 100 MeV below threshold and had different Z_0, Z_1

Fake 0^{++} data on the $a \approx 0.06$ fm ensemble

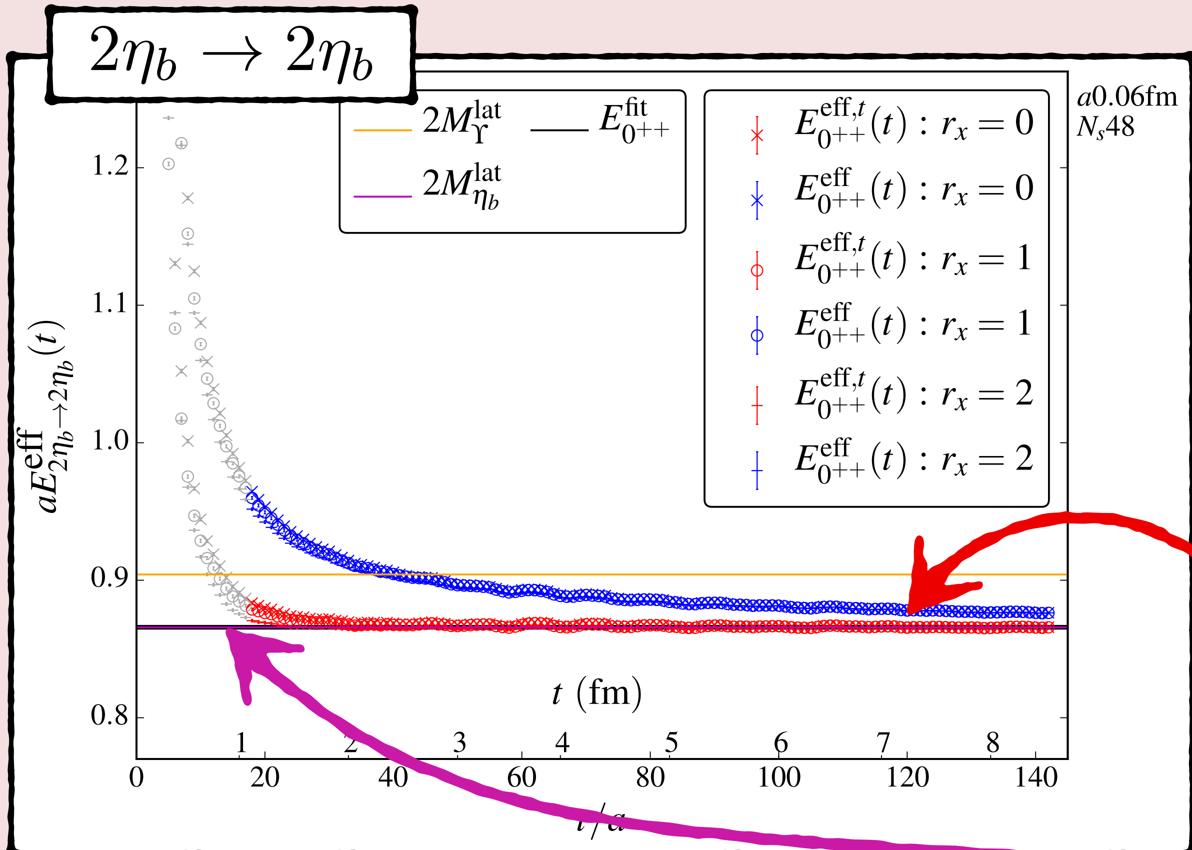
"What might you expect to see?"



If the new state was 100 MeV below threshold and had different Z_0, Z_1

If tetraquark exists we should see a fall below threshold and a clean signal as in blue curve!

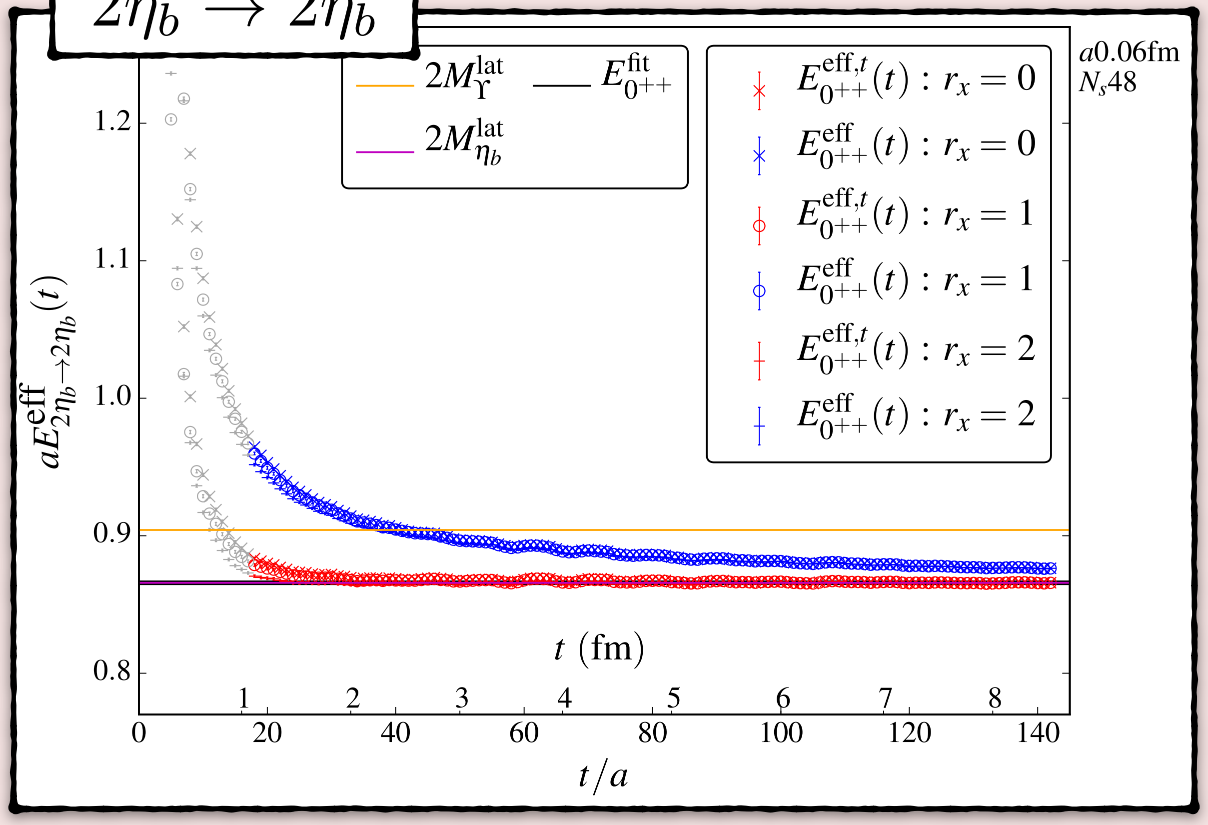
The 0^{++} data on the $a \approx 0.06$ fm ensemble



- For this talk, focus on red data points
- Then compare to $2\eta_b$ non-interacting threshold to determine binding

The 0^{++} data on the $a \approx 0.06$ fm ensemble

$2\eta_b \to 2\eta_b$



The 0^{++} data on the $a \approx 0.06$ fm ensemble

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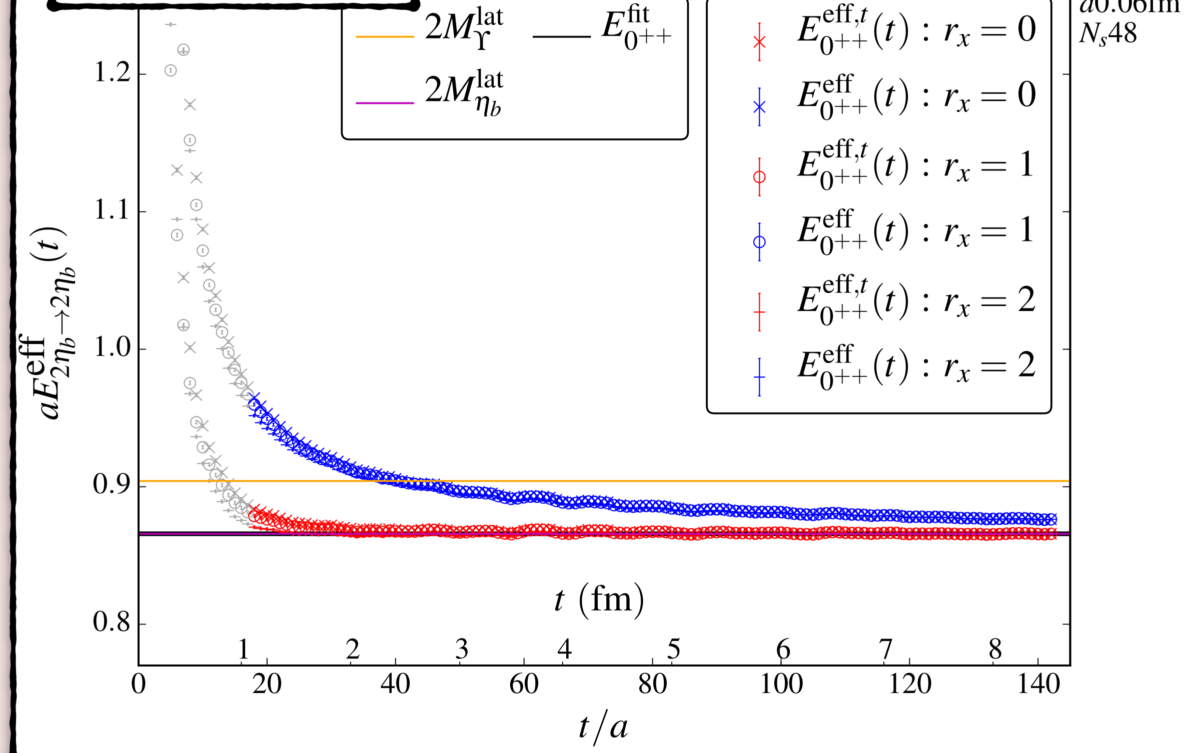
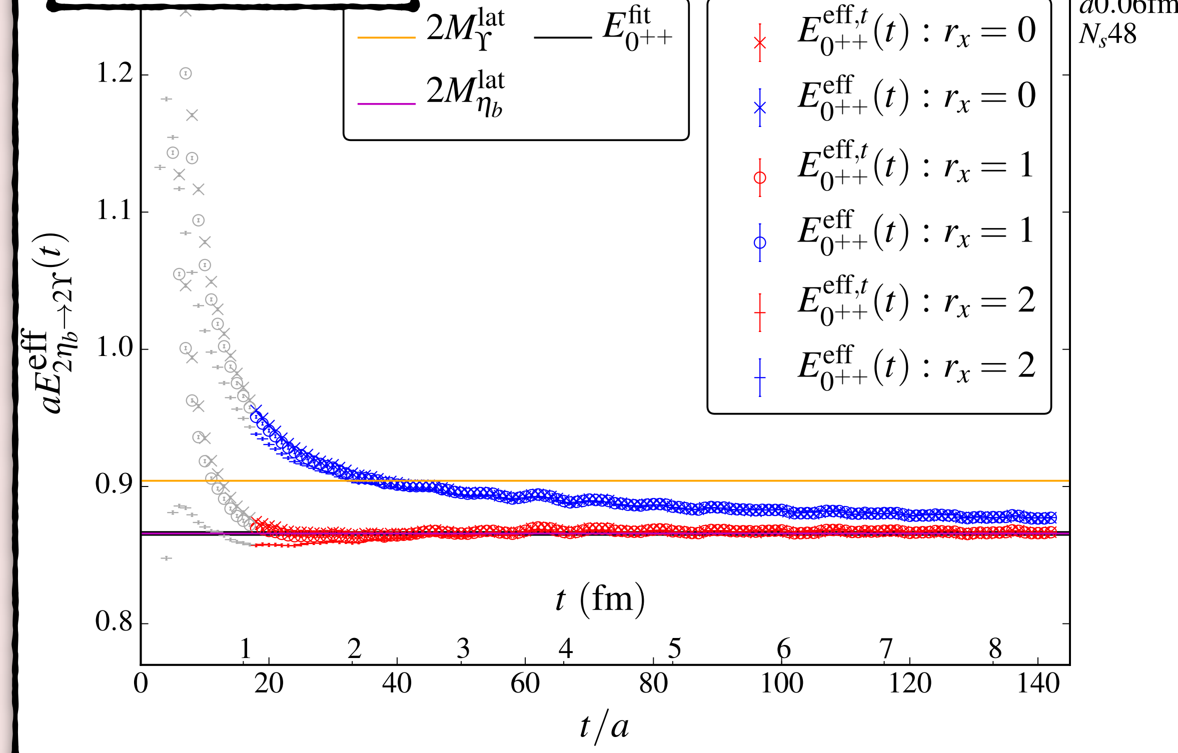
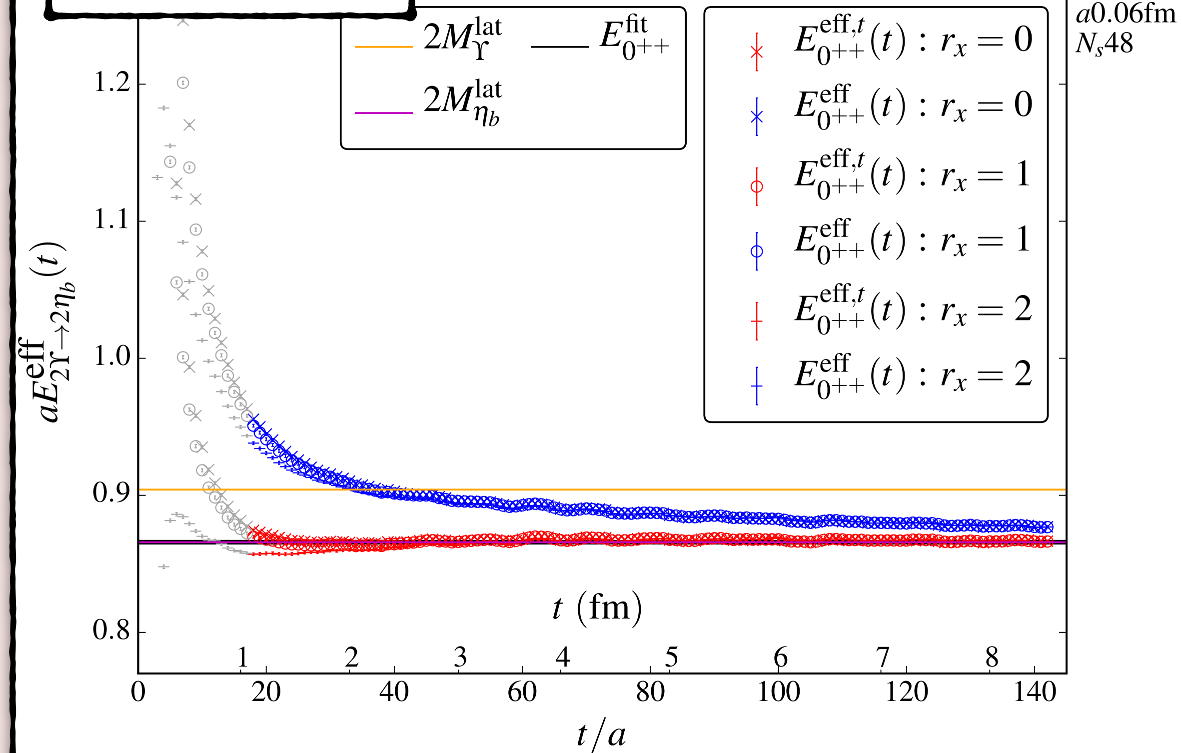


$2\eta_b \to 2\Upsilon$



$2\Upsilon \to 2\eta_b$



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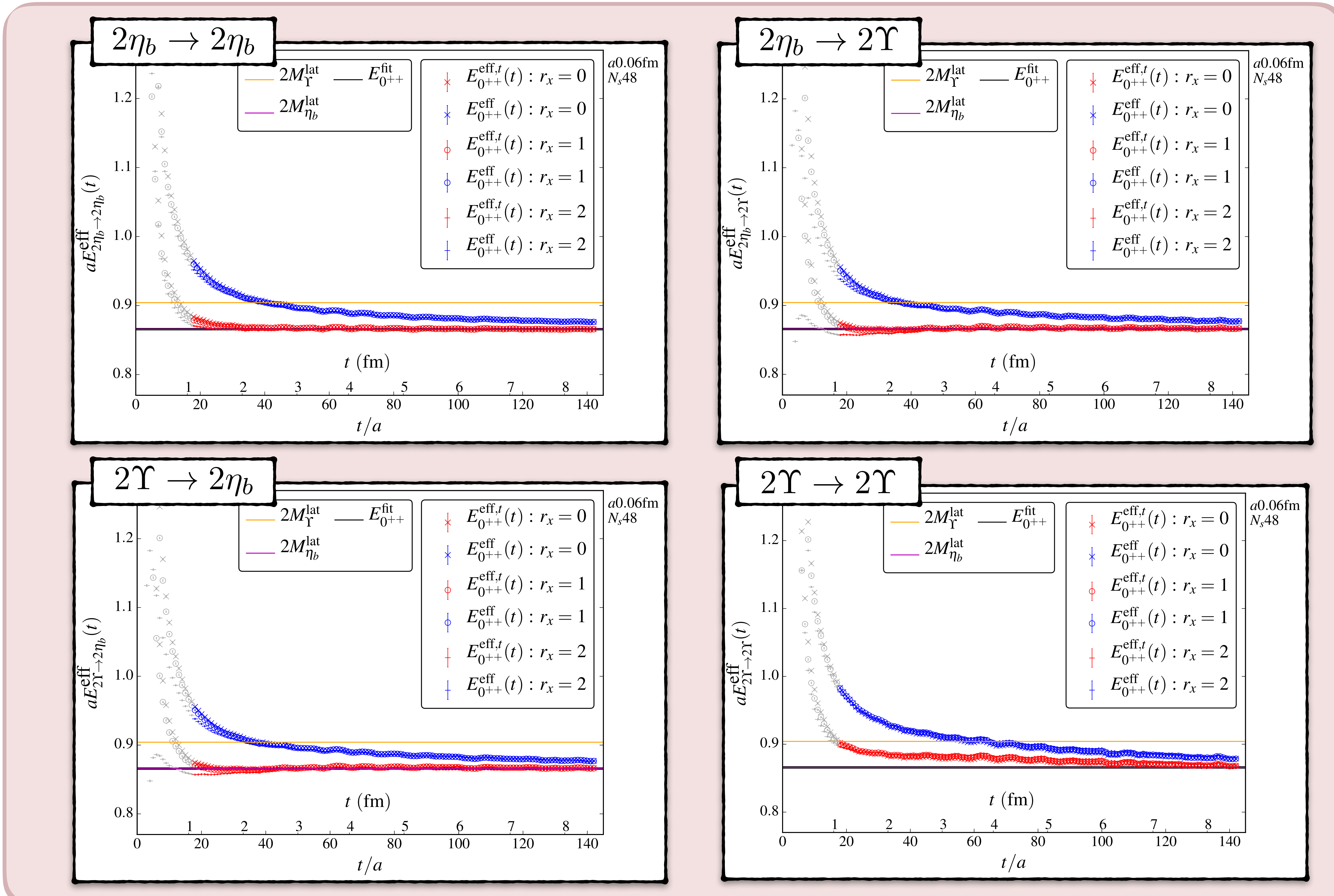
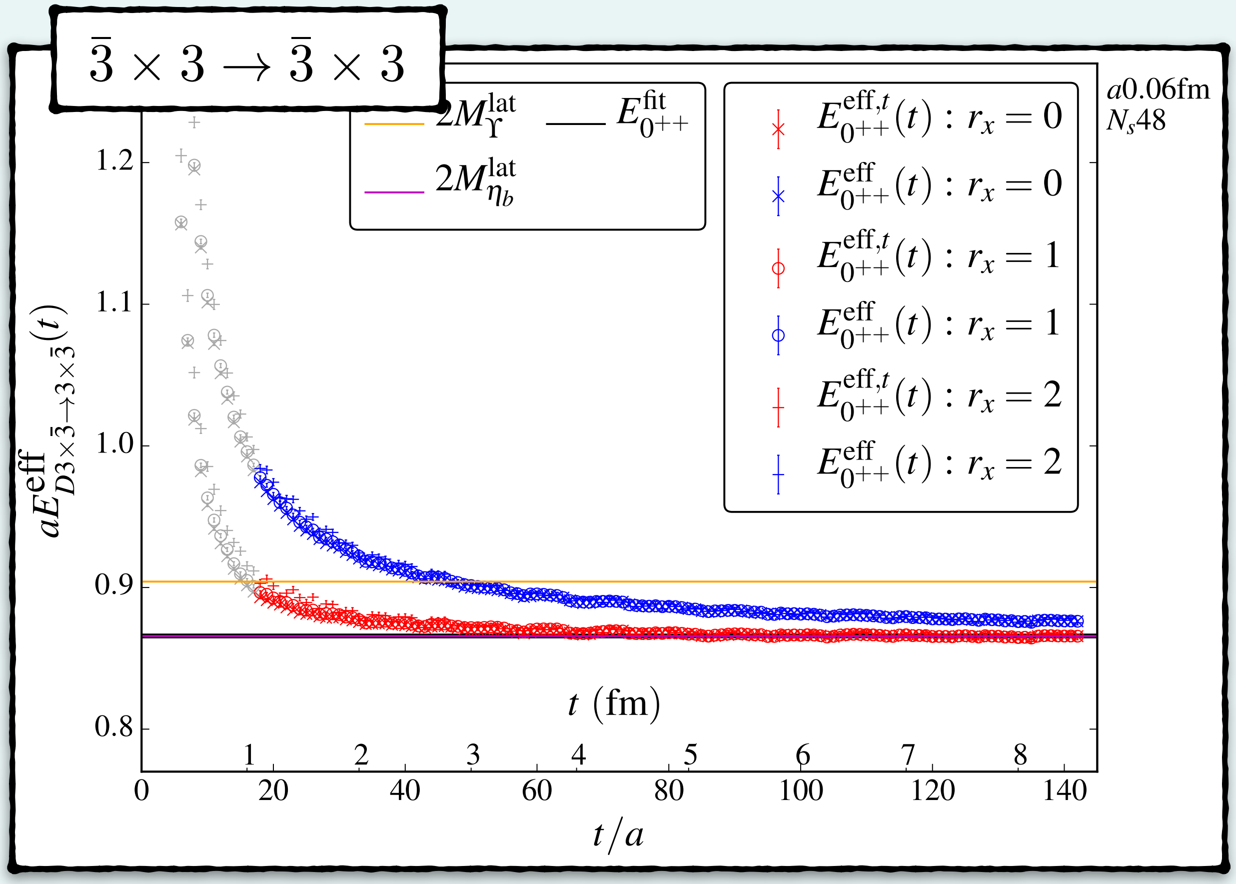
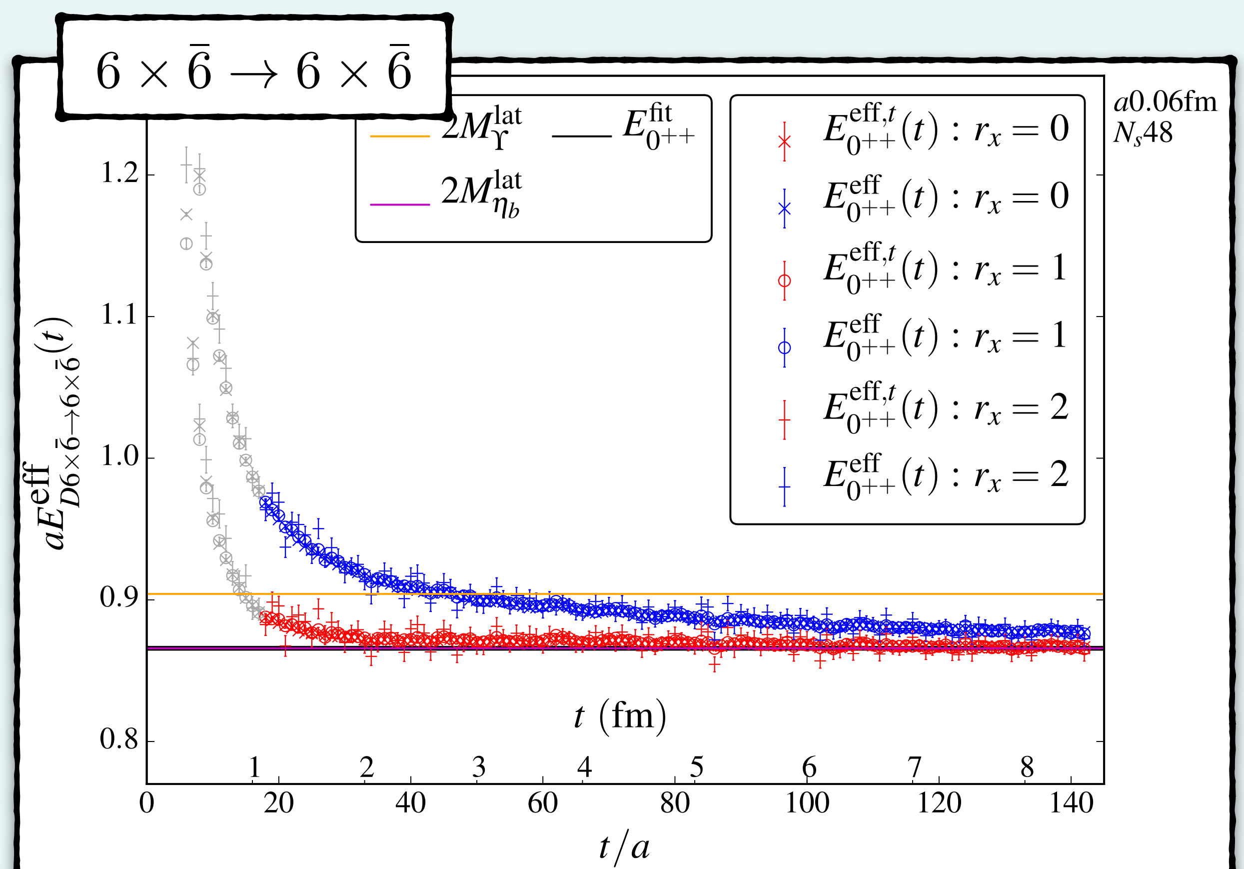
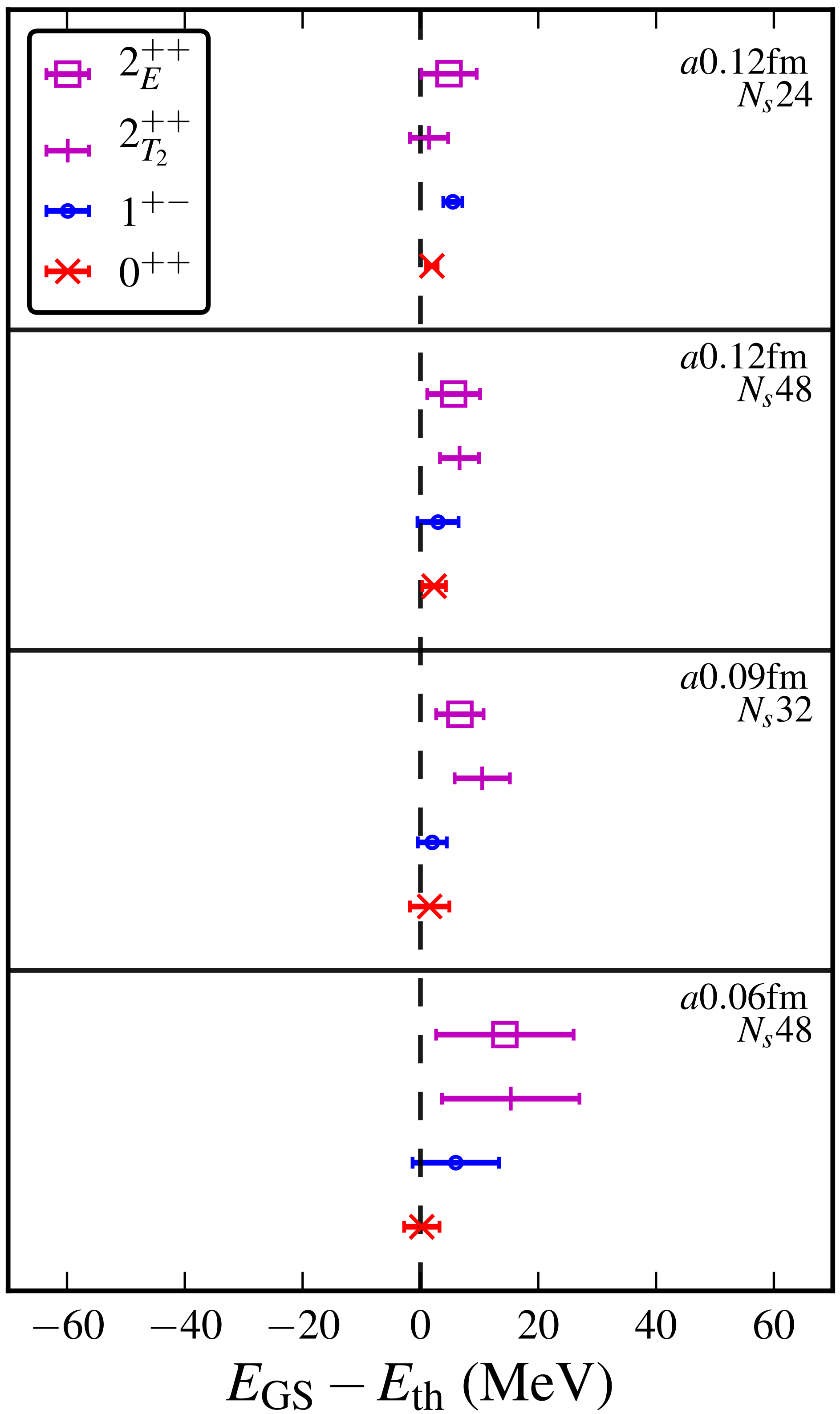


The 0^{++} data on the $a \approx 0.06$ fm ensemble

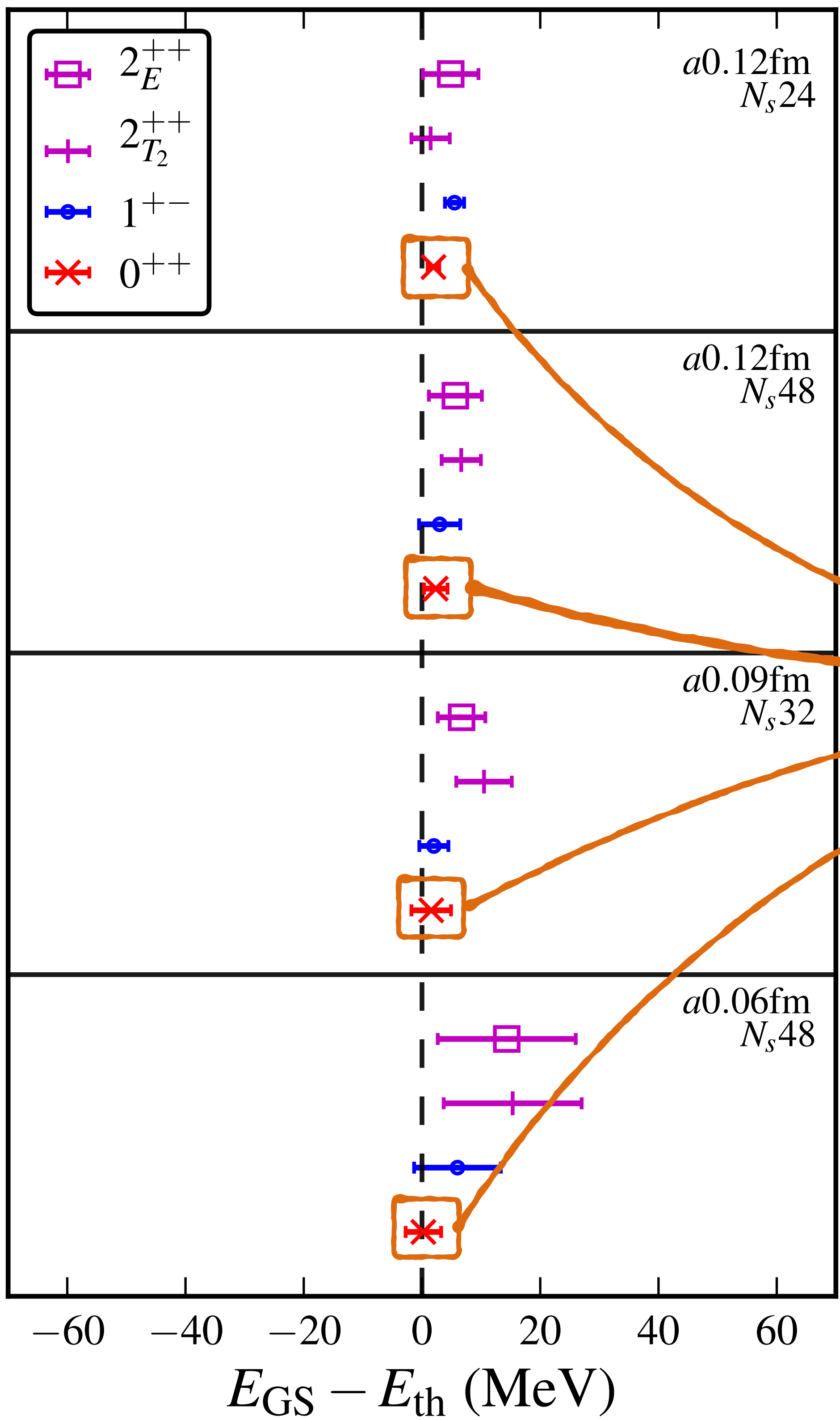




Summary of Energies from Lattice

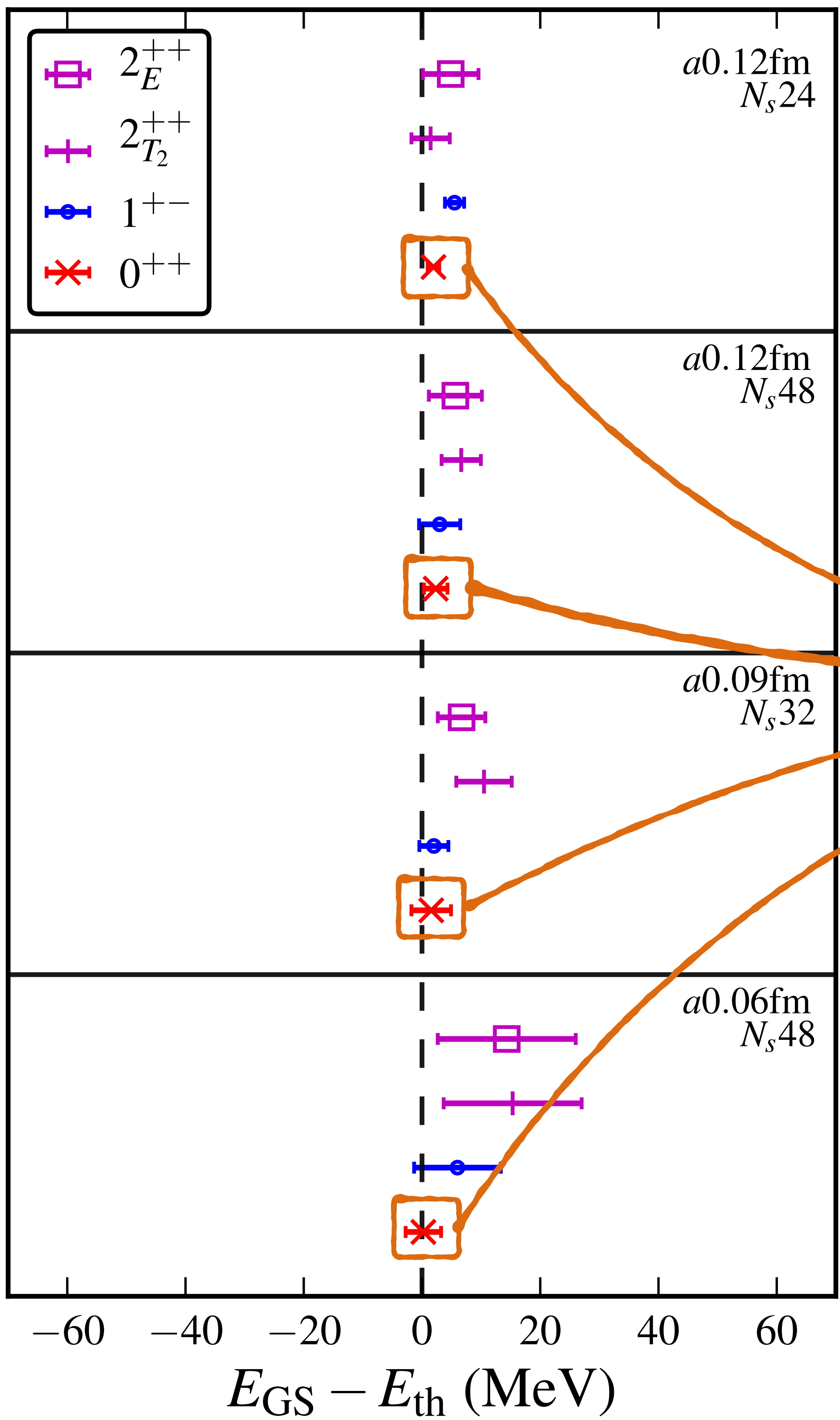


Summary of Energies from Lattice



No evidence of 0^{++} below $2\eta_b$ threshold

Summary of Energies from Lattice



No evidence of 0^{++} below $2\eta_b$ threshold

If you don't observe a process, need to determine a bound, e.g, proton decay.

Bound on 0^{++} $\mathbf{2b2\bar{b}}$ state to be stable

"How would it have missed?"

- If stable tetraquark exists, at a particular time t^*,

$$C(t^*) = |\langle 0|\mathcal{O}|4b\rangle|^2 e^{-aE_{4b}t^*} + |\langle 0|\mathcal{O}|2\eta_b\rangle|^2 e^{-aE_{2\eta_b}t^*}$$

$$= \quad Z_{4b}^2 \; e^{-aE_{4b}t^*} + \quad \tilde{Z}_{2\eta_b}^2 \, e^{-aE_{2\eta_b}t^*}$$

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Input Data

$$= \quad Z_{4b}^2 \; e^{-aE_{4b}t^*} + \quad \tilde{Z}_{2\eta_b}^2 \, e^{-aE_{2\eta_b}t^*}$$

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Input Data

Output Constraint

$$= \boxed{Z_{4b}^2 \; e^{-aE_{4b}t^*}} + \tilde{Z}_{2\eta_b}^2 \, e^{-aE_{2\eta_b}t^*}$$

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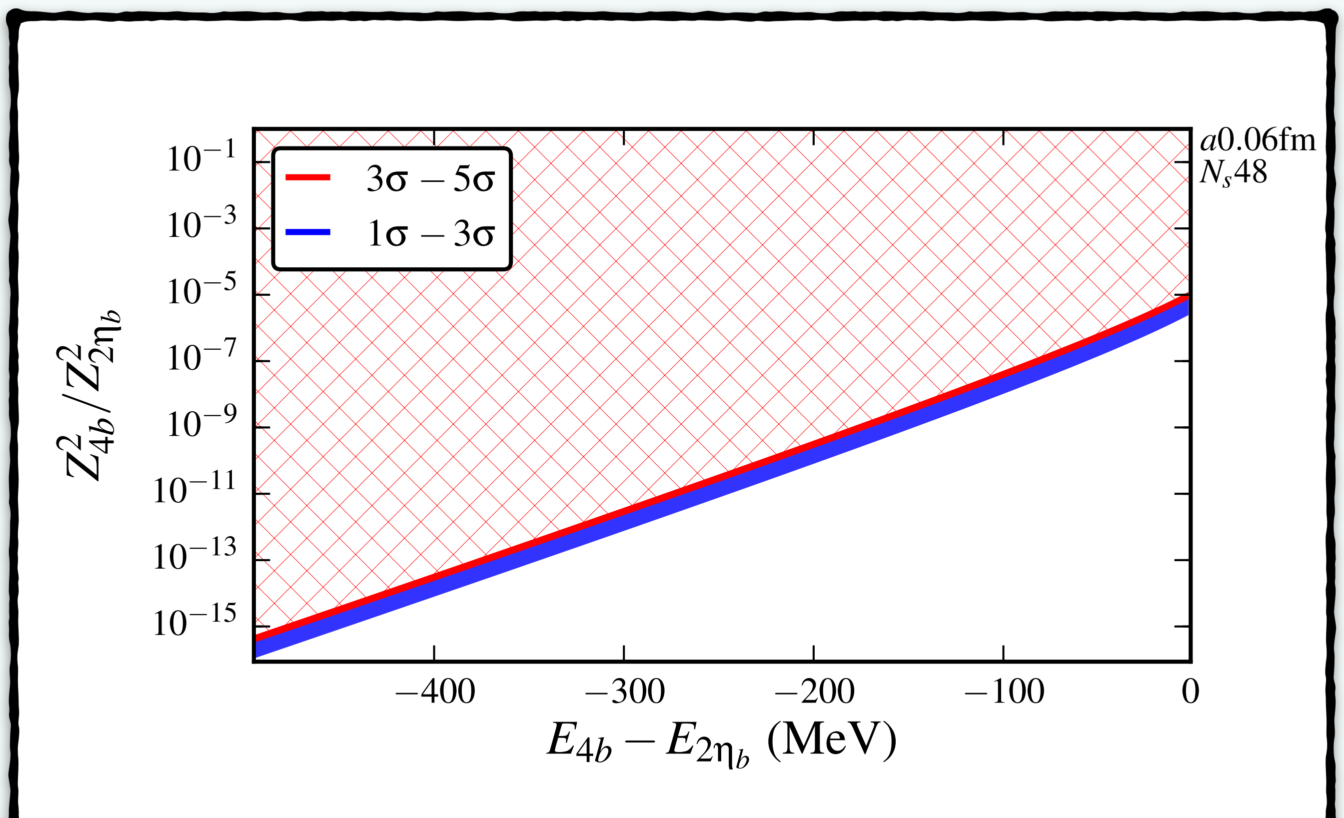
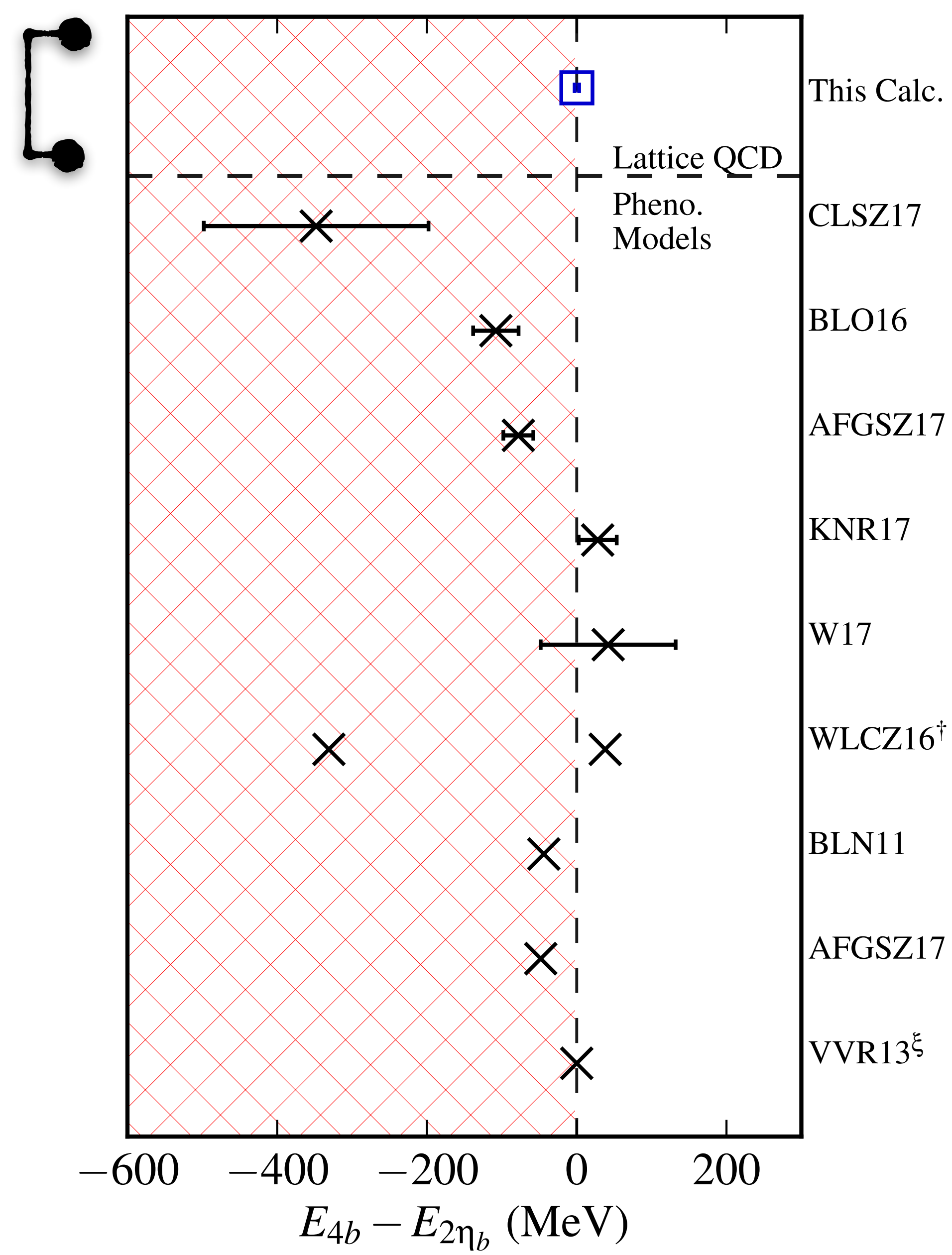
Output Constraint

$$= \boxed{Z_{4b}^2 \; e^{-aE_{4b}t^*}} + \tilde{Z}_{2\eta_b}^2 \, e^{-aE_{2\eta_b}t^*}$$



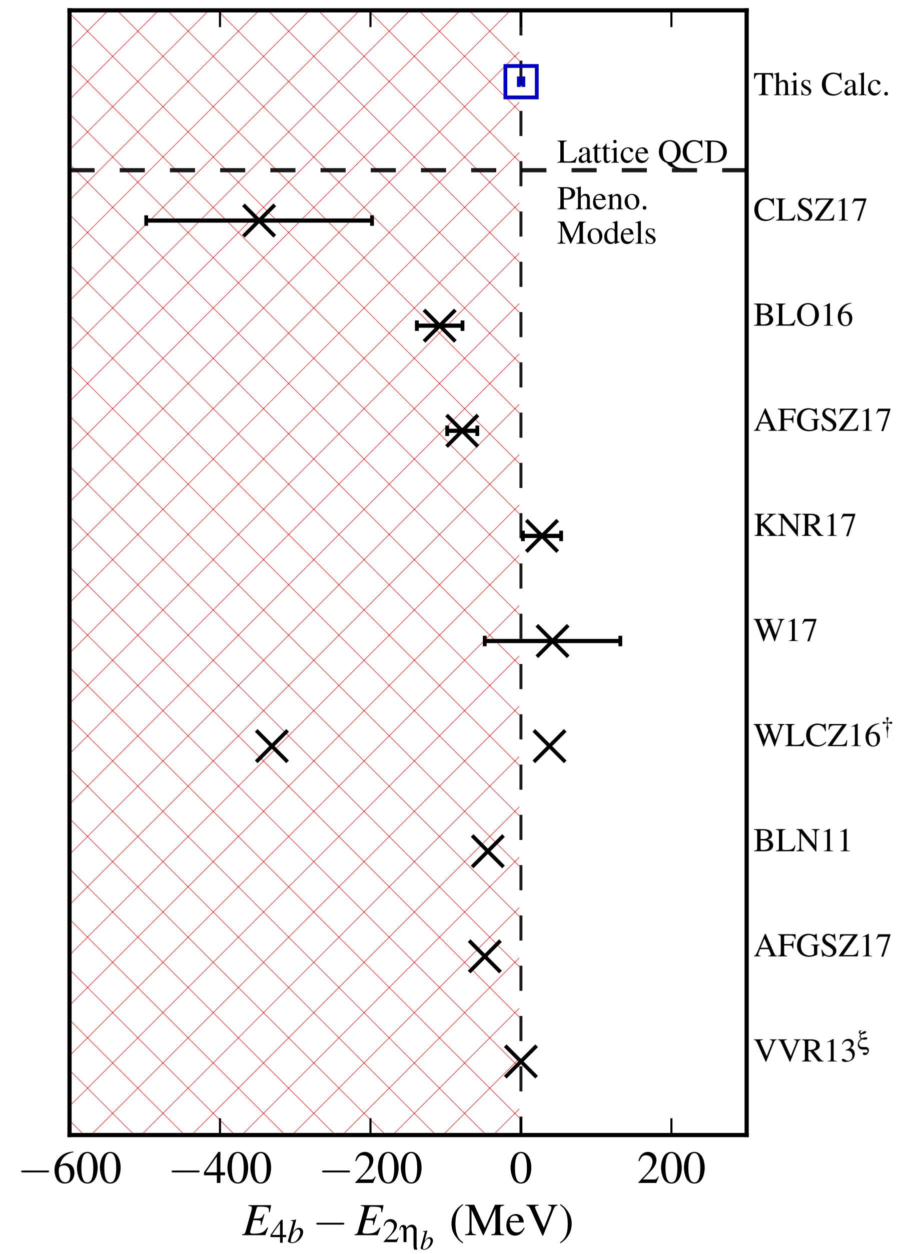
Summary

- In Summary, lattice QCD finds no evidence of a stable $2b2\bar{b}$ tetraquark



What The Models Need!

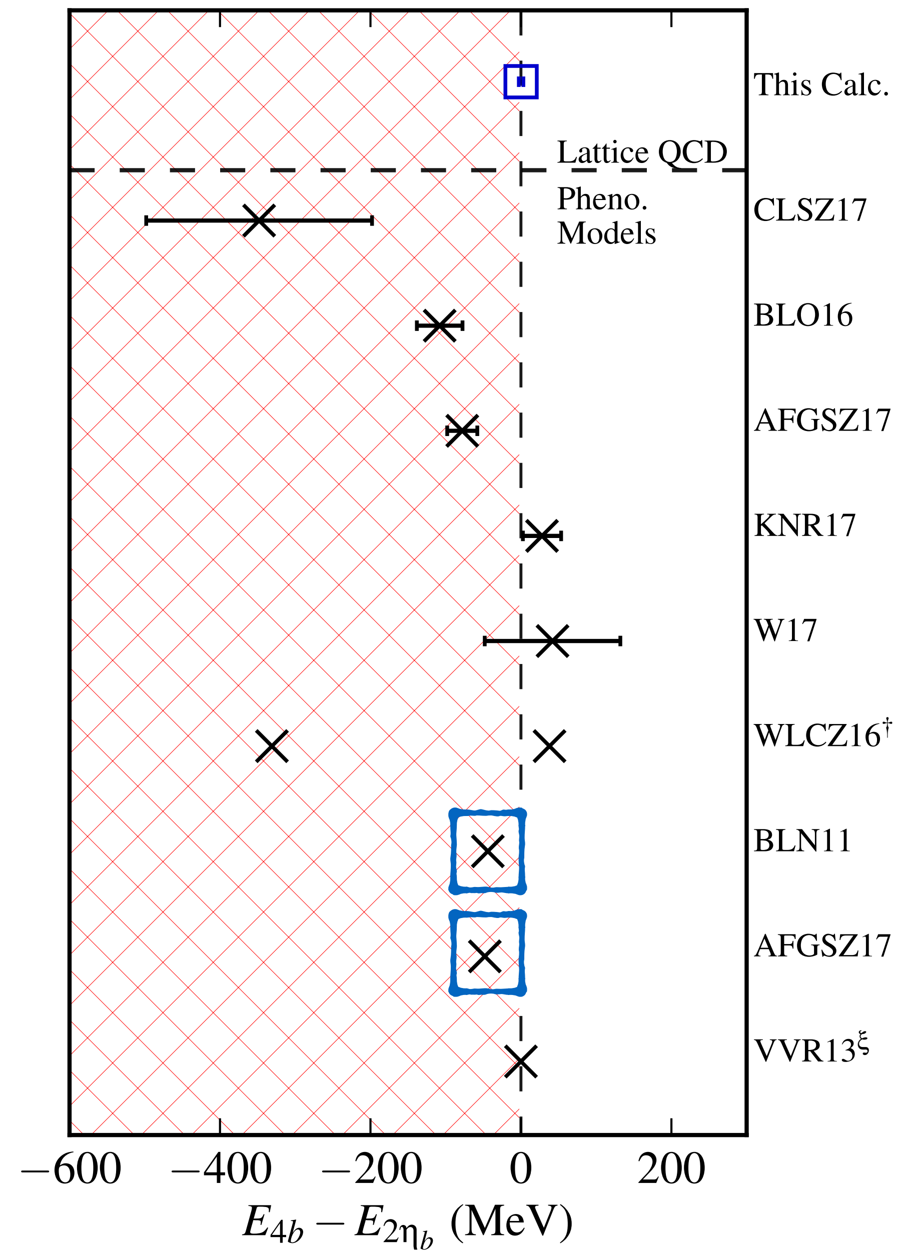
	Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?



What The Models Need!

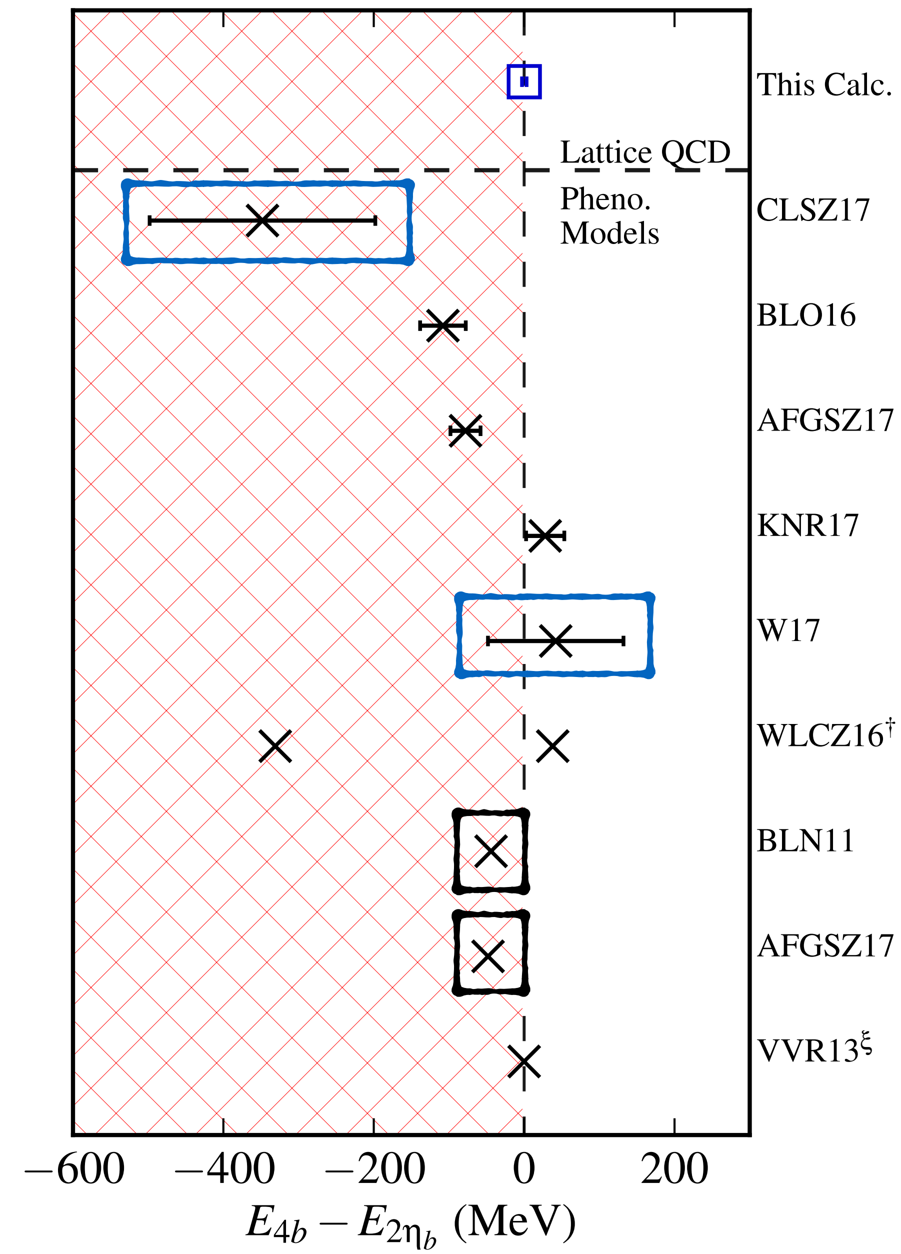
Diquarks

	Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?
1110.1867	✗	✓	✗
1710.0254	✗	✓	✓



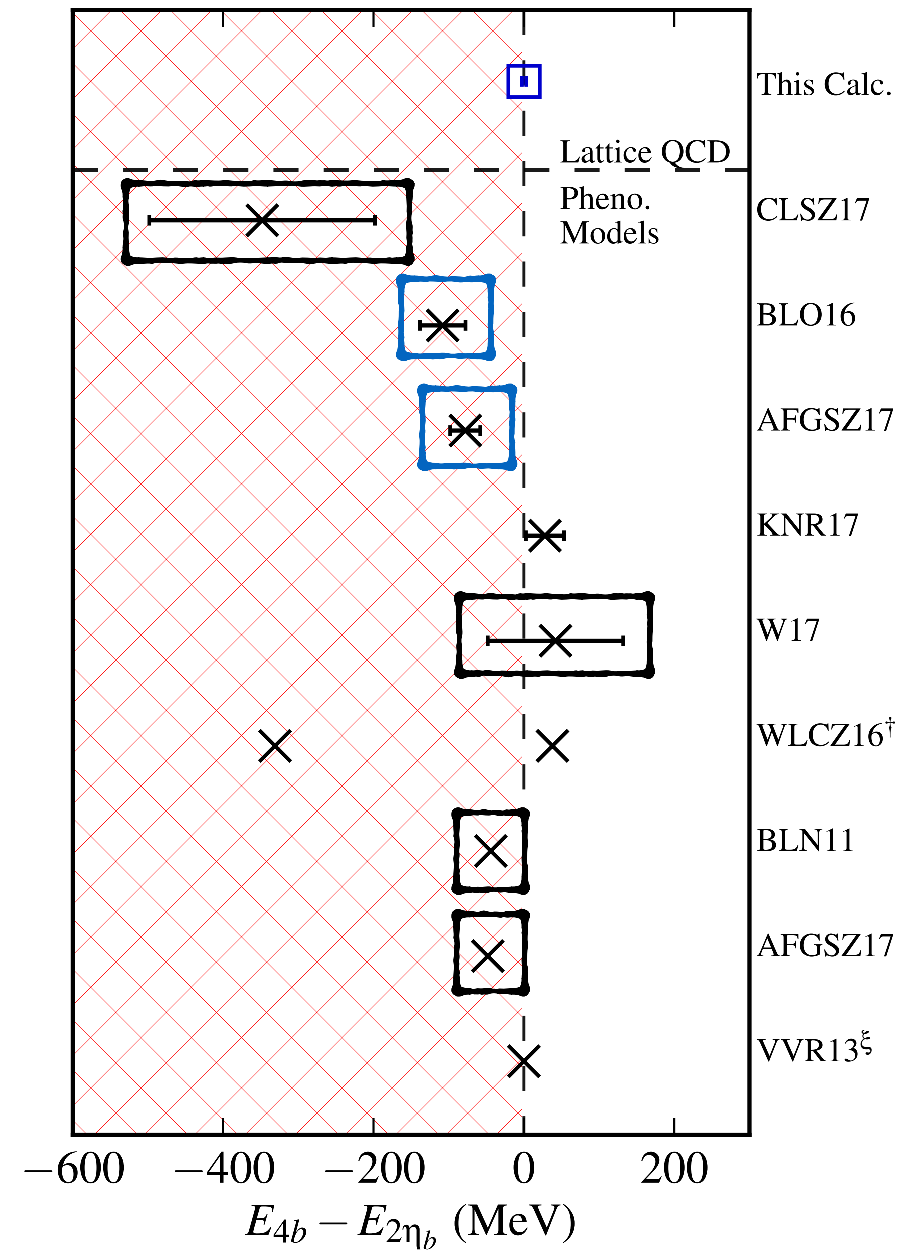
What The Models Need!

		Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?
Diquarks	1110.1867	✗	✓	✗
	1710.0254	✗	✓	✓
Sum-Rules	1605.01647	✓	✓	✗
	1701.04285	✓	✓	✗



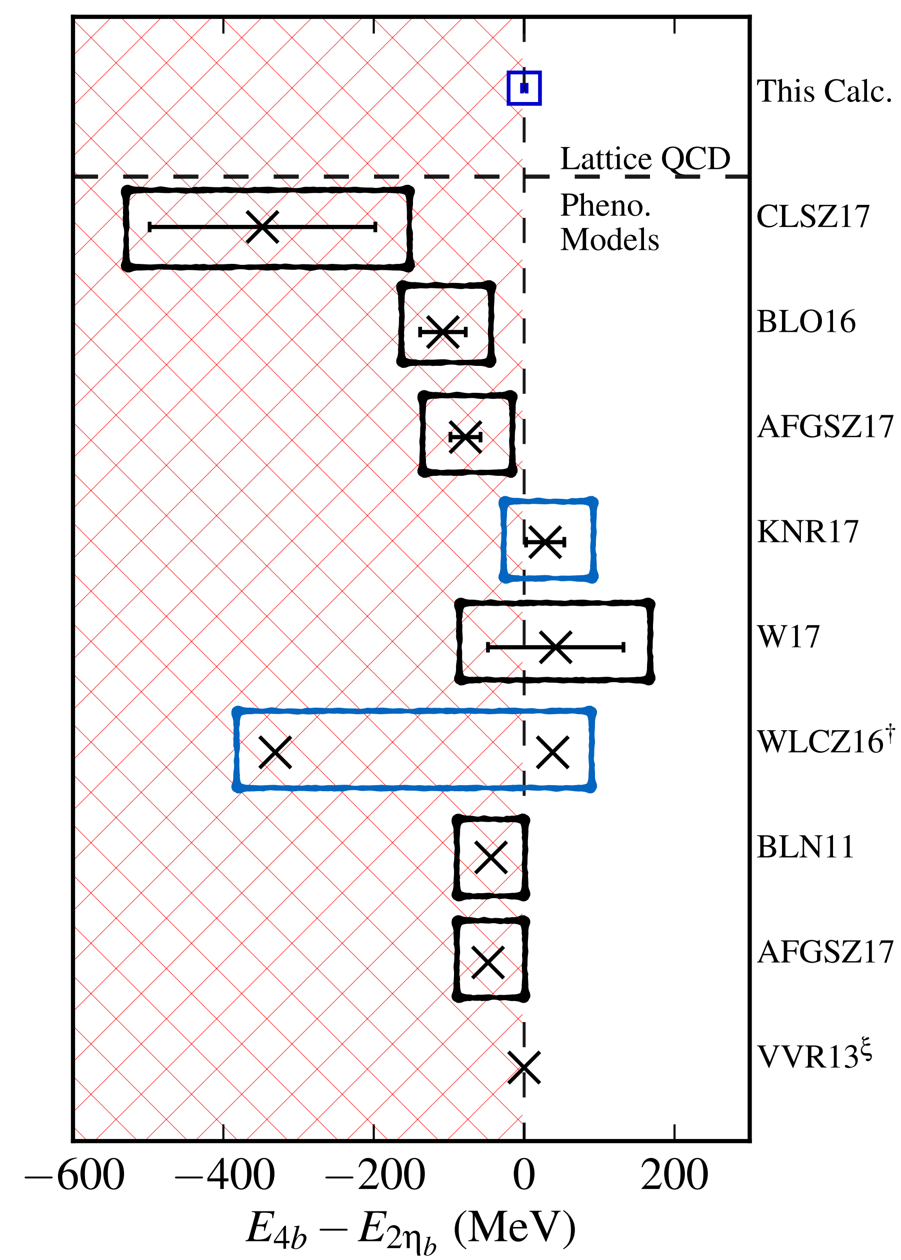
What The Models Need!

		Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?
Diquarks	1110.1867	✗	✓	✗
	1710.0254	✗	✓	✓
Sum-Rules	1605.01647	✓	✓	✗
	1701.04285	✓	✓	✗
Schrodinger Equation	1710.0254	✓	✗	✗
	1612.00012	✓	✗	✓



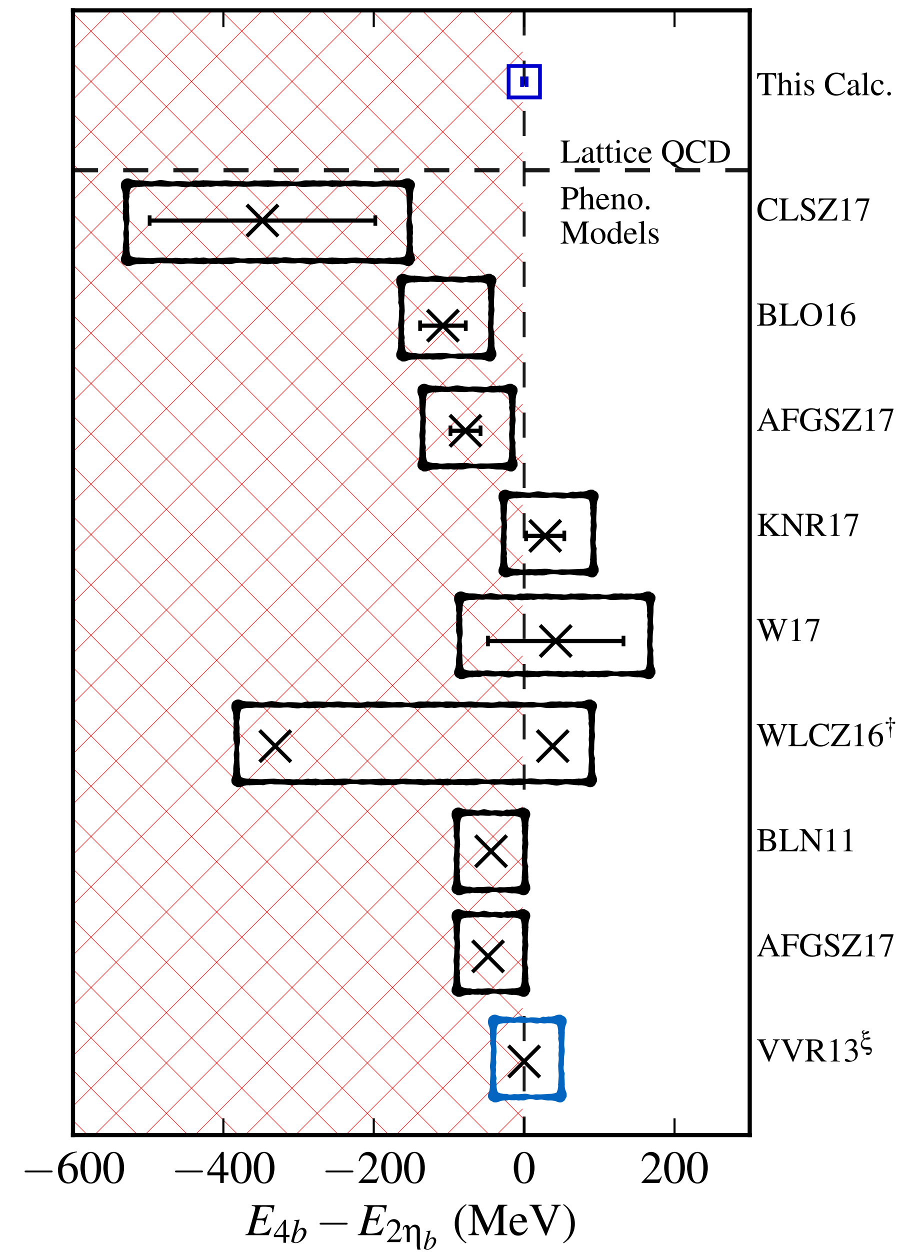
What The Models Need!

		Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?
Diquarks	1110.1867	✗	✓	✗
	1710.0254	✗	✓	✓
Sum-Rules	1605.01647	✓	✓	✗
	1701.04285	✓	✓	✗
Schrodinger Equation	1710.0254	✓	✗	✗
	1612.00012	✓	✗	✓
Pheno.	1605.01134	✓	✗	✗
	1611.00348	✓	✗	✓



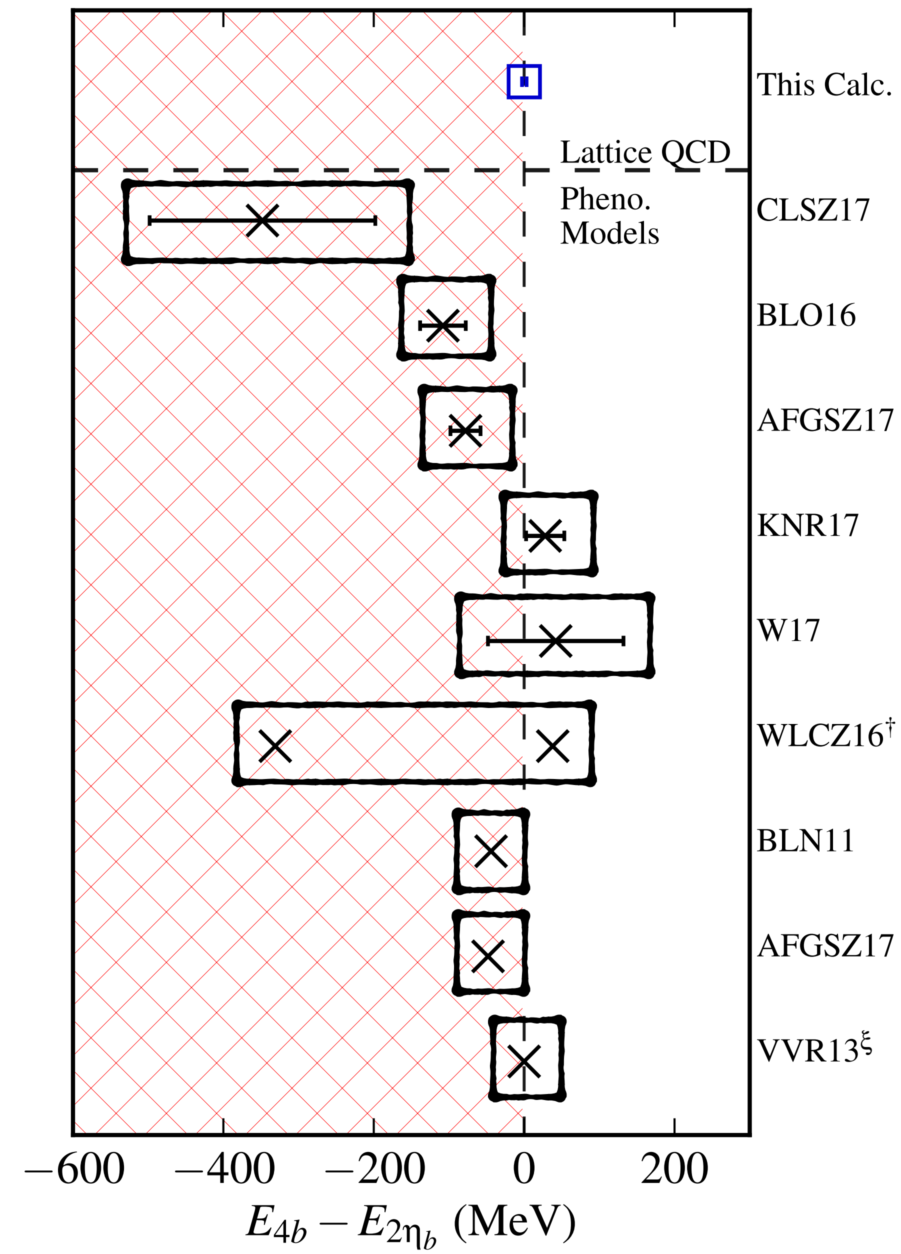
What The Models Need!

		Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?
Diquarks	1110.1867	✗	✓	✗
	1710.0254	✗	✓	✓
Sum-Rules	1605.01647	✓	✓	✗
	1701.04285	✓	✓	✗
Schrodinger Equation	1710.0254	✓	✗	✗
	1612.00012	✓	✗	✓
Pheno.	1605.01134	✓	✗	✗
	1611.00348	✓	✗	✓
String	1703.00783	✓	✓	✗

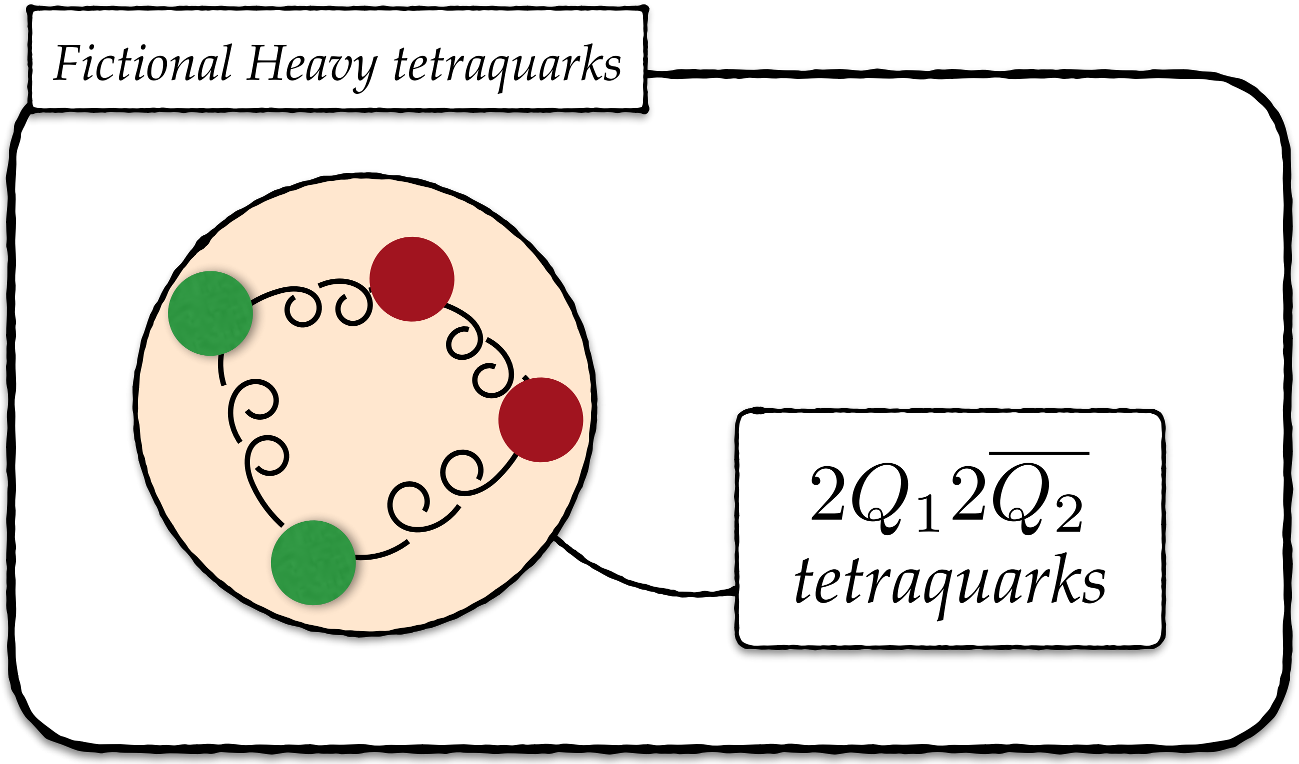


What The Models Need!

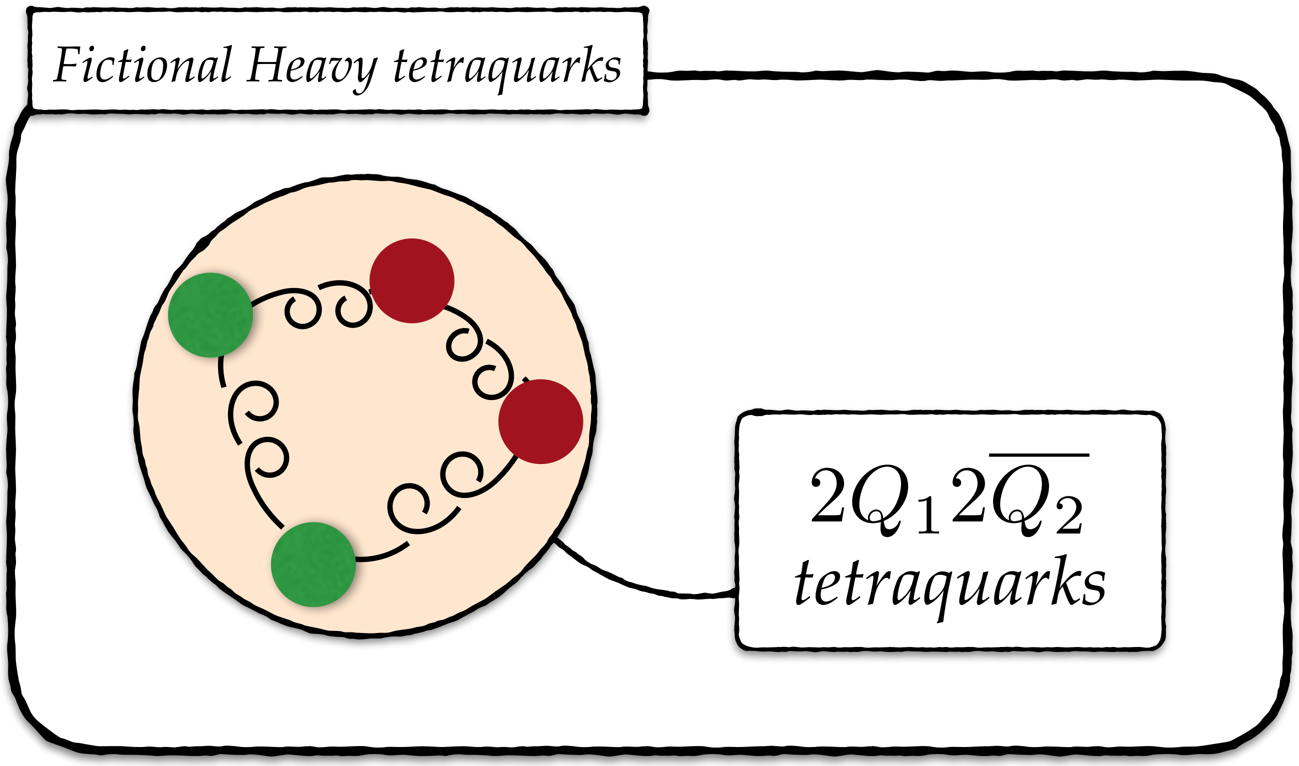
	N.B., The lattice is **NOT** a model	Have quarks as elementary particles?	Full 2×2 potential matrix (including mixing between different color components)?	Both Short and Long distance effects in Gluon exchange?
Diquarks	1110.1867	✗	✓	✗
	1710.0254	✗	✓	✓
Sum-Rules	1605.01647	✓	✓	✗
	1701.04285	✓	✓	✗
Schrodinger Equation	1710.0254	✓	✗	✗
	1612.00012	✓	✗	✓
Pheno.	1605.01134	✓	✗	✗
	1611.00348	✓	✗	✓
String	1703.00783	✓	✓	✗



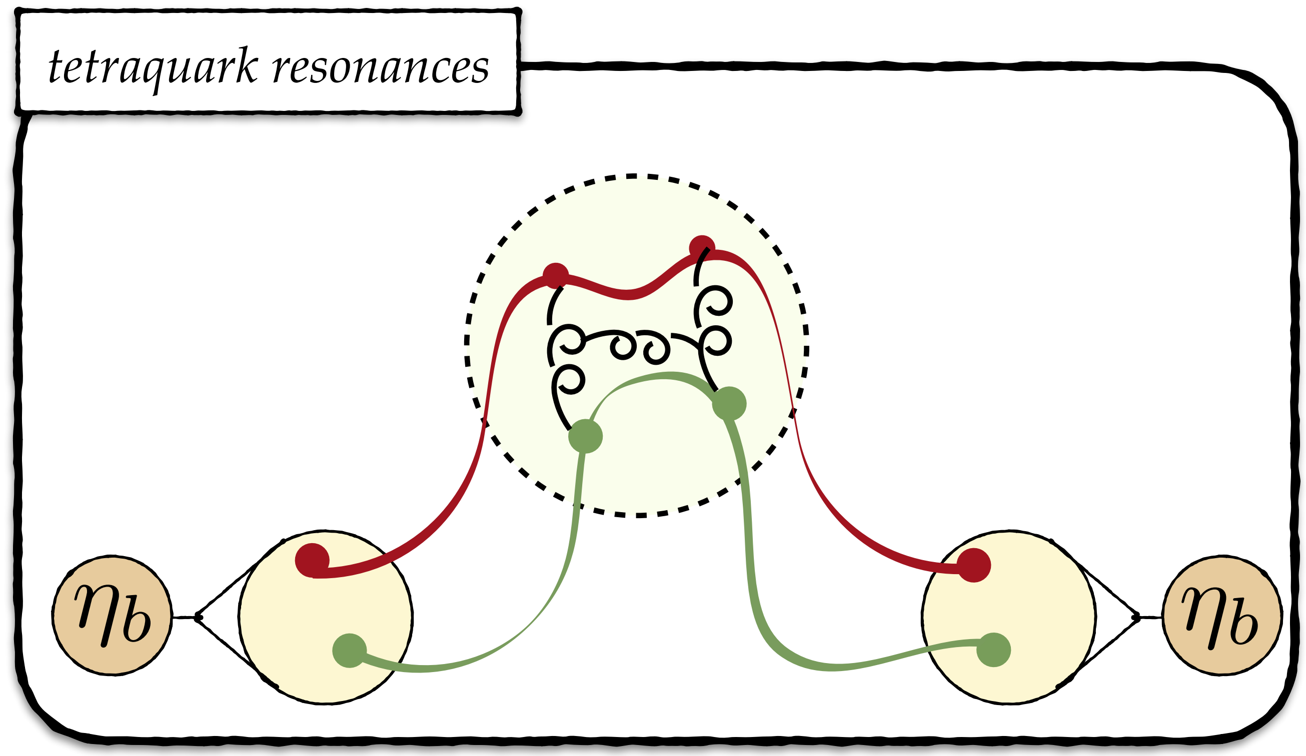
Future Work


Fictional Heavy tetraquarks
$2Q_1 2\overline{Q_2}$
tetraquarks

Future Work



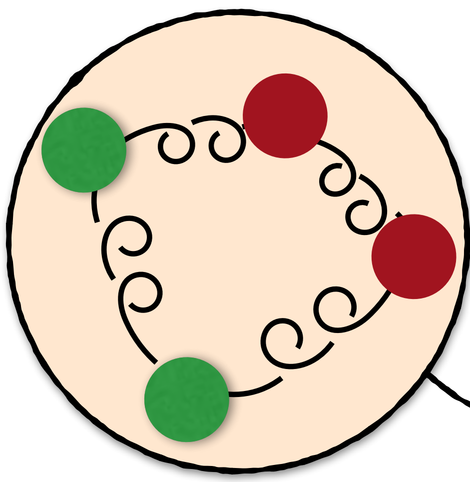
Fictional Heavy tetraquarks
$2Q_1 2\overline{Q_2}$
tetraquarks



tetraquark resonances
η_b
η_b

Future Work

Fictional Heavy tetraquarks



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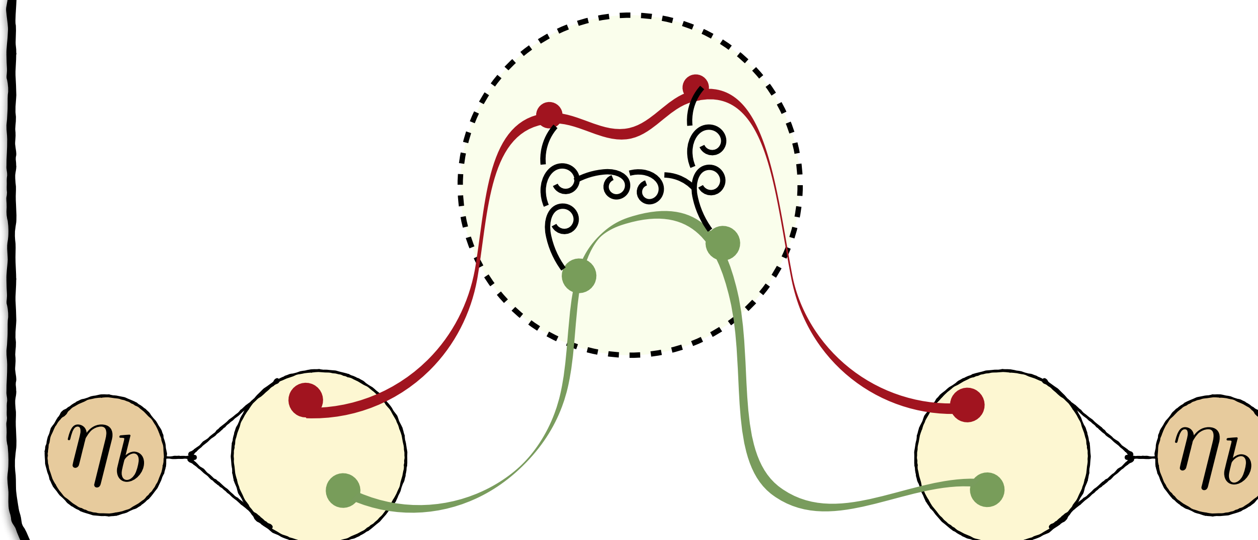
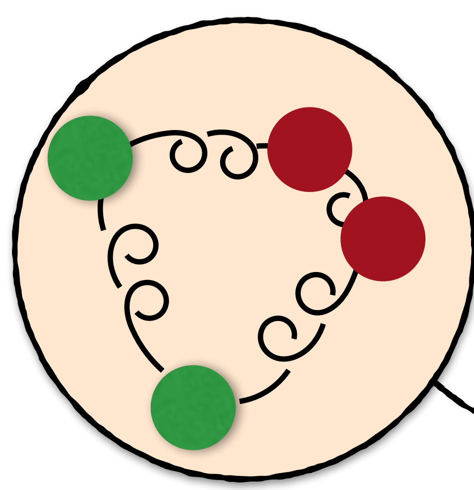
tetraquarks

tetraquark resonances



Stable Real tetraquarks



$bb\bar{u}\bar{d}$
tetraquarks

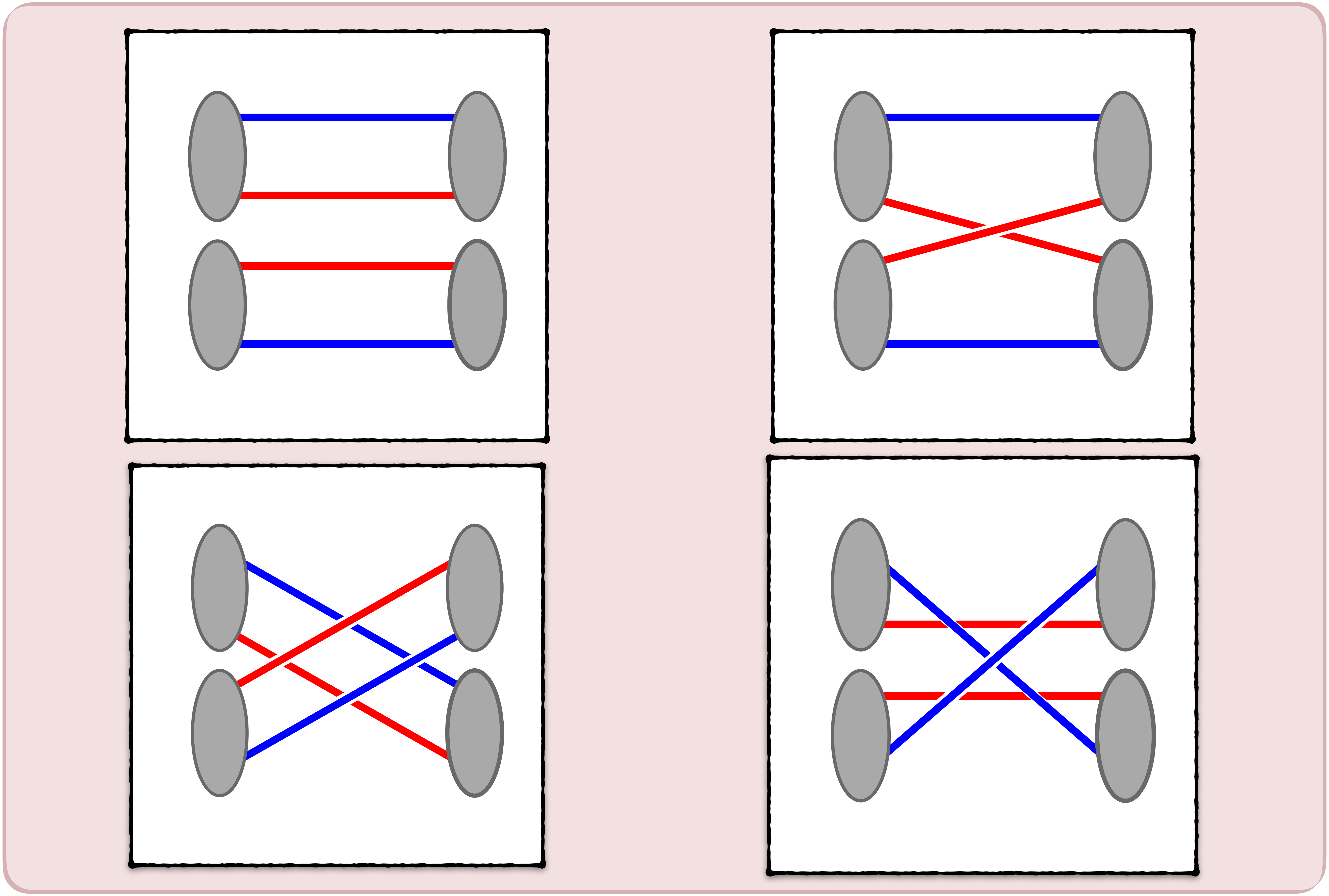
arxiv:1707.09575

Thank you!

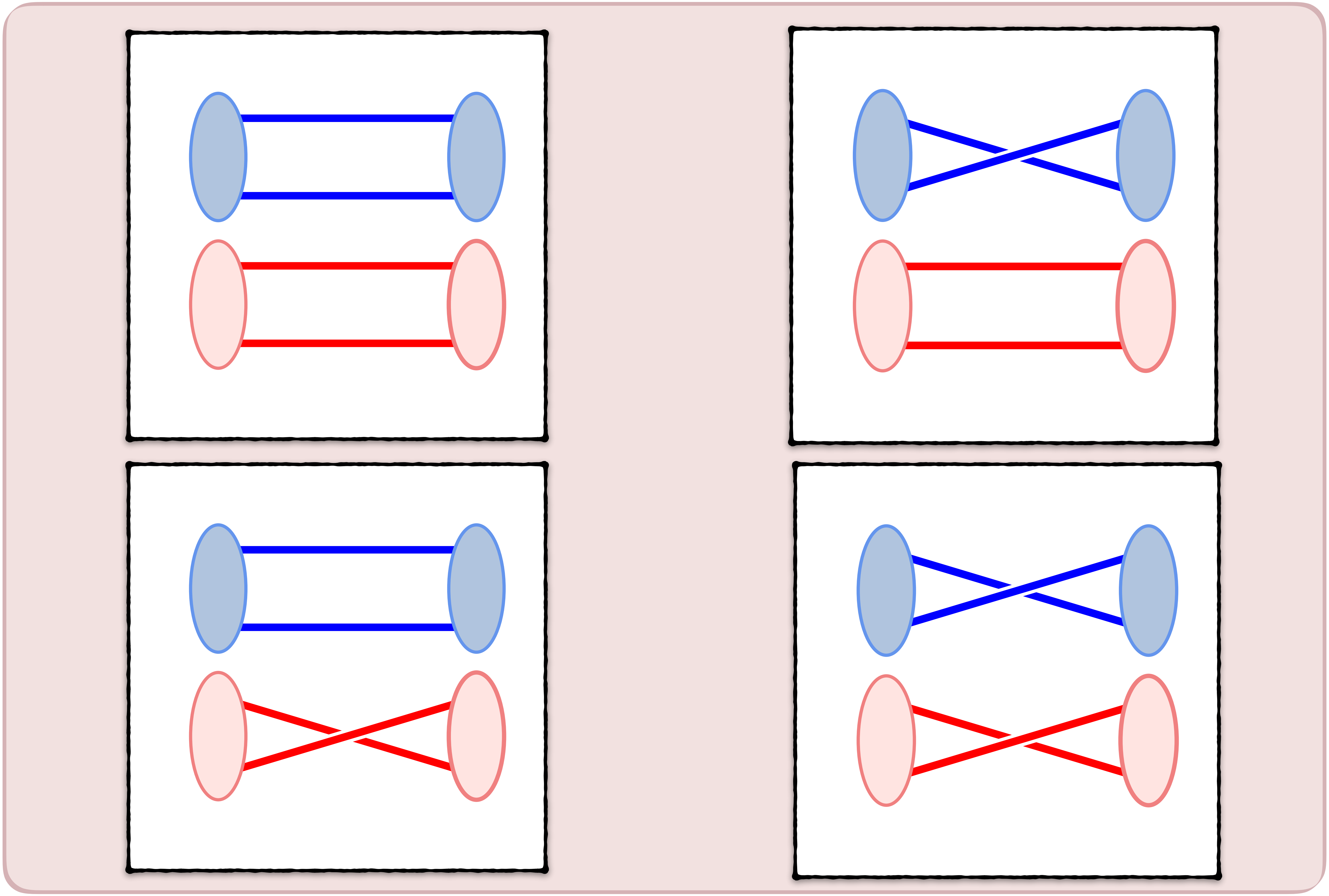
Thank You to Raul Briceno for slide template and pretty graphics!

Back-Up Slides

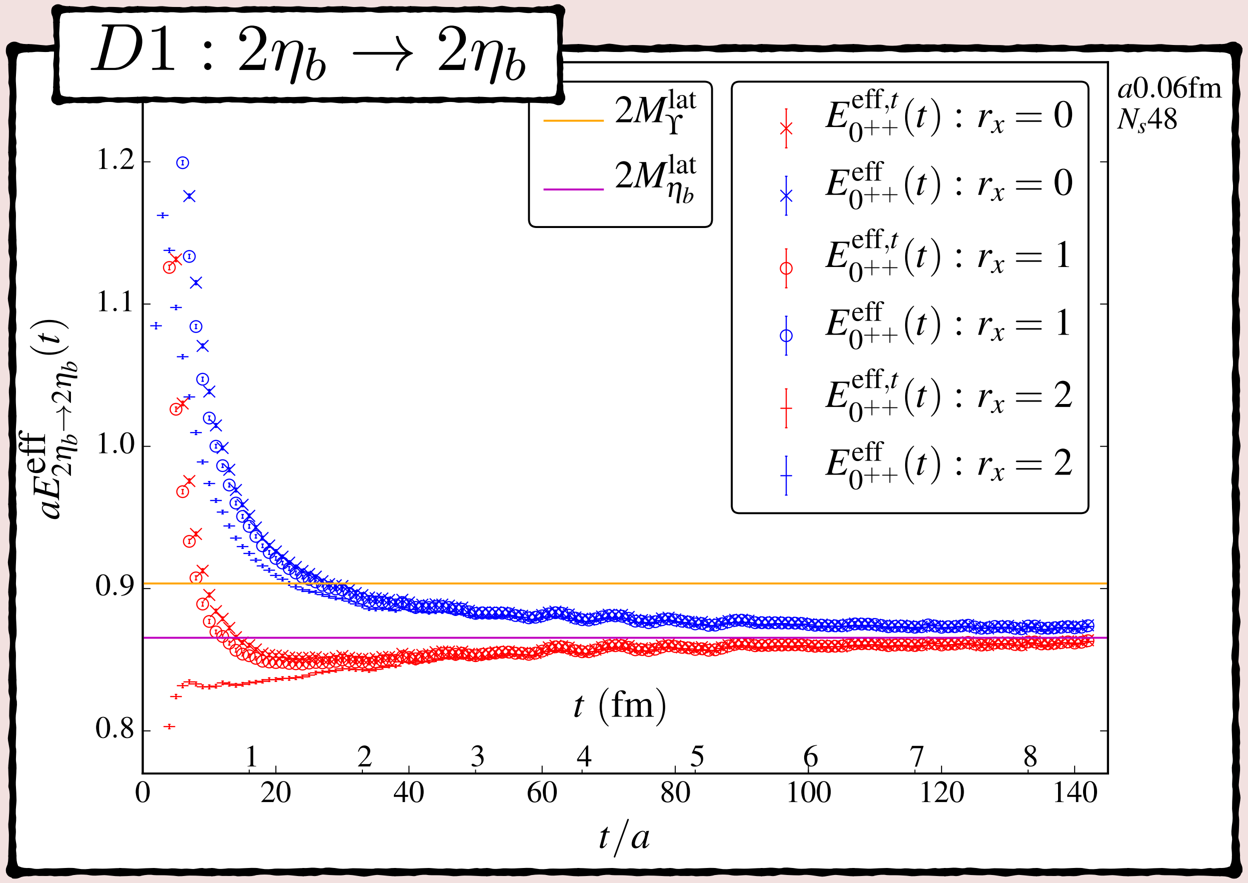
Two-Meson Wick Contractions

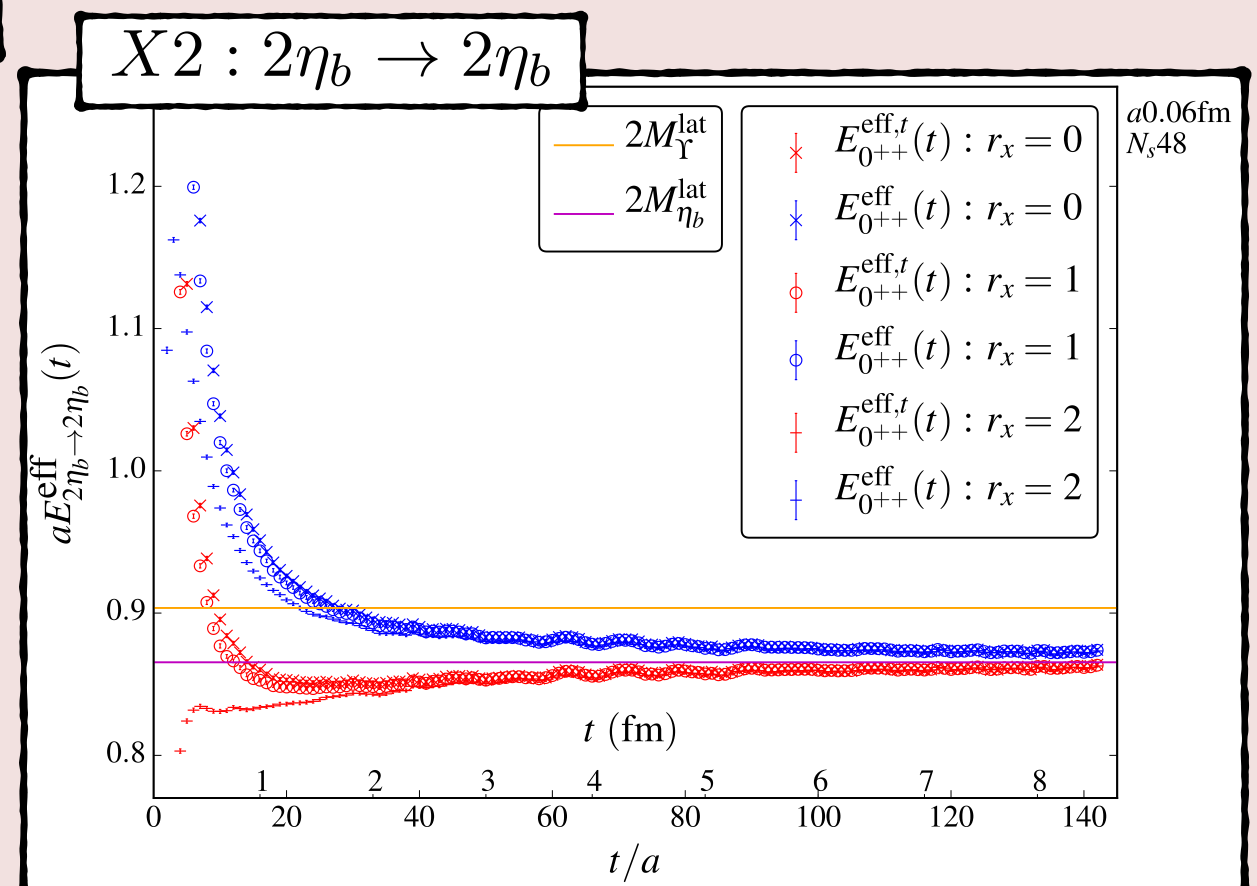


Diquark-Antidiquark Wick Contractions

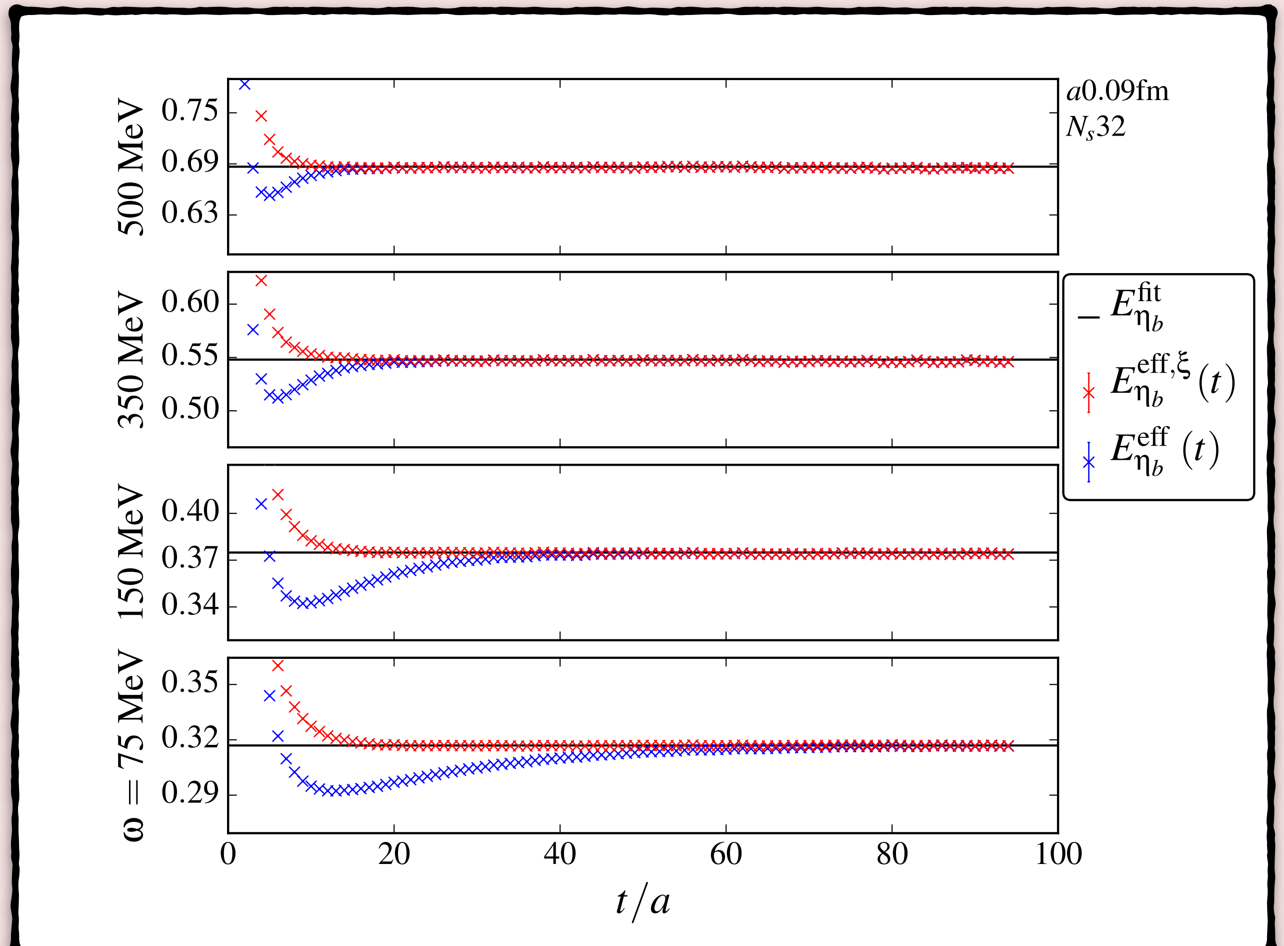


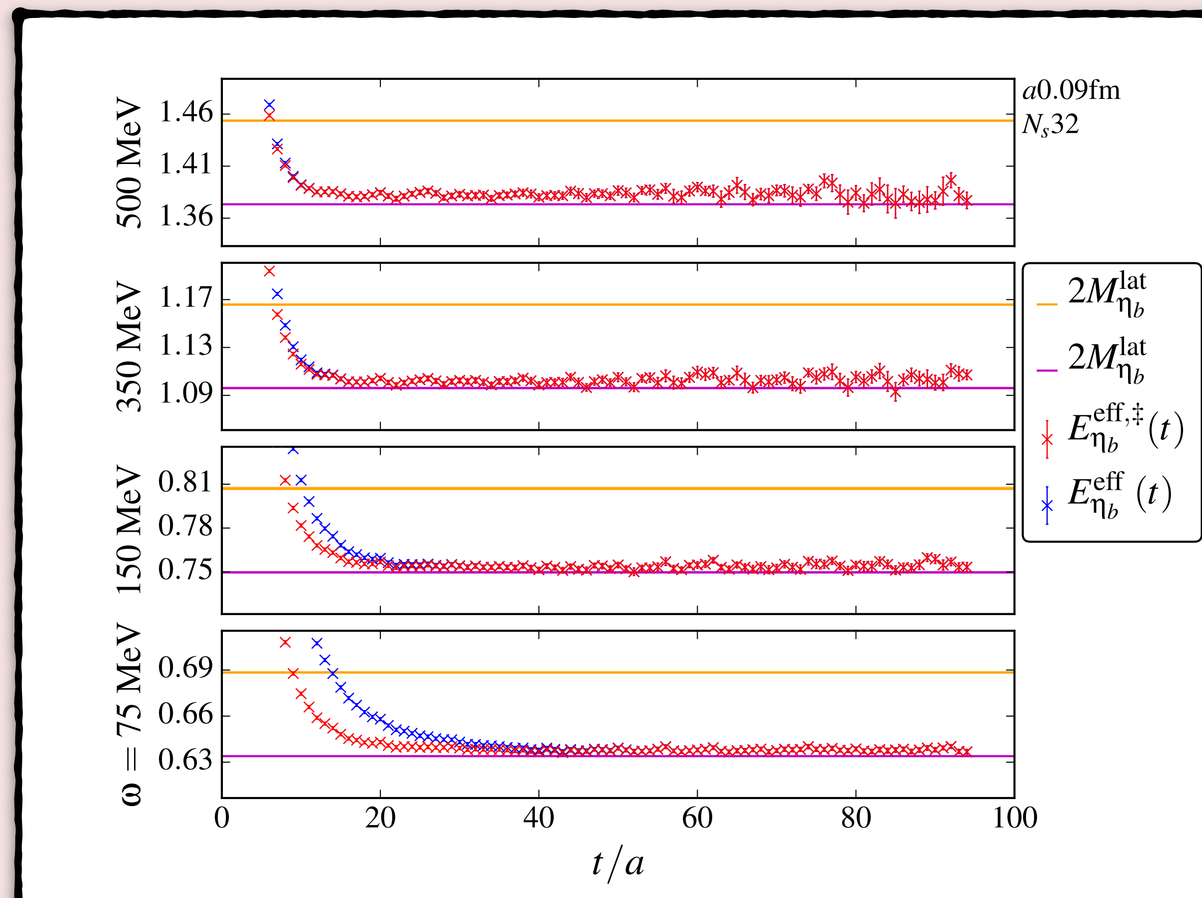
Individual Wick Contraction Correlator Data





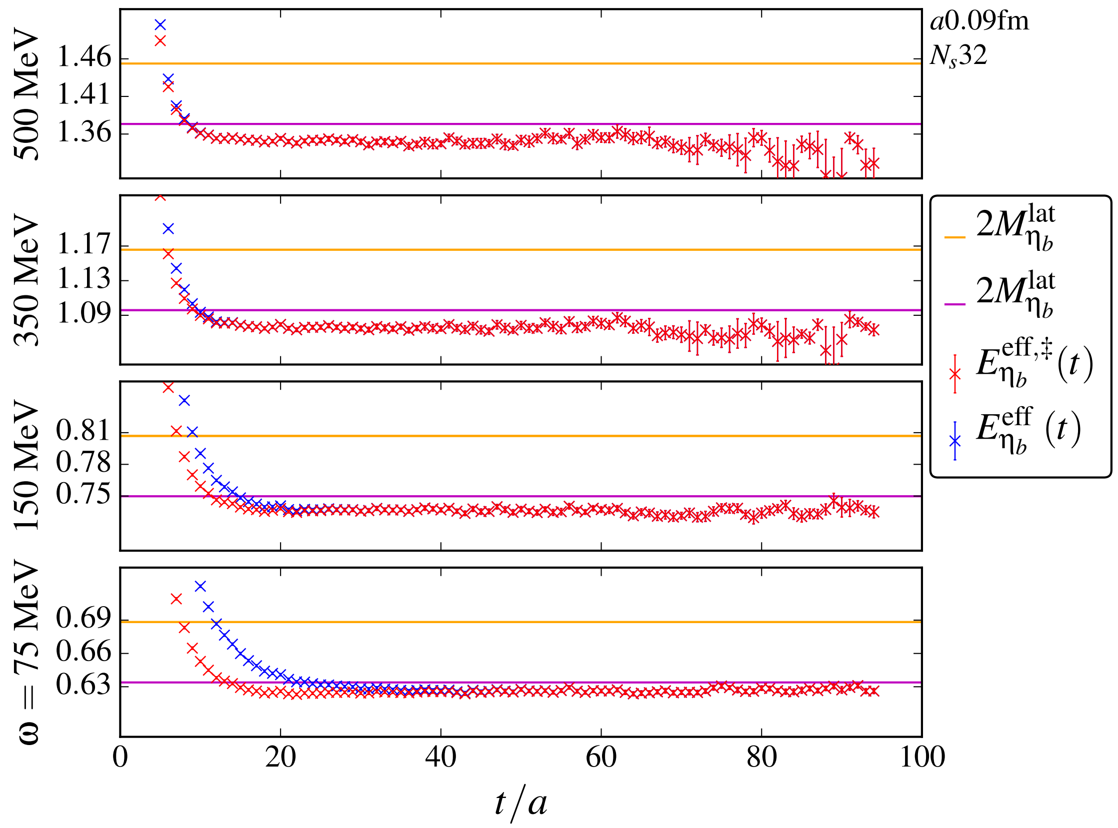
Correlator Data With Harmonic Oscillator





Individual Wick Contraction Correlator Data HO

$D1 : 2\eta_b \to 2\eta_b$



$X2 : 2\eta_b \to 2\eta_b$