Explore hadronic matrix elements in the timelike region: from HVPs to rare kaon decays

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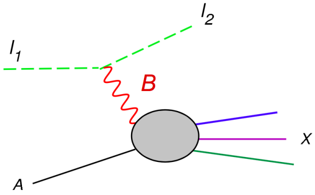
Workshop on Parton Distributions in Modern Era, Beijing, 07/14/2017

Introduction

Main topic of this workshop is DIS and PDF

- Lepton-hadron scattering process

$$\ell_1 + N(P) \to \ell_2 + X(P_X)$$



(B = g, W, Z)

- Hadronic current amplitude unkown

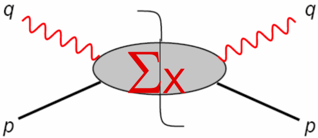
$$J^*_\mu(P,q) = \langle P_X | J^\dagger_\mu | P \rangle$$

Object of study:

- Probe the parton structure of the nucleon
- Study QCD dynamics at the confinement scale

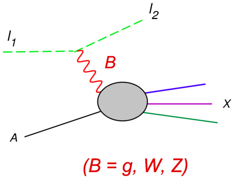
Hadronic tensor from cross section

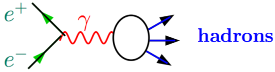
$$W_{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(P + q - P_X) \langle P | J_\mu | P_X \rangle \langle P_X | J^\dagger_\nu | P \rangle$$



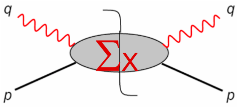
Start with a simpler case

Lepton-hadron scattering → e^+e^- scattering





Hadronic tensor for DIS → Hadronic vacuum polarization (HVP)



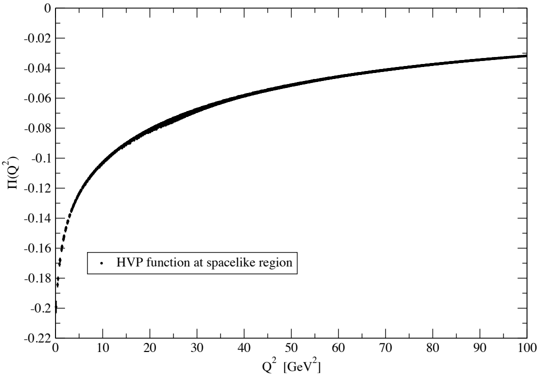
Hadronic vacuum polarization function

Hadronic vacuum polarization function

$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQx}\langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle \quad \Leftarrow$$

On lattice

- Discrete Fourier transform assigns the discrete momentum $Q_i = n_i(2\pi/L)$
- $q^2 = -\sum_i Q_i^2 < 0 \quad \Rightarrow \quad$ always spacelike



Analytic continuation method

Non-QCD state: photon is a superposition of a complete set of hadron states

- Analytic continuation method proposed by Ji & Jung, PRL 86 (2001) 208

$$\int dt\, e^{\omega t} \int d^3\vec{x}\, e^{i\vec{q}\vec{x}} \langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle = \int dt\, e^{\omega t} C_{\mu\nu}(\vec{q}, t)$$

- ω is photon energy, input by hand, thus can be tuned continuously

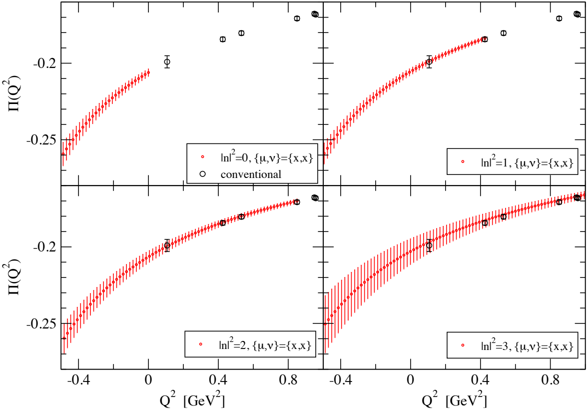
- Worry about $e^{\omega t}$ divergent for large t?
  $\Rightarrow$ $\langle 0|T\{J_\mu(t)J_\nu(0)\}|0\rangle$ exponentially decreases as $e^{-E_{\pi\pi}t}$

Subtract $\Pi(0)$ without Q^2 extrapolation

$$\Pi(-\omega^2) - \Pi(0) = -\int dt \left[\frac{e^{\omega t} - 1}{\omega^2} - \frac{t^2}{2}\right] C_{zz}(\vec{0}, t)$$

Tuning photon's energy ω

- Cover both spacelike and timelike
- Determine $\Pi(Q^2)$ at $Q^2 = 0$



XF, Hashimoto, Hotzel, Jansen, et al, PRD 88 (2013) 034505

Demonstration: Minkowski spacetime

- In Minkowski spacetime, in the framework of QCD+QED

$$\langle \gamma(k,\lambda)|J^M_\nu(0)|\Omega\rangle$$

- Using the LSZ reduction formula

$$\langle \gamma(k,\lambda)|J^M_\nu(0)|\Omega\rangle = i \lim_{k'\to k} \varepsilon^*_\mu(k,\lambda)\ k'^2 \int d^4x\ e^{-ik'x} \langle \Omega|\mathrm{T}\{A^M_\mu(x) J^M_\nu(0)\}|\Omega\rangle$$

where $A^M_\mu(x)$ is a photon field defined in Minkowski spacetime

- Treat QED part perturbatively:

$$\exp(iS_{\mathrm{int}}) = 1 + iS_{\mathrm{int}} + \cdots, \quad S_{\mathrm{int}} = e \int d^4x\ A^M_\mu(x) J^M_\mu(x)$$

- At $O(e)$ we have

$$\langle \gamma(k,\lambda)|J^M_\nu(0)|\Omega\rangle = i\ e\ \varepsilon^*_\mu(k,\lambda) \int d^4y\ e^{-iky} \langle \Omega|\mathrm{T}\{J^M_\mu(y) J^M_\nu(0)\}|\Omega\rangle$$

Demonstration: Euclidean spacetime

- In Euclidean spacetime, look at

$$\begin{aligned} M_{\mu\nu}(t_0,\vec{k}) &= \int d^3\vec{x}\; e^{-i\vec{k}\vec{x}}\langle\Omega|A^E_\mu(\vec{x},t_0)J^E_\nu(0)|\Omega\rangle \\ &\xrightarrow{t_0\to\infty} \sum_\lambda \langle\Omega|A^E_\mu(\vec{k})|\gamma(k,\lambda)\rangle e^{-\omega t_0}\langle\gamma(k,\lambda)|J^E_\nu(0)|\Omega\rangle \end{aligned}$$

  - $\omega < E_V$: neglect hadronic states, suppressed by $e^{-(E_V-\omega)t_0}$
  - Neglect three-photon states, suppressed by powers of QED coupling

- Employ pQED

$$M_{\mu\nu}(t_0,\vec{k}) = e\int dt \int d^3\vec{y}\; e^{-i\vec{k}\vec{y}} D_{\mu\rho}(\vec{k},t_0-t)\langle\Omega|J^E_\rho(\vec{y},t)J^E_\nu(0)|\Omega\rangle$$

- Photon propagator $\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}D_{\mu\nu}(k)e^{-ik_0t} = \int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{\delta_{\mu\nu}}{k_0^2+\vec{k}^2}e^{-ik_0t} = \frac{e^{-\omega|t|}}{2\omega}\delta_{\mu\nu}$

- We can write

$$M_{\mu\nu}(t_0,\vec{k}) \sim e\int dt\; e^{-\omega|t_0-t|}\int d^3\vec{y}\; e^{-i\vec{k}\vec{y}}\langle\Omega|J^E_\mu(\vec{y},t)J^E_\nu(0)|\Omega\rangle$$

Demonstration of the analytic continuation

- We get in Euclidean spacetime

$$\langle\gamma(k,\lambda)|J_\nu^E(0)|\Omega\rangle = e\,\varepsilon_\mu^*(\lambda)\int dt\, e^{\omega t}\int d^3\vec{y}\; e^{-i\vec{k}\vec{y}}\langle\Omega|J_\mu^E(\vec{y},t)J_\nu^E(0)|\Omega\rangle$$

- Remember that in Minkowski spacetime

$$\langle\gamma(k,\lambda)|J_\nu^M(0)|\Omega\rangle = i\,e\,\varepsilon_\mu^*(k,\lambda)\int d^4y\; e^{-iky}\langle\Omega|\mathrm{T}\{J_\mu^M(y)J_\nu^M(0)\}|\Omega\rangle$$

Thus the validity of analytic continuation method is demonstrated

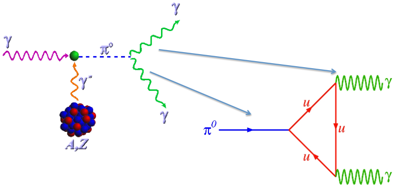
- This method is originally proposed by [Ji & Jung, PRL 86 (2001) 208, PRD 64 (2001) 034506]
- Besides for HVPs, analytic continuation has also been used for lat. calc. of charmonium radiative decay and $\pi^0 \to \gamma\gamma$

  Dudek & Edwards, PRL 97 (2006) 172001
  Cohen, Lin, Dudek, Edwards, PoS LAT (2008) 159
  XF, Aoki, Fukaya, Hashimoto et.al. PRL 109 (2012) 182001
  CLQCD, Chen et. al, EPJC 76 (2016) 358
  ...

Chiral anomaly at $m_u = m_d = 0$

Lattice QCD at $m_u = m_d \neq 0$

$\Rightarrow$ non-trivial test of chiral anomaly



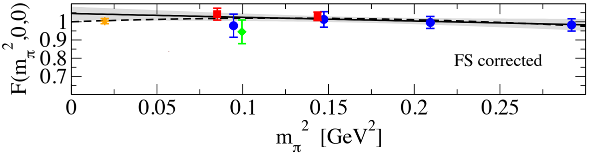
Non-QCD final state: $\langle \gamma(p_1, \lambda_1)\gamma(p_2, \lambda_2)|\pi^0(q)\rangle \Rightarrow$

$$\int d^4x\, e^{ip_1x} \langle 0|T\{J_\mu(x)J_\nu(0)\}|\pi^0(q)\rangle = \epsilon_{\mu\nu\alpha\beta}\, p_1^\alpha p_2^\beta F_{\pi^0\gamma\gamma}(m_\pi^2, p_1^2, p_2^2)$$

Analytic continuation:

$$\int dt\, e^{\omega t} \int d^3\vec{x}\, e^{i\vec{p}_1\vec{x}} \langle 0|T\{J_\mu(x)J_\nu(0)\}|\pi^0(q)\rangle$$

Lattice result for $F_{\pi^0\gamma\gamma}$, using overlap chiral fermion

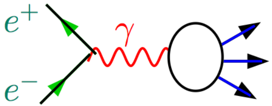


XF, Aoki, Fukaya, Hashimoto et.al. PRL 109 (2012) 182001

Explore the timelike region, above hadron production threshold

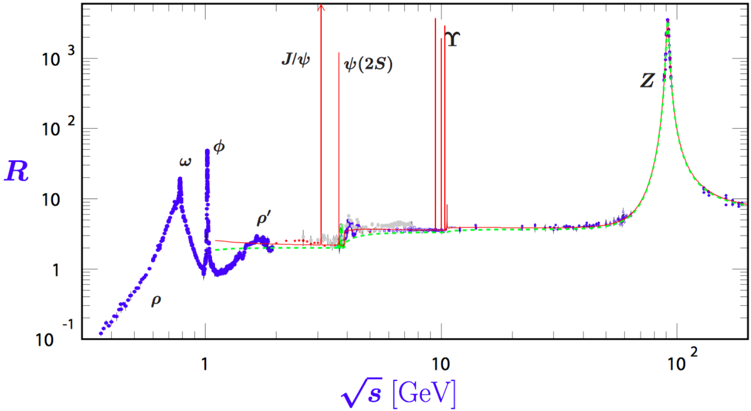
Highly non-trivial above hadron production threshold

BES III **experiment @BEPC**



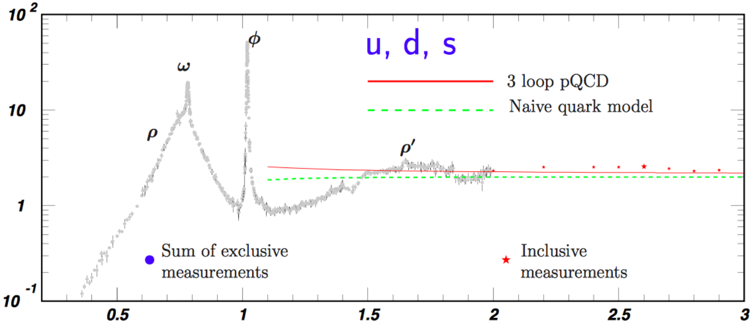
- Provide rich experimental data for low energy hadron physics

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



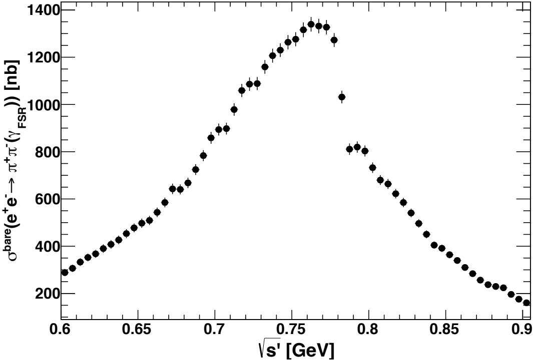
Require nonperturbative lattice QCD

In low energy region



- QCD asympototic freedom ⇒ pQCD fails in the low energy region
- QCD color confinement ⇒ hard to explain hadron spectrum e.g. resonance, bound state
- We need nonperturbative treatment – lattice QCD
  - Solve QCD from first-principals, don't rely on model assumption

$\sigma(e^+e^- \to \pi^+\pi^-)$ at 0.6 - 0.9 GeV, from [BES III, Phys. Lett. B753 (2016) 629]



- $\sqrt{s} < 0.9$ GeV, hadronic final state dominated by $\pi^+\pi^-$
- ρ resonant peak at 0.77 GeV
  - No essential difference for $e^+e^- \to \pi^+\pi^-$ at 0.6 GeV and 0.77 GeV
  - ρ resonance is a dynamical property for $\pi\pi$ scattering at 0.77 GeV

Timelike pion form factor

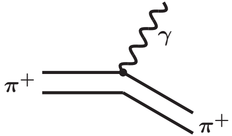
- From $\sigma(e^+e^- \to \pi^+\pi^-)$, timelike pion form factor can be extracted

$$\sigma(e^+e^- \to \pi^+\pi^-) = \sigma^0(e^+e^- \to \pi^+\pi^-)|F_\pi(s)|^2$$

  - σ^0: tree-level cross section by assuming $\pi^\pm$ as a point-like particle
  - $|F_\pi(s)|^2$ describes the internal EM structure for π

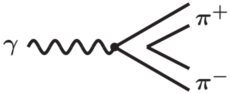
$\pi\pi$ scattering, ρ resonance, timelike pion form factor $\Rightarrow$ three in one

Spacelike form factor from $\langle\pi|J_\mu|\pi\rangle$



$\Rightarrow$ single hadron in i/f state, $F_\pi(q^2)$ is a real function, LQCD works well

Timelike form factor from $\langle\pi\pi|J_\mu|0\rangle$



$\Rightarrow$ multi-hadron in final state, $F_\pi(q^2)$ is complex, difficult for LQCD

$\pi\pi$ correlation function

Euclidean correlation functions are the primary objects in Lattice QCD

- Euclidean time $t \to it$, operator $O(t) = e^{Ht} O(0) e^{-Ht}$
- Single pion correlation function: $\pi(t) = \sum_{\mathbf{x}} \bar{u}\gamma_5 d(\mathbf{x}, t)$

$$C_\pi(t) = \langle 0|\pi^\dagger(t)\pi(0)|0\rangle = \sum_n e^{-E_\pi^n t} |\langle n|\pi|0\rangle|^2$$

  - At large t, only ground state $|n\rangle = |\pi\rangle$ saturates

- Two-pion correlation function

$$C_{\pi\pi}(t) = \langle 0|(\pi\pi)^\dagger(t)(\pi\pi)(0)|0\rangle = \sum_n e^{-E_{\pi\pi}^n t} |\langle \pi\pi, n|\pi\pi|0\rangle|^2$$

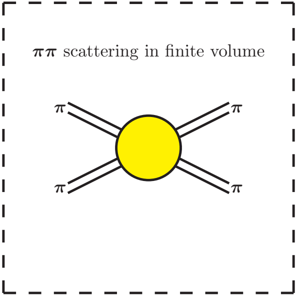
  - Lattice size is finite $\sim$ a few fm $\Rightarrow$ discrete $\pi\pi$ spectrum

What information can we obtain from discrete $E_{\pi\pi}$?

- If no interaction between $\pi\pi$, $E_{\pi\pi}$ is simply determined by lattice size L

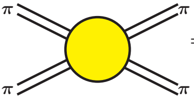
$$E_{\pi\pi}^{\text{free}} = \sqrt{m_\pi^2 + \vec{p}_1^{\,2}} + \sqrt{m_\pi^2 + \vec{p}_2^{\,2}}\,, \qquad \vec{p}_{1,2} = \frac{2\pi}{L}\vec{n}_{1,2}$$

- With interaction, $E_{\pi\pi}$ differs from $E_{\pi\pi}^{\text{free}}$
  - $\triangle E = E_{\pi\pi}^{n} - E_{\pi\pi}^{\text{free}} \neq 0$ contains the information of QCD interaction



$\Rightarrow \quad E_{\pi\pi}$

$\pi\pi$ scattering in infinite volume



$\Rightarrow \langle \pi\pi, \text{out} | \pi\pi, \text{in} \rangle = e^{2i\delta}$

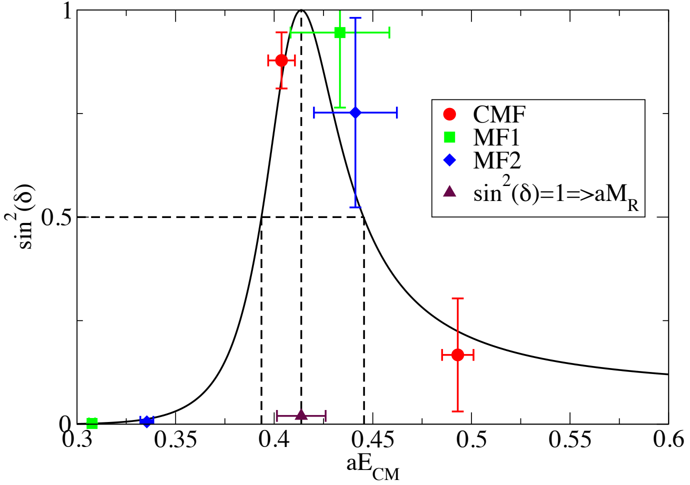
- Lüscher establish a relation between $E_{\pi\pi}$ and $\delta(E)$

$$\delta(k) = n\pi - \phi(q), \quad n \in \mathbb{Z}, \quad q = \frac{k}{2\pi/L}, \quad E_{\pi\pi} = 2\sqrt{m_\pi^2 + k^2}$$

Lattice results for the $I = 1$, P-wave $\pi\pi$ scattering phase

use **3** Lorentz frames; at each frame, calculate lowest **2** $E_{\pi\pi}$

$\Rightarrow$ 6 scattering phase to map out the resonance region



XF, K. Jansen, D. Renner, PRD83 (2011) 094505

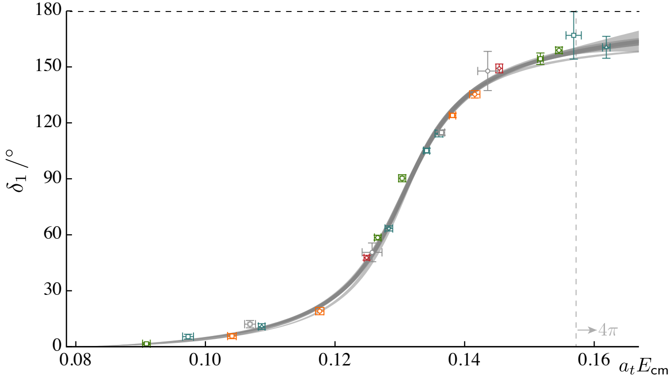
(one of the early work on $\rho \to \pi\pi$)

Recent lattice results

Sophisticated calculation of $I = 1$ $\pi\pi$ scattering at $m_\pi = 236$ MeV

[Hadron Spectrum Collaboration, 1507.02599]

- 1) distillation smearing 2) variational method
- 3) anisotropic lattice $a_t = a_s/3.5$ 4) multi-channel approach



Matrix element with multi-hadron state

$\langle \pi(\vec{p})\pi(-\vec{p})|J_\mu|\Omega\rangle$, final state involves multi-hadrons

Maiani-Testa no-go theorem '91:
LQCD can't explore the transition involving multi-hadrons

- Lattice results → physical ones, requiring lattice size $L \to \infty$
- For a large, quasi-infinite volume, $\vec{p} = \frac{2\pi}{L}\vec{n} \to 0$
- Ground state dominance at large t leads to

$$|\pi(\vec{p})\pi(-\vec{p})\rangle \Rightarrow |\pi(\vec{0})\pi(\vec{0})\rangle \Rightarrow A_{\gamma^* \to \pi\pi}\Big|_{E_{\pi\pi}=2m_\pi}$$

- Dense discrete energy level → hard to extract excited state
- Only information of $A_{\gamma^* \to \pi\pi}$ at $E_{\pi\pi} = 2m_\pi$

Solution

- We must work in the finite volume [Lellouch, Lüscher, '00]
- Finite-size correction must be treated properly

Finite volume approach (I)

$$\int_V d^3x \, \langle 0|J_\mu(x,t)J_\mu(0)|0\rangle_V = \sum_n |\langle \pi\pi, n|J_\mu|0\rangle_V|^2 e^{-E^n_{\pi\pi} t} + \cdots$$

From the correlation function of vector current operator J_μ, we can get

- $E^n_{\pi\pi}$: can be used to determine δ
- $|\langle \pi\pi, n|J_\mu|0\rangle_V|^2$: What information can we obtain?

Recall Lüscher's formula

$$\phi(q) + \delta(k) = n\pi$$

- $V \to \infty$, finite volume sums $\to$ infinite volume integrals
  [Lin, Martinelli, Sachrajda, Testa, '01]

$$\sum_n \to \int dE \, \rho_V(E), \quad \rho_V(E) = \frac{dn}{dE} = \frac{q\phi'(q) + k\delta'(k)}{4\pi k^2} E$$

$\rho_V(E)$ can be viewed as the density of states at energy E

Finite volume approach (II)

- Finite volume

$$\int_V d^3x \, \langle 0|J_\mu(x,t)J_\mu(0)|0\rangle_V \xrightarrow{V\to\infty} \int dE\, \rho_V(E) |\langle \pi\pi, E|J_\mu|0\rangle_V|^2 e^{-Et}$$

- Infinite volume: spectral representation for vector-current correlator

$$C_\infty(t) = \int d^3x \, \langle 0|J_\mu(x,t)J_\mu(0)|0\rangle = \int_{2m_\pi}^{\infty} dE\, f(E) e^{-Et}$$

$$f(E) = \frac{1}{6\pi^2}\frac{k^3}{E}|F_\pi(s)|^2, \quad s = E^2 = 4(m_\pi^2 + k^2)$$

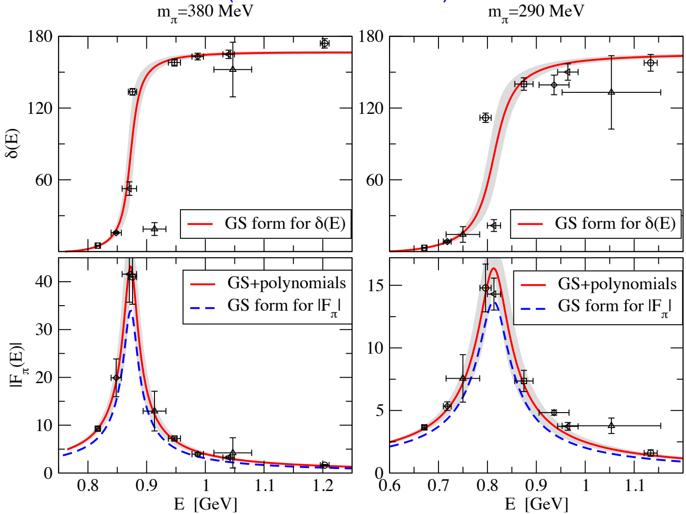
- Final relation

$$|F_\pi(s)|^2 = \frac{3\pi s}{2k^5}\left(q\frac{\partial\phi(q)}{\partial q} + k\frac{\partial\delta(k)}{\partial k}\right)|\langle \pi\pi, E|J_\mu|0\rangle_V|^2$$

- Result is correct, but demonstration is not very satisfactory
  - Equivalent integral does not mean equivalent integrand
  - Integral $\int_{2m_\pi}^{\infty}$ runs over inelastic region
- New demonstration [Christ, XF, Martinelli, Sachrajda, '15]

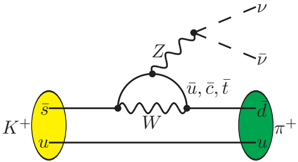
Timelike pion form factor

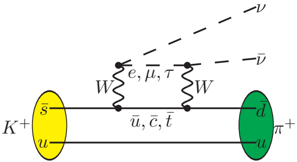
$$|F_\pi(s)|^2 = \frac{3\pi s}{2k^5}\left(q\frac{\partial\phi(q)}{\partial q} + k\frac{\partial\delta(k)}{\partial k}\right)|\langle\pi\pi, E|J_\mu|0\rangle_V|^2$$



XF, Aoki, Hashimoto, Kaneko, PRD91 (2015) 054504

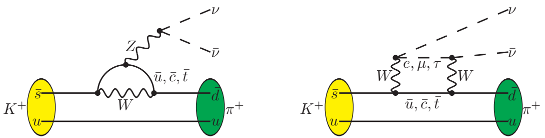
Rare Kaon decays





$$\underbrace{\langle \pi\nu\bar{\nu}|H_W(x)\,H_W(0)|K\rangle}_{\text{Rare Kaon decay}} \quad \text{vs} \quad \underbrace{\langle N(P)|J_\mu(x)\,J_\nu^\dagger(0)|N(P)\rangle}_{\text{Deep inelastic scattering}}$$

$K^+ \to \pi^+ \nu\bar{\nu}$: Experiment vs Standard model



Z-exchange

W-W

$K^+ \to \pi^+ \nu\bar{\nu}$: largest contribution from top quark loop, thus theoretically clean

$$\mathcal{H}_{eff} \sim \frac{G_F}{\sqrt{2}} \cdot \underbrace{\frac{\alpha_{\rm EM}}{2\pi \sin^2\theta_W} \lambda_t X_t(x_t)}_{\mathcal{N} \sim 2\times10^{-5}} \cdot (\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A}$$

Probe the new physics at scales of $\mathcal{N}^{-\frac{1}{2}} M_W = O(10\ \text{TeV})$

Past experimental measurement is 2 times larger than SM prediction

$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} \qquad \text{[BNL E949, '08]}$$

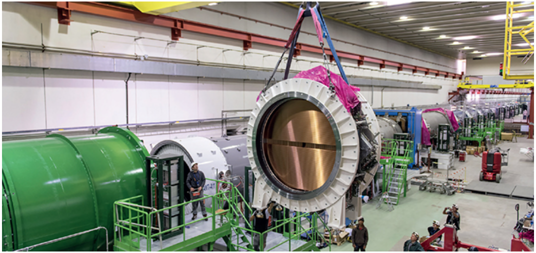
$$\text{Br}(K^+ \to \pi^+ \nu\bar{\nu})_{\text{SM}} = 9.11 \pm 0.72 \times 10^{-11} \qquad \text{[Buras et. al., '15]}$$

but still consistent with > 60% exp. error

New experiment

New generation of experiment: NA62 at CERN

- aims at observation of $O(100)$ events [2014-2018]
- 10%-precision measurement of $\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})$



Latest results reported at FPCP 2017

- Detector installation completed in 09.2016
- 5% of 2016 data $\Rightarrow$ no event yet
- Full 2016 data $\Rightarrow$ $O(1)$ events

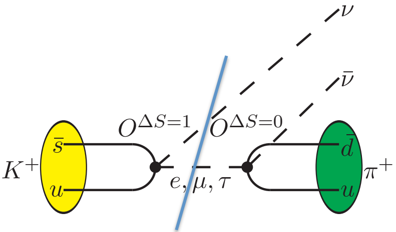
2^{nd}-order weak interaction and bilocal matrix element

Hadronic matrix element for the 2^{nd}-order weak interaction

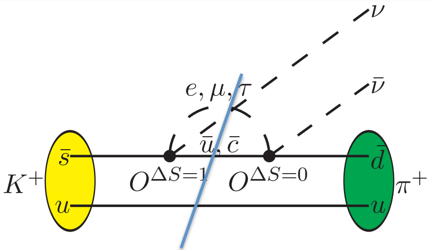
$$\int_{-T}^{T} dt\, \langle \pi^+ \nu\bar{\nu} | T\left[Q_A(t) Q_B(0)\right] | K^+ \rangle$$
$$= \sum_n \left\{ \frac{\langle \pi^+ \nu\bar{\nu} | Q_A | n\rangle \langle n | Q_B | K^+\rangle}{M_K - E_n} + \frac{\langle \pi^+ \nu\bar{\nu} | Q_B | n\rangle \langle n | Q_A | K^+\rangle}{M_K - E_n} \right\} \left(1 - e^{(M_K - E_n)T}\right)$$

- For $E_n > M_K$, the exponential terms exponentially vanish at large T
- For $E_n < M_K$, the exponentially growing terms must be removed
- $\sum_n$: principal part of the integral replaced by finite-volume summation
  - possible large finite volume correction when $E_n \to M_K$

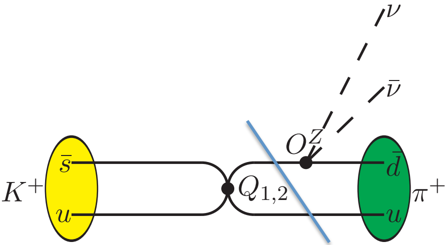
[Christ, XF, Martinelli, Sachrajda, PRD 91 (2015) 114510]



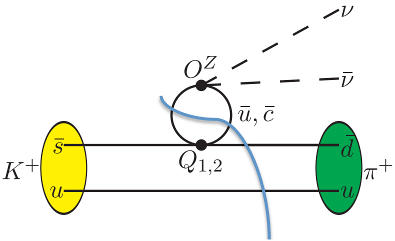
$$K^+ \to \ell^+\nu \quad \& \quad \ell^+ \to \pi^+\bar{\nu}$$



$$K^+ \to \pi^0\ell^+\nu \quad \& \quad \pi^0\ell^+ \to \pi^+\bar{\nu}$$



$$K^+ \xrightarrow{H_W} \pi^+ \quad \& \quad \pi^+ \xrightarrow{V_\mu} \pi^+$$

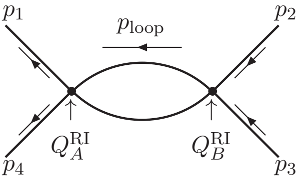
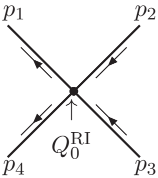


$$K^+ \xrightarrow{H_W} \pi^+\pi^0 \quad \& \quad \pi^+\pi^0 \xrightarrow{A_\mu} \pi^+$$

[Christ, XF, Portelli, Sachrajda, PRD 93 (2016) 114517]

SD divergence appears in $Q_A(x)Q_B(0)$ when $x \to 0$

- Introduce a counter term $X \cdot Q_0$ to remove the SD divergence

$\langle\{Q_A Q_B\}^{\mathrm{RI}}\rangle\Big|_{p_i^2=\mu_0^2} =$  $- \; X(\mu_0, a)\times$  $= \; 0$

The coefficient X is determined in the RI/(S)MOM scheme

- The bilocal operator in the $\overline{\mathrm{MS}}$ scheme can be written as

$$\left\{\int d^4x\, T[Q_A^{\overline{\mathrm{MS}}}(x)\, Q_B^{\overline{\mathrm{MS}}}(0)]\right\}^{\overline{\mathrm{MS}}}$$

$$= Z_A Z_B \left\{\int d^4x\, T[Q_A^{\mathrm{lat}}\, Q_B^{\mathrm{lat}}]\right\}^{\mathrm{lat}} + \left(-X^{\mathrm{lat}\to\mathrm{RI}} + Y^{\mathrm{RI}\to\overline{\mathrm{MS}}}\right) Q_0(0)$$

  - $X^{\mathrm{lat}\to\mathrm{RI}}$ is calculated using NPR and $Y^{\mathrm{RI}\to\overline{\mathrm{MS}}}$ calculated using PT

Results for charm quark contribution

Charm quark contribution P_c

$$P_c = P_c^{\mathrm{SD}} + \delta P_{c,u}$$

NNLO QCD [Buras, Gorbahn, Haisch, Nierste, JHEP 0611 (2006) 002]:

$$P_c^{\mathrm{SD}} = 0.365(12)$$

Phenomenological ansatz [Isidori, Mescia, Smith, NPB 718 (2005) 319]

$$\delta P_{c,u} = 0.040(20)$$

Lattice results @$m_\pi = 420$ MeV, $m_c = 860$ MeV

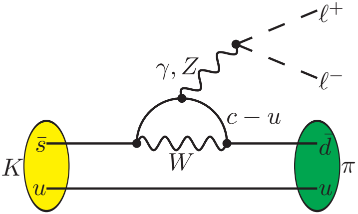
[Bai, Christ, XF, Lawson, Portelli, Sachrajda, PRL 118 (2017) 252001]

$$P_c = 0.2529(\pm 13)_{\mathrm{stat}}(\pm 32)_{\mathrm{scale}}(-45)_{\mathrm{FV}}$$
$$P_c - P_c^{\mathrm{SD}} = 0.0040(\pm 13)_{\mathrm{stat}}(\pm 32)_{\mathrm{scale}}(-45)_{\mathrm{FV}}$$

- As a smaller m_c is used, P_c is also smaller
- Cancellation in W-W and Z-exchange diag. leads to small $P_c - P_c^{\mathrm{SD}}$
- Important to perform the calculation at physical m_π and m_c

CP conserving decay: $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$

- Involve both γ- and Z-exchange diagram, but γ-exchange is much larger



- Unlike Z-exchange, the γ-exchange diagram is LD dominated
  - By power counting, loop integral is quadratically UV divergent
  - EM gauge invariance reduces divergence to logarithmic
  - $c - u$ GIM cancellation further reduces log divergence to be UV finite

Lattice calculation strategy

Focus on γ-exchange

- Hadronic part of decay amplitude is described by a form factor

$$\begin{aligned} T^{\mu}_{+,S}(p_K,p_\pi) &= \int d^4x\, e^{iqx}\langle \pi(p_\pi)|T\{J^{\mu}_{em}(x)\mathcal{H}^{\Delta S=1}(0)\}|K^+/K_S(p_K)\rangle \\ &= \frac{G_F M_K^2}{(4\pi)^2} V_{+,S}(z)\left[z(k+p)^\mu - (1-r_\pi^2)q^\mu\right] \end{aligned}$$

with $q = p_K - p_\pi$, $z = q^2/M_K^2$, $r_\pi = M_\pi/M_K$

The target for lattice QCD is to calculate the form factor $V_{+,S}(z)$

- Lattice calculation strategy:

  [Christ, XF, Portelli, Sachrajda, PRD 92 (2015) 094512]

  - Use conserved vector current to protect the EM gauge invariance
  - Use charm as an active quark flavor to maintain GIM cancellation

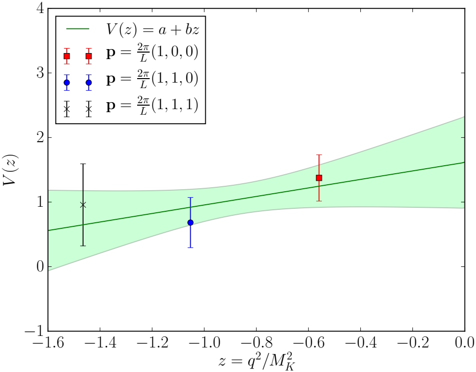
First exploratory calculation on $K^+ \to \pi^+ \ell^+ \ell^-$

Use $24^3 \times 64$ ensemble, $N_{\text{conf}} = 128$
[Christ, XF, Jütter, Lawson, Portelli, Sachrajda, PRD 94 (2016) 114516]

$a^{-1} = 1.78$ GeV, $m_\pi = 430$ MeV
$m_K = 625$ MeV, $m_c = 530$ MeV

Momentum dependence of $V_+(z)$

$$V_+(z) = a_+ + b_+ z$$
$$\Rightarrow \quad a_+ = 1.6(7), \; b_+ = 0.7(8)$$



$K^+ \to \pi^+ e^+ e^-$ data + phenomenological analysis: $a_+ = -0.58(2)$, $b_+ = -0.78(7)$
[Cirigliano, et. al., Rev. Mod. Phys. 84 (2012) 399]

$$V_j(z) = a_j + b_j z + \underbrace{\frac{\alpha_j r_\pi^2 + \beta_j (z - z_0)}{G_F M_K^2 r_\pi^4}}_{K \to \pi\pi\pi} \underbrace{\left[1 + \frac{z}{r_V^2}\right]}_{F_V(z)} \underbrace{\left[\phi(z/r_\pi^2) + \frac{1}{6}\right]}_{\text{loop}}, \quad j = +, S$$

- Experimental data only provide $\frac{d\Gamma}{dz} \Rightarrow$ square of form factor $|V_+(z)|^2$
- Need phenomenological knowledge to determine the sign for a_+, b_+

Summary

- HVPs at spacelike momenta
  $\Rightarrow$ standard lattice QCD calculation
- Ji & Jung's analytic continuation method
  $\Rightarrow$ allows to probe the region below hadron production threshold
- Lüscher's finite-volume method
  $\Rightarrow$ allows to probe the region above hadron production threshold
  - Timelike pion form factor
  - Rare Kaon decays
- For "Deep" inelastic scattering
  $\Rightarrow$ a challenging task as highly above hadron production threshold
  - Keh-Fei's talk on hadronic tensor
  - Huey-Wen's talk on quasi-PDF