Renormalization of quasi-PDFs

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Based on works done with Tomomi Ishikawa, Jian-Wei Qiu and Shinsuke Yoshida

Parton Distributions in Modern Era
PKU, Beijing, Jul. 15th, 2017

Outline

I. Introduction

II. One-loop calculation

III. Power counting

IV. Renormalization

V. Summary

PDFs

➢ **Spin-averaged quark distribution**

$$f_{q/p}(x,\mu^2) = \int \frac{d\xi_-}{4\pi} e^{-ix\xi_- P_+} \langle P | \overline{\psi}(\xi_-)\, \gamma_+ \exp\left\{ -ig \int_0^{\xi_-} d\eta_- A_+(\eta_-) \right\} \psi(0) | P \rangle$$

- Important quantity in QCD

Crucial to determine PDFs nonperturbatively from first principle

General method

- Good "Lattice cross section": $\tilde{\sigma}_{\mathrm{E}}^{\mathrm{Lat}}(\tilde{x}, 1/a, P_z)$
  - ① Calculable on Euclidean lattice QCD YQM, Qiu, 1404.6860, 1412.2688
  - ② UV and IR safe perturbatively (renormalizable)
  - ③ CO divergence: factorizable (similar to DIS cross section)

$$\tilde{\sigma}_{\mathrm{M}}(\tilde{x}, \tilde{\mu}^2, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_{i/h}(x, \mu^2)\, \mathcal{C}_i\left(\frac{\tilde{x}}{x}, \tilde{\mu}^2, \mu^2, P_z\right) + O(1/\tilde{\mu}^2)$$

- The last condition relates "Lattice cross section" to PDFs

- Candidates of good "lattice cross section"
  - Quasi-PDFs Ji, 2013
  - Pseudo-PDFs Radyushkin, 2017
  - Conserved currents Liu et. al.
  - …

Coordinate-space quasi-quark PDFs

➢ Coordinate space definition

$$\tilde{F}_{q/p}(\xi_z, \tilde{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z} \langle h(p)|\overline{\psi}_q(\xi_z)\, \frac{\gamma_z}{2} \Phi^{(f)}_{n_z}(\{\xi_z, 0\})\, \psi_q(0)|h(p)\rangle$$

- **Momentum-space quasi-PDFs: hard to well define**

➢ Conjecture of all-orders renormalization

$$\tilde{F}^R_{i/p}(\xi_z, \tilde{\mu}^2, p_z) = e^{-C_i|\xi_z|} Z^{-1}_{wi} Z^{-1}_{vi}\, \tilde{F}^b_{i/p}(\xi_z, \tilde{\mu}^2, p_z)$$

Ishikawa, YQM, Qiu, Yoshida, 1609.02018
Chen, Ji, Zhang, 1609.08102
Constantinou, H. Panagopoulos, 1705.11193
…

- **Rigorous proof is needed!**

Proof: Importance and difficulty

➢ Why proof is important?

- All-order proof of factorization needs multiplicative renormalization   YQM, Qiu, 1404.6860, 1412.2688
- Whether mixing with other operators under renormalization? A close set of operators are needed

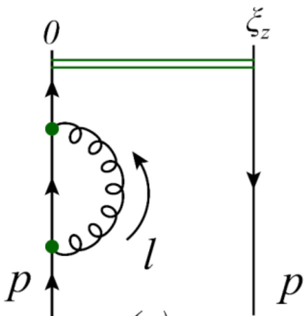
➢ Why proof is difficult

- Because of z-direction dependence, Lorentz symmetry is broken, hard to exhaust all possible UV divergences
- Renormalization of composite operator is needed

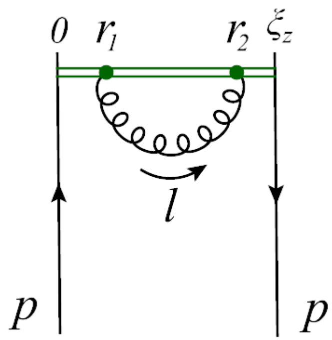
Broken of Lorentz symmetry

➢ Identifying UV divergences

- Renormalization of QCD in covariant gauge: only from 4-dimensional loop integration, all components become large
- Quasi-PDFs: 3-dimensional integration as while as 4-dimensional integration can generate UV divergences



(c)



(a)

$$l^\mu = \bar{l}^\mu + l_z n_z^\mu$$

UV: 4-D integration

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 (p-l)^2}$$

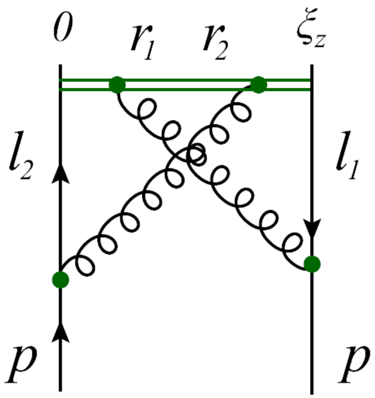
UV: 3-D integration

$$\int \frac{d^3 \bar{l}}{l^2} = \int \frac{d^3 \bar{l}}{\bar{l}^2 - l_z^2}$$

Broken of Lorentz symmetry con.

➢ Hard to identify all UV regions

- **Need to consider 3-D and 4-D integrations for each loop**



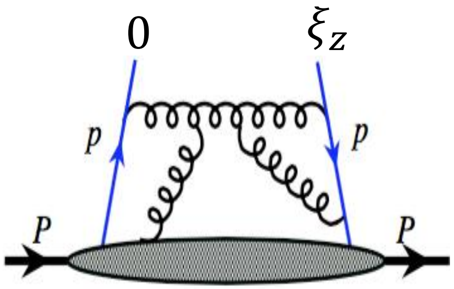
① l_1: **3-D**, l_2: **3-D**
② l_1: **3-D**, l_2: **4-D**
③ l_1: **4-D**, l_2: **3-D**
④ l_1: **4-D**, l_2: **4-D**

- **A n-loop diagram, to identify all possible UV divergences, needs consider 2^n different cases!**

Composition operator renormalization

➢ Quasi-quark PDF in $\mathrm{A_z} = 0$ gauge: no gauge link

$$\tilde{F}_{q/p}(\xi_z, \tilde{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z} \langle h(p)|\overline{\psi}_q(\xi_z)\, \frac{\gamma_z}{2}\, \psi_q(0)|h(p)\rangle$$



- Renormalization of quark field $\bar{\psi}_q$ and ψ_q: taking care by renormalized QCD Lagrangian
- Renormalization of the bi-local operator as a whole: still needs to study

➢ Comparison: Quark PDF in $\mathrm{A_+} = 0$ gauge

- Similar for quark field renormalization
- Renormalization of the bi-local operator as a whole: needed!
- It is this renormalization that mixes quark PDF with gluon PDF

Proof using auxiliary field

➢ Replace gauge link by auxiliary field

$$O(x,y)=\overline{\psi}(x)\Gamma L(x,y)\psi(y)$$

$$O(x,y)=\overline{\psi}(x)\Gamma Q(x)\overline{Q}(y)\psi(y)$$

Ji, Zhang, Zhao, 1706.08962
See Ji's and Zhang's talks

- Nonlocal operator becomes a bi-local operator
- Renormalization is taking care by the renormalization of full Lagrangian

$$O(z_2,z_1)=Z_{\bar{j}}Z_j O_R(z_2,z_1)$$

➢ Comments

✓ Well-established procedure, go back to 1980s Dorn (1986)
✓ Simple and elegant
? Depends on the correctness of an all order proof of the renormalizability of HQET Bagan, Gosdzinsky, 9305297
? Renormalization of composite operator is not considered

Keys for a rigorous proof

Ishikawa YQM, Qiu, Yoshida, 1707.03107

- **Working in Feynman gauge**
  - Because renormalization of QCD Lagrangian in Feynman gauge is well known

- **Key to prove the renormalization: show that UV divergences are local in space-time**
  - Nontrivial conclusion! E.g. UV divergences for normal PDFs are non-local in "–" direction
  - The most difficult part in our proof
  - One can guess this, but a rigorous proof is badly needed

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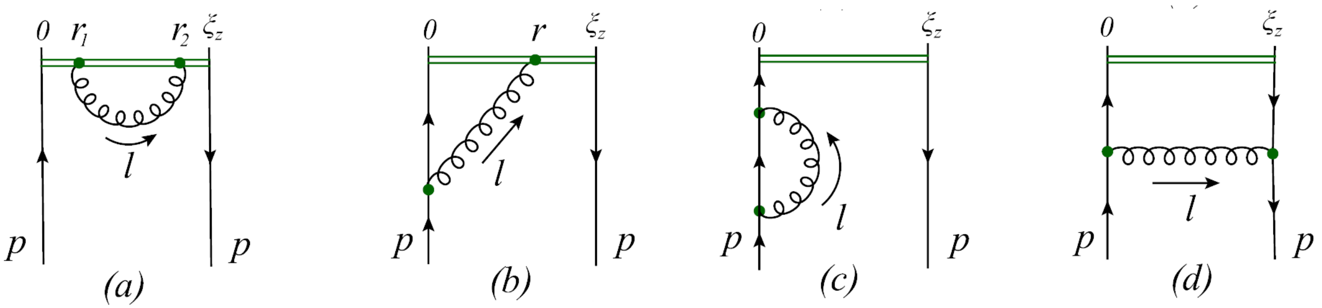
One-loop diagrams: quark in a quark

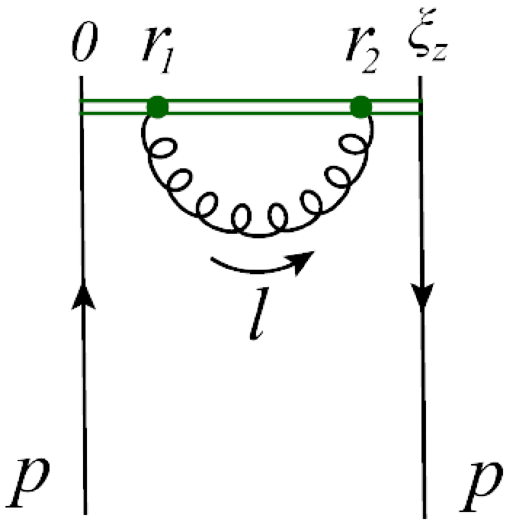
➢ Quasi quark PDFs at one loop level



- Will demonstrate that UV divergences are local in space-time, which is significantly different from normal PDFs

- Note: normal PDFs, UV divergences from the regio$(l_+, l_-, l_\perp) \sim (1, \lambda^2, \lambda)$ with $\lambda \to \infty$, nonlocal in '-' direction in coordinate space.
- Thus, renormalization of normal PDFs is a convolution, while renormalization of quasi-PDFs is multiplicative factor

Fig.1 (a)



(a)

$$M_{1a} = \frac{e^{ip_z\xi_z}}{p_z}\frac{1}{N_c}\mathrm{Tr}_c[T^aT^a]\int_a^{\xi_z-2a}dr_1\int_{r_1+a}^{\xi_z-a}dr_2$$
$$\times\int\frac{d^4l}{(2\pi)^4}e^{-ip_z\xi_z}\,e^{il_z(r_2-r_1)}\left(\frac{-ig_{\mu\nu}}{l^2}\right)$$
$$\times(-ig_sn_z^\mu)(-ig_sn_z^\nu)\mathrm{Tr}\left[\frac{1}{2}\not{p}\,\frac{1}{2}\gamma_z\right]$$
$$=\frac{\alpha_sC_F}{4i\pi^3}\int_a^{\xi_z-2a}dr_1\int_{r_1+a}^{\xi_z-a}dr_2\int d^4l\frac{e^{il_z(r_2-r_1)}}{l^2}$$

- **Cutoff "a" between fields along gaugelink**
- **Conclusion independent of regulators**

$$\int\frac{d^3\bar{l}}{l^2}=\int\frac{d^3\bar{l}}{\bar{l}^2-l_z^2}\qquad d^4l=d^3\bar{l}\,dl_z\qquad l^2=\bar{l}^2-l_z^2$$
$$=\int d^3\bar{l}\left(\frac{1}{\bar{l}^2}+\frac{l_z^2}{(\bar{l}^2-l_z^2)\bar{l}^2}\right)\qquad\int dl_z e^{il_z(r_2-r_1)}=2\pi\delta(r_2-r_1)$$

- **First term vanishes because $r_1\neq r_2$, thus 3D integration is finite**

Fig. 1(a) cont.

- Fix 3D, l_z integration is finite
- UV divergent only if all 4 components of l^μ go to infinity

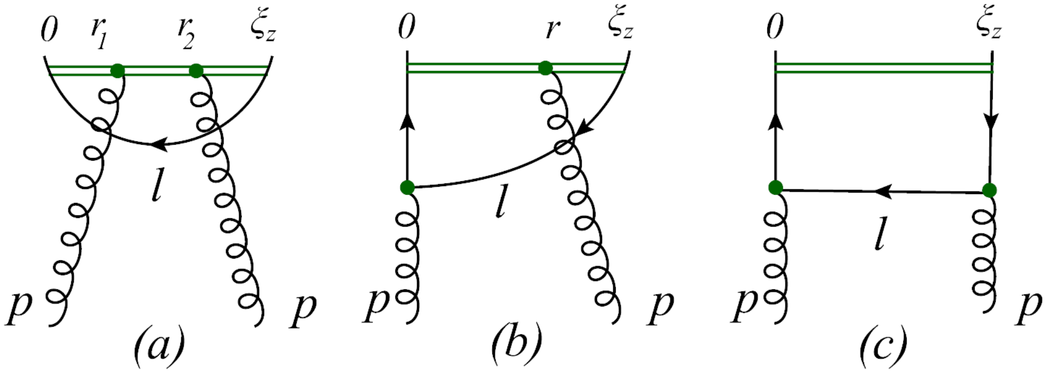
$$M_{1a} \stackrel{\text{div}}{=} -\frac{\alpha_s C_F}{\pi}\frac{|\xi_z|}{a} + \frac{\alpha_s C_F}{\pi}\ln\frac{|\xi_z|}{a}$$

- At this order, UV divergences only come from the region where all loop momenta go to infinity, thus localized in coordinate space.
- Will show next: this behavior remains true up to all order in perturbation theory.

$$\begin{aligned} M^{(1)} &\stackrel{\text{div}}{=} M_{1a} + 2\times M_{1b} + 2\times\frac{1}{2}M_{1c} + M_{1d} \\ &= \frac{\alpha_s C_F}{\pi}\left(-\frac{|\xi_z|}{a} + 2\ln\frac{|\xi_z|}{a} - \frac{1}{4\epsilon}\right). \end{aligned}$$

One-loop diagrams: quark in a gluon

➢ Gluon to quark



$$M_{2a} \propto \int_0^{\xi_z} dr_1 \int_{r_1}^{\xi_z} dr_2 \int d^4l\, e^{-il_z\xi_z} \frac{l_z}{l^2}$$
$$= \frac{\xi_z^2}{2} \int dl_z\, e^{-il_z\xi_z}\, l_z \int d^3\bar{l} \left(\frac{1}{\bar{l}^2} + \frac{l_z^2}{(\bar{l}^2 - l_z^2)\bar{l}^2} \right)$$

- **UV divergence from 3-D $\propto \delta'(\xi_z)$, vanishes for finite ξ_z**

One-loop diagrams: quark in a gluon con.

➢ Finite term

$$
\begin{aligned}
&\frac{\xi_z^2}{2}\int dl_z\, e^{-il_z\xi_z}\, l_z \int d^3\bar{l}\frac{l_z^2}{(\bar{l}^2-l_z^2)\bar{l}^2}\\
\propto&\frac{\xi_z^2}{2}\int dl_z\, e^{-il_z\xi_z}\,\frac{l_z^3}{|l_z|}\\
=&\frac{2i}{\xi_z},
\end{aligned}
$$

- **Divergent as $\xi_z \to 0$**
- **Result in bad large $\tilde{x}$ behavior in momentum space**

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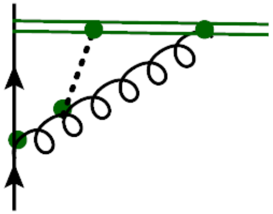
Divergence index

➢ UV divergence at higher loops

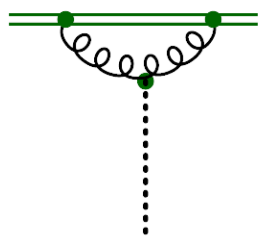


- Construct higher-loop diagrams from lower-loop diagrams by adding gluons to it
- Define divergence index ω_3 (ω_4) for 3D (4D) integration
- Using $\Delta\omega_3$ $(\Delta\omega_4)$ to denote divergence index changes for 3D (4D) integration

➢ Condition for renormalizability

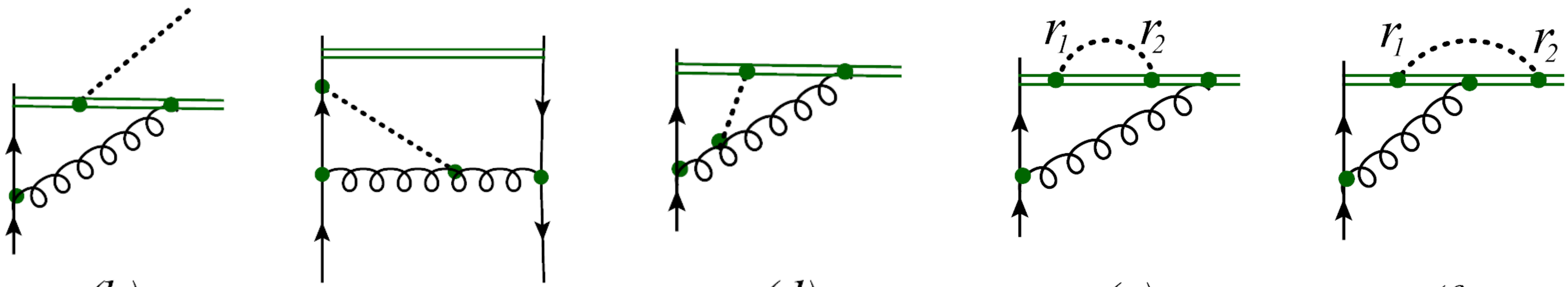
- Finite number of divergent topologies
- Sufficient condition: $\Delta\omega_3 \leq 0$ and $\Delta\omega_4 \leq 0$ for all cases, but not a necessary condition

Cases I-V



(a)

Case I: Only one end of the gluon is attached to parton lines of a loop, like Fig. 3(a). In this case we have one more propagator and one more QCD vertex in the loop, which results in $\Delta\omega_3 = -1$ and $\Delta\omega_4 = -1$. Thus a linear divergence can be changed to a logarithmic divergence, and a logarithmic divergent diagram is changed to be finite.



(b) (c) (d) (e) (f)

II. $\Delta\omega_3 = 0$, $\Delta\omega_4 = -1$

III. $\Delta\omega_3 = -1$, $\Delta\omega_4 = 0$

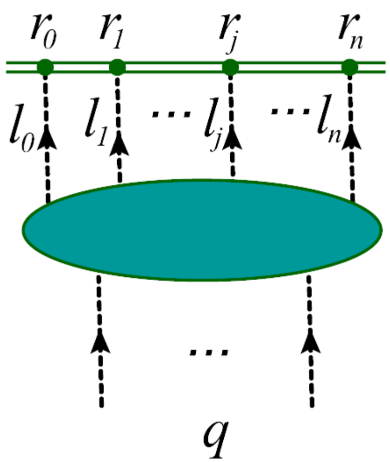
IV. $\Delta\omega_3 = 0$, $\Delta\omega_4 = 0$

V. $\Delta\omega_3 = 1$, $\Delta\omega_4 = 0$

- $\Delta\omega_3 > 0$ **for case V, may result in infinite topologies of UV div.**
- **Dangerous for the renormalizability**

GLI diagram

➢ **Gauge-link-irreducible (GLI) diagram**



- **Diagram is connected no matter how many cuts are applied on the gauge link, or remove it**
- **Similar as the terminology 1PI**

$$l_0 = q - l_1 - \cdots - l_n$$

- **Can be generated from one-loop diagrams combined with insertions in Cases I, III, IV, all of which has $\Delta\omega_3 \leq 0$ and $\Delta\omega_4 \leq 0$**
- **Thus superficial UV divergence index $\omega \leq 1$**

Finiteness of 3-D integration for GLI

➢ Dependence on l_j

$$e^{iq_z r_0}\prod_{j=1}^{n}\int_{r_{j-1}+a}^{r_{max}-a} dr_j \int \frac{d^4 l_j}{(2\pi)^4} e^{il_{jz}(r_j-r_0)}\mathcal{M}(q,l_1,\cdots,l_n)$$

- Numerator in M: decompose to $\bar{l}_j$ and l_{jz}
- Denominator in M:

$$\frac{1}{(l_j+k)^2} = \frac{1}{\Delta - 2k_z l_{jz} - l_{jz}^2}$$

$$= \frac{1}{\Delta} + \frac{2k_z l_{jz}}{\Delta^2} + \frac{(\Delta+4k_z^2+2k_z l_{jz})l_{jz}^2}{(\Delta-2k_z l_{jz}-l_{jz}^2)\Delta^2}$$

$$\Delta = (\bar{l}_j+\bar{k})^2 - k_z^2$$

- Last term: finite for integration of $\bar{l}_j$

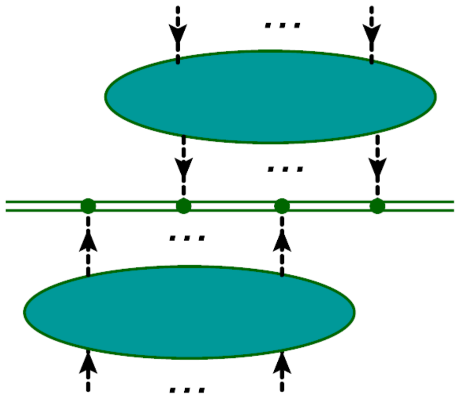
➢ UV divergence from integration of $\bar{l}_j$

- l_{jz} dependence is factorized out, vanish for finite $r_j - r_0$

$$\int dl_{jz} e^{il_{jz}(r_j-r_0)} l_z^m \propto \delta^{(m)}(r_j-r_0)$$

Quasi-PDFs: UV divergences local

- A non-GLI diagram made up by 2 GLI dia.



- Superficial UV divergence index $\omega \leq 2$
- For each GLI sub-diagram, similar argue for GLI diagram. UV finite if any 3-D integration is applied

- Easily generate to any non-GLI diagram:
  - Overall UV divergence, obtained by fixing "z" component of any loop momentum, eventually vanishes after the integration of this "z" component
  - UV divergences of quasi-PDFs: from the region whether all loop momenta become large → local in space-time
  - As $\Delta\omega_4 \leq 0$ for all cases: finite div. topology, renormalizable

PDFs: UV divergences non-local

➢ 3-D' (l_- and $l_\perp$) integration of PDFs

$$\frac{1}{(l+k)^2} = \frac{1}{\hat{\Delta} + 2l_+(l_- + k_-)} \qquad \hat{\Delta} = 2k_+(l_- + k_-) - (\vec{l}_\perp + \vec{k}_\perp)^2$$

$$= \frac{1}{\hat{\Delta}} - \frac{2(l_- + k_-)l_+}{\hat{\Delta}^2} + \frac{4(l_- + k_-)^2 l_+^2}{(\hat{\Delta} + 2l_+(l_- + k_-))\hat{\Delta}^2}$$

- Similar argue as quasi-PDFs: l_+ is factorized in the first two terms ,vanish under 3-D' integration
- But the last term is still UV divergent under 3-D' integration

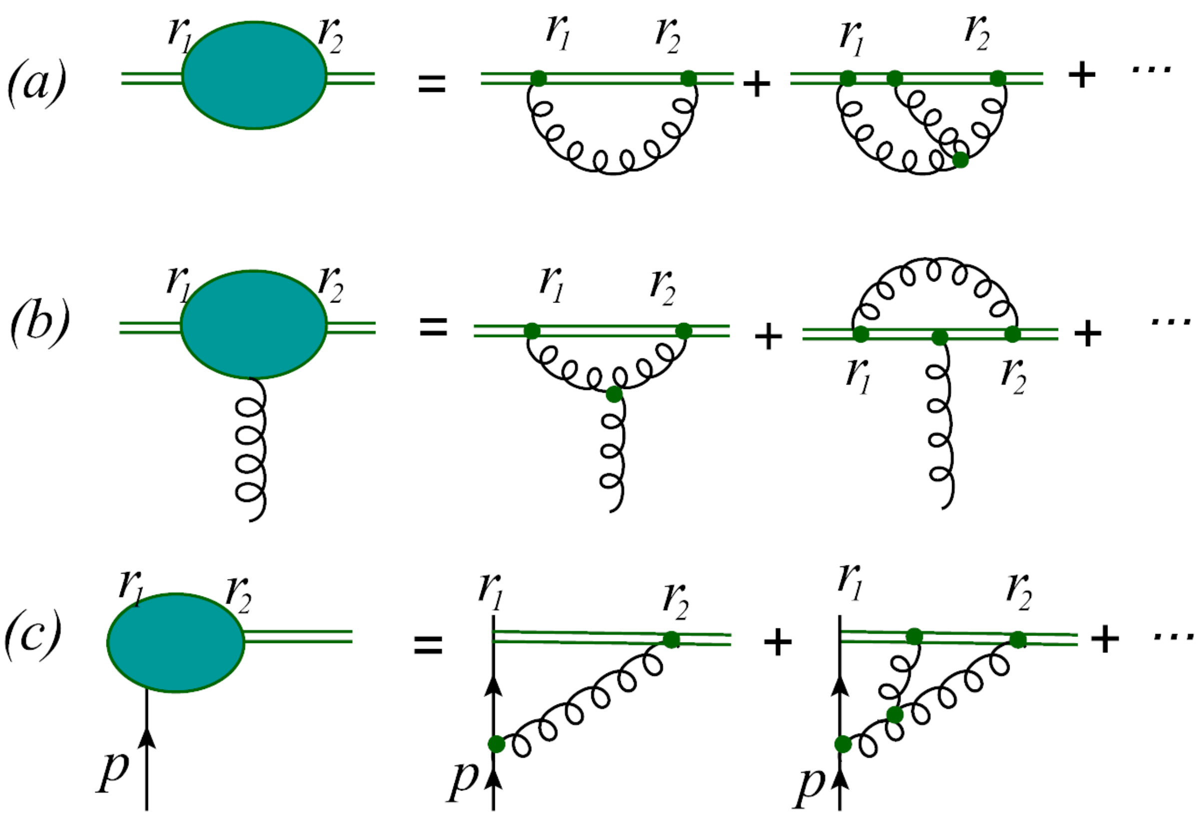
➢ UV divergent region and non-locality

$$(l_+, l_-, \vec{l}_\perp) \sim (1, \lambda^2, \lambda) \text{ as } \lambda \to \infty$$

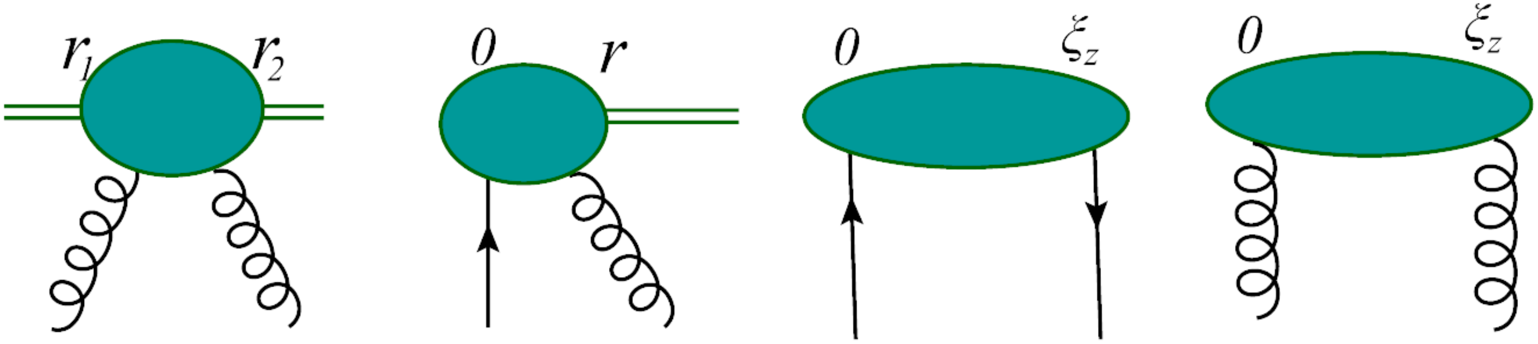
$$l_- l_+ \sim l_\perp^2 \sim \lambda^2$$

- Non-local in "-" direction in space-time

UV divergent topologies



UV finite topologies



- **The last diagram: no mixing between quasi-quark PDF and quasi-gluon PDF**

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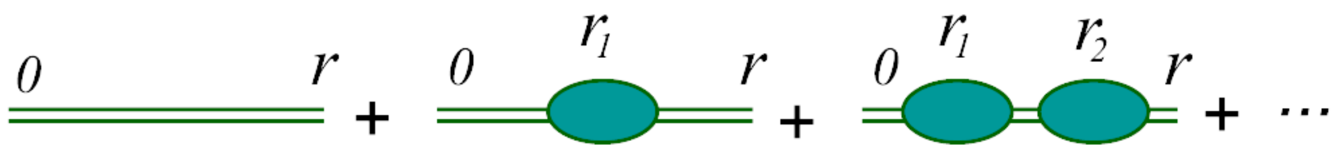
Power divergence

➢ Renormalization



$$1 + c \int_0^r dr_1 + c^2 \int_0^r dr_1 \int_{r_1}^r dr_2 + \cdots$$
$$= \mathcal{P} e^{\, c \int_0^r dr'} = e^{\, c\, r}\,,$$

- It is allowed to introduce an overall factor $e^{-c|\xi_z|}$ to remove all power UV divergences

➢ Interpretation

- Mass renormalization of test particle

Dotsenko, Vergeles, NPB (1980)

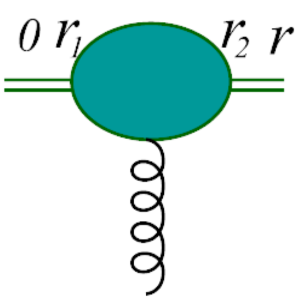
Log divergence related to gaugelink

Dotsenko, Vergeles, NPB (1980)

➢ Log div. from gaugelink self energy

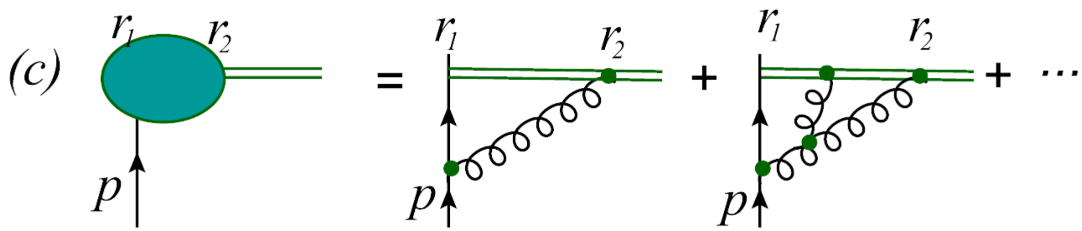
- Besides power divergence, there are also logarithmic UV divergences
- It is known that these divergences can be removed by a "wave function" renormalization of the test particle, Z_{wq}^{-1}.

➢ Log div. from gluon-gaugelink vertex



- Logarithmic UV: can be absorbed by the coupling constant renormalization of QCD.

UV from vertex correction



➢ Remove UV div. at fixed order

- The most dangerous UV diagram, may mix with other operators
- Locality of UV divergence: no dependence on $r_2 - r_1$ or p
- UV divergence is proportional to quark-gaugelink vertex at lowest order, with a constant coefficient
- A constant counter term is able to remove this UV divergence.

➢ Renormalization to all-orders

- Using bookkeeping forests subtraction method, the net effect is to introduce a constant multiplicative renormalizaton factor $Z_{vq}^{-1}$for the quark-gaugelink vertex.

Renormalization

Ishikawa YQM, Qiu, Yoshida, 1707.03107

- Using renormalized QCD Lagrangian:

  - All UV divergences (too all orders) can be removed by the following renormalization

$$\tilde{F}^{R}_{i/p}(\xi_z, \tilde{\mu}^2, p_z) = e^{-C_i|\xi_z|} Z_{wi}^{-1} Z_{vi}^{-1} \tilde{F}^{b}_{i/p}(\xi_z, \tilde{\mu}^2, p_z)$$

- Renormalization: multiplicative factor, not mix with other operators

  - Significantly different from normal PDFs

- Quasi quark PDF is indeed a "good lattice cross section"

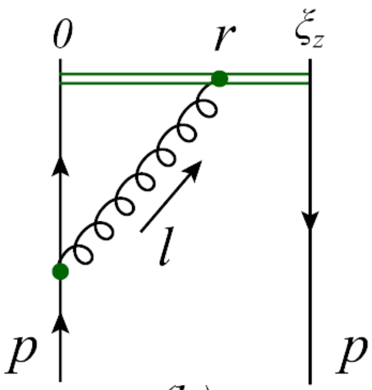
Summary

- "Lattice cross section" = hadronic matrix elements that are calculabe + renormalizable + factorizable
- Candidates of good "Lattice cross section": Quasi-PDFs, Pseudo-PDFs, Conserved currents, ...
- We show that quasi-PDFs are renormalizable, and thus are good "Lattice cross section"
- Find good "lattice cross section" for other nonperturbative quantities: GPDs, TMDs, ...

Thank you!

Back up!

Fig. (b)



(b)

$$M_{1b} = -\, g_s^2 C_F \int_a^{\xi_z - a} dr \int \frac{d^4 l}{(2\pi)^4} \times e^{i l_z r} \frac{\mathrm{Tr}[\not{p}\gamma_z(\not{p} - \not{l})\gamma_z]}{4 p_z l^2 (p-l)^2},$$

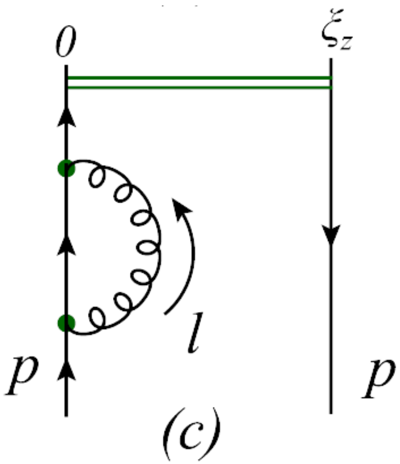
A potential UV divergence in 3D integration:

$$\int \frac{l^\mu \, d^3 \bar{l}}{l^2 (p-l)^2}$$

- **finite because potential divergent integral is an odd function in $\bar{l}^\mu$**
- **UV divergent only if all 4 components of l^μ go to infinity**
- **Cutoff “a” is enough to regularize UV divergence**

$$M_{1b} \overset{\text{div}}{=} \frac{\alpha_s C_F}{2\pi} \ln \frac{|\xi_z|}{a}$$

Fig. (c)

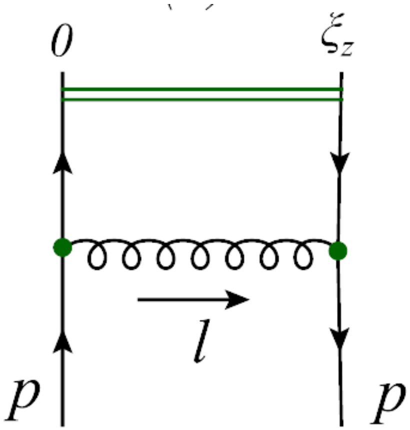


$$M_{1c} = \frac{i g_s^2 \mu_r^{2\epsilon}}{2p_z}(1-\epsilon)C_F \int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{Tr}[\not{p}\gamma_z \not{p}(\not{p}-\not{l})]}{p^2 l^2 (p-l)^2}$$
$$= i g_s^2 \mu_r^{2\epsilon} C_F (1-\epsilon) \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 (p-l)^2},$$

- **UV divergent only if all 4 components of l^μ go to infinity**

$$M_{1c} \overset{\text{div}}{=} -\frac{\alpha_s C_F}{4\pi}\frac{1}{\epsilon}$$

Fig.1 (d)



(d)

$$M_{1d} = -ig_s^2 C_F e^{ip_z\xi_z} \int \frac{d^4k}{(2\pi)^4} e^{-ik_z\xi_z} \times \frac{\mathrm{Tr}[\not{p}\gamma_\mu \not{k}\gamma_z \not{k}\gamma^\mu]}{4p_z k^4 (p-k)^2}.$$

Because of the oscillation factor $e^{-i\,k_z\,\xi_z}$, k_z cannot go to infinity

$$M_{1d} \overset{\text{div}}{=} 0$$