

Spectra of heavy-light mesons in a relativistic model

Jing-bin Liu(刘景斌)

liujb@ihep.ac.cn

arXiv: 1507.08372, 1605.05550

Institute of High Energy Physics

October 29, 2017
© CCNU, Wuhan

Outline

- ① Motivation
- ② The Relativistic Model
- ③ Solving the Wave Equation
- ④ Numerical Results and Discussion
- ⑤ Summary

Non-relativistic Potential

According to the conventional constituent-quark model, the mesons can be seen as a composition of a quark and an antiquark. The quark potential depends on the spin structure of the interaction.

$$\Gamma \otimes \Gamma = \begin{cases} \text{Scalar,} & 1 \otimes 1 \\ \text{Pseudo-scalar,} & \gamma_5 \otimes \gamma^5 \\ \text{Vector,} & \gamma_\mu \otimes \gamma^\mu \\ \text{Pseudo-vector,} & \gamma_\mu \gamma_5 \otimes \gamma^\mu \gamma^5 \\ \text{Tensor,} & \sigma_{\mu\nu} \otimes \sigma^{\mu\nu} \end{cases}$$

Vector:

$$\begin{aligned} V_V = & V(r) + \frac{1}{8} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \Delta V(r) - \frac{1}{2m_1 m_2} \left\{ \{ \vec{p}_1 V(r) \vec{p}_2 - \vec{p}_1 \vec{r} \frac{V'(r)}{r} \vec{r} \vec{p}_2 \} \right\} \\ & + \frac{2}{3m_1 m_2} \Delta V(r) (\vec{s}_1 \cdot \vec{s}_2) + \frac{1}{3m_1 m_2} \left(\frac{V'(r)}{r} - V''(r) \right) \left(3(\vec{s}_1 \cdot \hat{\vec{r}})(\vec{s}_2 \cdot \hat{\vec{r}}) - \vec{s}_1 \cdot \vec{s}_2 \right) \\ & + \frac{V'(r)}{2r} \left[\frac{(\vec{r} \times \vec{p}_1) \cdot \vec{s}_1}{m_1^2} - \frac{(\vec{r} \times \vec{p}_2) \cdot \vec{s}_2}{m_2^2} + \frac{2}{m_1 m_2} ((\vec{r} \times \vec{p}_1) \cdot \vec{s}_2 - (\vec{r} \times \vec{p}_2) \cdot \vec{s}_1) \right], \end{aligned}$$

Non-relativistic Potential

Scalar:

$$V_S = S(r) - \frac{S'(r)}{2r} \left(\frac{(\vec{r} \times \vec{p}_1) \cdot \vec{s}_1}{m_1^2} - \frac{(\vec{r} \times \vec{p}_2) \cdot \vec{s}_2}{m_2^2} \right) - \frac{1}{2m_1^2} \{\{\vec{p}_1 S(r) \vec{p}_1\}\} - \frac{1}{2m_2^2} \{\{\vec{p}_2 S(r) \vec{p}_2\}\},$$

Pseudo-scalar:

$$V_P = \frac{1}{3m_1 m_2} \Delta P(r) (\vec{s}_1 \cdot \vec{s}_2) - \frac{1}{3m_1 m_2} \left(\frac{P'(r)}{r} - P''(r) \right) \left(3(\vec{s}_1 \cdot \hat{\vec{r}})(\vec{s}_2 \cdot \hat{\vec{r}}) - \vec{s}_1 \cdot \vec{s}_2 \right),$$

Pseudo-vector:

$$V_A = -4A(r)(\vec{s}_1 \cdot \vec{s}_2) + O(1/m^2),$$

Tensor:

$$V_T = 4T(r)(\vec{s}_1 \cdot \vec{s}_2) + O(1/m^2),$$

with the notation

$$\{\{\vec{p}_i F \vec{p}_j\}\} = \frac{1}{4} (\vec{p}_i F \vec{p}_j + \vec{p}_j F \vec{p}_i + \vec{p}_i \vec{p}_j F + \vec{p}_j \vec{p}_i F).$$

Relativized Quark Model

"Mesons in a relativized quark model with chromodynamics",
 S. Godfrey and N. Isgur,
 Phys. Rev. D32, 189 (1985).

- The smearing function:

$$\tilde{V}(r) = \int d^3\mathbf{r}' \rho_{ij}(\mathbf{r} - \mathbf{r}') V(r'),$$

with

$$\begin{aligned}\rho_{ij}(\mathbf{r} - \mathbf{r}') &= \frac{\sigma_{ij}^3}{\pi^{3/2}} e^{-\sigma_{ij}^2(\mathbf{r} - \mathbf{r}')^2}, \\ \sigma_{ij}^2 &= \sigma_0^2 \left(\frac{1}{2} + \frac{1}{2} \left(\frac{4m_1 m_2}{(m_1 + m_2)^2} \right)^4 \right) + s_0^2 \left(\frac{2m_1 m_2}{m_1 + m_2} \right)^2\end{aligned}$$

- $m \leftrightarrow E$ ambiguity:

$$\frac{\tilde{V}_i(r)}{m_1 m_2} \rightarrow \left(\frac{m_1 m_2}{E_1 E_2} \right)^{1/2+\epsilon_i} \frac{\tilde{V}_i(r)}{m_1 m_2} \left(\frac{m_1 m_2}{E_1 E_2} \right)^{1/2+\epsilon_i}$$

Relativized Quark Model

TABLE II. The parameters of soft-QCD spectroscopy.

| Masses ^a | |
|--|---------------------------------------|
| $\frac{1}{2}(m_u + m_d) = 220$ | MeV |
| $m_s = 419$ | MeV |
| $m_c = 1628$ | MeV |
| $m_b = 4977$ | MeV |
| Potentials | |
| $b = 0.18$ | GeV ² |
| $\alpha_s^{\text{critical}} = 0.60$ | |
| $\Lambda = 200$ | MeV |
| $c = -253$ | MeV |
| Relativistic effects (see Appendix A) | |
| Smearing: | $\sigma_0 = 1.80$ GeV |
| | $s = 1.55$ |
| $m \leftrightarrow E$ ambiguity: | $\epsilon_c = -0.168$ |
| | $\epsilon_t = +0.025$ |
| | $\epsilon_{\text{sol}(\nu)} = -0.035$ |
| | $\epsilon_{\text{so}(S)} = +0.055$ |

- ① Motivation
- ② The Relativistic Model
- ③ Solving the Wave Equation
- ④ Numerical Results and Discussion
- ⑤ Summary

Bethe-Salpeter Equation

"A Relativistic Equation for Bound-State Problems",
 H. A. Bethe and E. E. Salpeter,
 Phys. Rev. 84, 1232 (1951).

The Bethe-Salpeter Equation for the quark-antiquark system can be written as:

$$(\not{p}_1 - m_1)\chi(p_1, p_2)(\not{p}_2 + m_2) = \frac{1}{(2\pi)^4} \int d^4 p'_1 d^4 p'_2 \bar{K}(p_1, p_2; p'_1, p'_2)\chi(p'_1, p'_2),$$

where p_1 and p_2 relate to the total momentum P and the relative momentum p as follows:

$$\begin{aligned} p_1 &= \alpha_1 P - p, \quad \alpha_1 = \frac{m_1}{m_1 + m_2}, \\ p_2 &= \alpha_2 P + p, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}. \end{aligned}$$

Using the energy-momentum conservation, we have

$$(\not{p}_1 - m_1)\chi(p, P)(\not{p}_2 + m_2) = \int \frac{d^4 p'}{(2\pi)^4} \bar{K}(p, p', P)\chi(p', P).$$

Bethe-Salpeter Equation

Here we choose the interaction kernel as the standard Coulomb-plus-linear form, which is one-gluon-exchange (OGE) dominant at short distances with linear confinement at long distances. If one applies the instantaneous approximation, the kernel can be written as:

$$\overline{K}(p, p', P) = \gamma^{(1)} \cdot \gamma^{(2)} V_v(-\mathbf{k}^2) + V_s(-\mathbf{k}^2),$$

where the transferred momentum \mathbf{k} is defined as:

$$\mathbf{k} = \mathbf{p} - \mathbf{p}'.$$

Since the interaction kernel $\overline{K}(p, p', P)$ is no longer dependent on p'^0 , we can perform the integration over p'^0 .

The projection operators are defined as

$$\Lambda_{\pm}^{(i)} = \frac{1}{2}(1 \pm h_i),$$

$$h_i = \frac{H_i}{\omega_i},$$

$$\omega_i = \sqrt{\mathbf{p}^2 + m_i^2}, \quad i = 1, 2$$

Bethe-Salpeter Equation

where

$$\begin{aligned} H_1(\mathbf{p}) &= -\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p} + \beta^{(1)} m_1, \\ H_2(\mathbf{p}) &= \boldsymbol{\alpha}^{(2)} \cdot \mathbf{p} + \beta^{(2)} m_2. \end{aligned}$$

Applying the projection operators, the four coupled equation can be written as

$$\begin{aligned} (E - \omega_1 - \omega_2) \phi_{++}(\mathbf{p}) &= -2\pi i \Gamma_{++}(\mathbf{p}), \\ (E + \omega_1 + \omega_2) \phi_{--}(\mathbf{p}) &= 2\pi i \Gamma_{--}(\mathbf{p}), \\ \phi_{+-}(\mathbf{p}) &= 0, \\ \phi_{-+}(\mathbf{p}) &= 0, \end{aligned}$$

where

$$\begin{aligned} \phi_{\pm\pm} &= \Lambda_{\pm}^{(1)} \Lambda_{\pm}^{(2)} \phi, \quad \Gamma_{\pm\pm} = \Lambda_{\pm}^{(1)} \Lambda_{\pm}^{(2)} \Gamma, \\ \phi(\mathbf{p}) &= \int dp^0 \psi(p^0, \mathbf{p}), \\ \Gamma(\mathbf{p}) &= \int \frac{d^3 k}{(2\pi)^4} \gamma_0^{(1)} \gamma_0^{(2)} \left[V_v(-\mathbf{k}^2) \gamma^{(1)} \cdot \gamma^{(2)} - V_s(-\mathbf{k}^2) \right] \phi(\mathbf{p} + \mathbf{k}). \end{aligned}$$

The Equivalent Equation

After Fourier transformation into coordinate space, we can prove that the four coupled equation are equivalent to:

$$\left(H_1 + H_2 + \frac{1}{2}(h_1 + h_2)U - E \right) \phi(\mathbf{r}) = 0,$$

$$(h_1 - h_2)\phi(\mathbf{r}) = 0,$$

with

$$U(\mathbf{r}) = U_1(\mathbf{r}) + U_2(\mathbf{r}),$$

$$U_1(\mathbf{r}) = V_v(r) + \beta^{(1)}\beta^{(2)}V_s(r),$$

$$U_2(\mathbf{r}) = -\frac{1}{2}[\boldsymbol{\alpha}^{(1)} \cdot \boldsymbol{\alpha}^{(2)} + (\boldsymbol{\alpha}^{(1)} \cdot \hat{\mathbf{r}})(\boldsymbol{\alpha}^{(2)} \cdot \hat{\mathbf{r}})]V_v(r).$$

Furthermore,

$$(h_1 - h_2)\phi = 0 \Leftrightarrow \phi = \frac{1}{2}(h_1 + h_2)\varphi.$$

The Equivalent Equation

Then we have

$$\left(H_1 + H_2 + \frac{1}{2}(h_1 + h_2)U - E \right) \frac{1}{2}(h_1 + h_2)\varphi = 0,$$

which is equivalent to

$$\left(\frac{1}{2}(1 + h_1 h_2)(\omega_1 + \omega_2) + \frac{1}{2}(h_1 + h_2)U \frac{1}{2}(h_1 + h_2) - \frac{1}{2}(h_1 + h_2)E \right) \varphi = 0.$$

Noticing the relations

$$\begin{aligned} (1 + h_1 h_2) \frac{1}{2}(h_1 + h_2) &= h_1 + h_2, \\ (1 + h_1 h_2)h_{1,2} &= h_1 + h_2, \end{aligned}$$

we can have two different equivalent forms, they are

$$\frac{1}{2}(1 + h_1 h_2) \left(\omega_1 + \omega_2 + \frac{1}{2}(h_1 + h_2)U \frac{1}{2}(h_1 + h_2) - \frac{1}{2}(h_1 + h_2)E \right) \varphi = 0$$

and

$$\left(\omega_1 + \omega_2 + \frac{1}{2}(h_1 + h_2)U \frac{1}{2}(h_1 + h_2) - \frac{1}{2}(h_1 + h_2)E \right) \frac{1}{2}(1 + h_1 h_2)\varphi = 0.$$

The Equivalent Equation

Here we guess a form that is equivalent to the above two equations by employing the common part of them, that is

$$\left(\omega_1 + \omega_2 + \frac{1}{2}(h_1 + h_2)U \frac{1}{2}(h_1 + h_2) - \frac{1}{2}(h_1 + h_2)E \right) \psi = 0.$$

- Left multiplying by $\frac{1}{2}(1 + h_1 h_2)$,
- Taking take $\psi = \frac{1}{2}(1 + h_1 h_2)\varphi$.

The above equation is equivalent to the instantaneous Bethe-Salpeter equation. Analogously, we can obtain the two other equivalent equations

$$\left(\omega_1 + \omega_2 + \frac{1}{2}(h_1 + h_2)U \frac{1}{2}(h_1 + h_2) - h_1 E \right) \psi = 0,$$

$$\left(\omega_1 + \omega_2 + \frac{1}{2}(h_1 + h_2)U \frac{1}{2}(h_1 + h_2) - h_2 E \right) \psi = 0.$$

Foldy-Wouthuysen Transformation

The original Hamiltonian can be written in the form

$$H = \beta m + \mathcal{E} + \mathcal{O},$$

where \mathcal{O} is the “odd” operator and \mathcal{E} is the “even” operator.

$$\tilde{H} = e^{iS} H e^{-iS}$$

$$e^A H e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A^{(n)}, B],$$

$$[A^{(0)}, B] = B, [A^{(n+1)}, B] = [A, [A^{(n)}, B]]$$

$$S = -\frac{i\beta}{2m} \mathcal{O}$$

According to the Foldy-Wouthuysen transformation, the transformed Hamiltonian reads

$$\tilde{H} = \beta m + \mathcal{E} + \frac{\beta}{2m} \mathcal{O}^2 + \frac{1}{8m^2} [[\mathcal{O}, \mathcal{E}], \mathcal{O}] - \frac{\beta}{8m^3} \mathcal{O}^4 + \dots$$

The Effective Hamiltonian

Now we consider performing the Foldy-Wouthuysen transformation on the equivalent equation. the odd and even operators in the above equation are

$$\begin{aligned}\mathcal{E} = & -E \left(\frac{m_1}{\omega_1} - 1 \right) \beta^{(1)} +_{\Delta} \omega_1 + \omega_2 \\ & + \frac{1}{2} \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right) U_2 \frac{1}{2} \left(\frac{m_1}{\omega_1} \beta^{(1)} + h_2 \right) + h.c. \\ & + \frac{1}{2} \left(\frac{m_1}{\omega_1} \beta^{(1)} + h_2 \right) U_1 \frac{1}{2} \left(\frac{m_1}{\omega_1} \beta^{(1)} + h_2 \right) + \frac{1}{2} \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right) U_1 \frac{1}{2} \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{O} = & -E \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right) + \frac{1}{2} \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right) U_1 \frac{1}{2} \left(\frac{m_1}{\omega_1} \beta^{(1)} + h_2 \right) + h.c. \\ & + \frac{1}{2} \left(\frac{m_1}{\omega_1} \beta^{(1)} + h_2 \right) U_2 \frac{1}{2} \left(\frac{m_1}{\omega_1} \beta^{(1)} + h_2 \right) + \frac{1}{2} \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right) U_2 \frac{1}{2} \left(-\frac{\boldsymbol{\alpha}^{(1)} \cdot \mathbf{p}}{\omega_1} \right),\end{aligned}$$

where *h.c.* stands for Hermitian conjugate.

The Effective Hamiltonian

$$H = H_0 + H', \quad H' = H'_1 + H'_a + H'_b.$$

The leading order Hamiltonian H_0 and the subleading Hamiltonian H' to order $1/m_Q^2$ can be written as

$$H_0 = \omega_1 + \omega_2 + \frac{1}{2} (1 + h_2) \overline{U}_1 \frac{1}{2} (1 + h_2),$$

$$H'_1 = -\frac{1}{2} \left\{ \frac{\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}}{m_1}, \tilde{U}_2 \right\},$$

$$H'_a = \frac{1}{4} \frac{\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}}{m_1} \tilde{U}_1 \frac{\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}}{m_1} + \frac{1}{8} \left\{ \frac{\mathbf{p}^2}{m_1}, \overline{U}_1 \right\},$$

$$H'_b = \frac{1}{4E} \left(\tilde{U}_2 \frac{1}{2} (1 - h_2) \tilde{U}_1 \frac{\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}}{m_1} + h.c. \right) - \frac{1}{4E} \left(\tilde{U}_2 \frac{1}{2} (1 - h_2) \frac{\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p}}{m_1} \overline{U}_1 + h.c. \right),$$

the interaction potentials $\overline{U}_1(\mathbf{r})$, $\tilde{U}_1(\mathbf{r})$ and $\tilde{U}_2(\mathbf{r})$ in the above equations are defined as:

$$\overline{U}_1(\mathbf{r}) = V_v(r) + \beta^{(2)} V_s(r),$$

$$\tilde{U}_1(\mathbf{r}) = V_v(r) - \beta^{(2)} V_s(r),$$

$$\tilde{U}_2(\mathbf{r}) = -\frac{1}{2} [\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\alpha}^{(2)} + (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{r}})(\boldsymbol{\alpha}^{(2)} \cdot \hat{\mathbf{r}})] V_v(r).$$

- ① Motivation
- ② The Relativistic Model
- ③ Solving the Wave Equation
- ④ Numerical Results and Discussion
- ⑤ Summary

$$\Omega(p)j_l(kr)Y_{lm}(\hat{\mathbf{r}}) = \Omega(k)j_l(kr)Y_{lm}(\hat{\mathbf{r}})$$

$\Omega(p)$ is a pseudo-differential operator function and $\Omega(k)$ is a normal function, p and k stand for the modules of momentum operator p and momentum k , respectively.

$$\begin{aligned}\Omega(p)\psi(\mathbf{r}) &= \Omega(p) \int d^3\mathbf{r}' \delta^3(\mathbf{r} - \mathbf{r}')\psi(\mathbf{r}') \\ &= \int d^3\mathbf{r}' \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Omega(k) e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \psi(\mathbf{r}').\end{aligned}$$

The exponential factor $e^{i\mathbf{k}\cdot\mathbf{r}}$ can be decomposed into series of spherical harmonics

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{lm} i^l j_l(kr) Y_{lm}^*(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{r}}),$$

where j_l is the l -th order spherical Bessel function, $Y_{lm}(\hat{\mathbf{r}})$ is the spherical harmonics. With the normalization conditions of the spherical Bessel function and the spherical harmonics, the eigenequation of $\Omega(p)$ can be easily verified.

Solving the Wave Equation

It is easy to verify that the operators

$$\{H_0, \mathbf{j}^2, j_z, K, S_z\}$$

are a set of mutually commuting operators, where $\mathbf{j} = \mathbf{L} + \mathbf{S}^{(2)}$, $\mathbf{S}^{(2)} = \frac{1}{2}\boldsymbol{\Sigma}^{(2)}$, $K = \beta^{(2)}(\boldsymbol{\Sigma}^{(2)} \cdot \mathbf{L} + 1)$, $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}^{(1)}$. Then the eigenequation associated with H_0 can be written as:

$$H_0 \Psi_{n,k,j,m_j,s}^{(0)}(\mathbf{r}) = E_{n,k,j}^{(0)} \Psi_{n,k,j,m_j,s}^{(0)}(\mathbf{r}),$$

where

$$\Psi_{n,k,j,m_j,s}^{(0)}(\mathbf{r}) = \begin{pmatrix} g_{n,l,j}(r) y_{j,l}^{m_j}(\theta, \varphi) \\ i f_{n,l,j}(r) y_{j,2j-l}^{m_j}(\theta, \varphi) \end{pmatrix} \chi_s$$

with $k = \pm(j + 1/2)$, for $l = j \pm 1/2$,

$$y_{j,l}^m = \begin{pmatrix} k_{j,l,m}^+ Y_l^{m-1/2} \\ k_{j,l,m}^- Y_l^{m+1/2} \end{pmatrix}, \quad k_{j,l,m}^\pm = \begin{cases} +\sqrt{\frac{l \pm m + 1/2}{2l+1}}, & j = l + 1/2 \\ \mp \sqrt{\frac{l \mp m + 1/2}{2l+1}}, & j = l - 1/2. \end{cases}$$

Solving the Wave Equation

The spinor part of wave function can be expanded in terms of the spherical Bessel function in the limited space:

$$\Psi = \sum_{i=1}^N g_i \psi_i^A + \sum_{\alpha=1}^N f_{\alpha} \psi_{\alpha}^B,$$

where the orthonormalized basis can be written as

$$\begin{aligned}\psi_i^A &= \frac{1}{N_i^A} j_{l_A} \left(\frac{a_i^A r}{L} \right) \begin{pmatrix} y_{j,l_A}^{m_j} \\ 0 \end{pmatrix}, \\ \psi_{\alpha}^B &= \frac{i}{N_{\alpha}^B} j_{l_B} \left(\frac{a_{\alpha}^B r}{L} \right) \begin{pmatrix} 0 \\ y_{j,l_B}^{m_j} \end{pmatrix}.\end{aligned}$$

In the representation of $\{\psi_i^A, \psi_{\alpha}^B\}$, the operator H_0 has its matrix form

$$H_0 = \begin{pmatrix} <\omega_1 + \omega_2>_{ij} & \\ & <\omega_1 + \omega_2>_{\alpha\beta} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} < H_a >_{ij} & < H_b >_{i\beta} \\ < H_c >_{\alpha j} & < H_d >_{\alpha\beta} \end{pmatrix}.$$

Solving the Wave Equation

We have

$$\begin{aligned} <\omega_1 + \omega_2>_{ij} &= \left[\omega_1\left(\frac{a_i^A}{L}\right) + \omega_2\left(\frac{a_i^A}{L}\right) \right] \delta_{ij}, \\ <\omega_1 + \omega_2>_{\alpha\beta} &= \left[\omega_1\left(\frac{a_\alpha^B}{L}\right) + \omega_2\left(\frac{a_\alpha^B}{L}\right) \right] \delta_{\alpha\beta}, \end{aligned}$$

here we define a symbolic notation

$$\langle \phi(r) \rangle_{m, l_A; n, l_B} = \int_0^L dr r^2 j_{l_A}\left(\frac{a_m^A r}{L}\right) \phi(r) j_{l_B}\left(\frac{a_n^B r}{L}\right),$$

then we have

$$\begin{aligned} < H_a >_{ij} &= \frac{1}{N_i^A N_j^A} \left(\frac{m_2}{\omega_2\left(\frac{a_i^A}{L}\right)} + 1 \right) \left(\frac{m_2}{\omega_2\left(\frac{a_j^A}{L}\right)} + 1 \right) < V_v + V_s >_{i, l_A; j, l_A} \\ &+ \frac{1}{N_i^A N_j^A} \frac{1}{\omega_2\left(\frac{a_i^A}{L}\right)} \frac{1}{\omega_2\left(\frac{a_j^A}{L}\right)} \left\langle \left(\frac{k+1}{r} + \frac{d}{dr} \right)^{\dagger} (V_v - V_s) \left(\frac{k+1}{r} + \frac{d}{dr} \right) \right\rangle_{i, l_A; j, l_A}, \end{aligned}$$

Solving the Wave Equation

$$\begin{aligned} < H_b >_{i\beta} &= \frac{1}{N_i^A N_\beta^B} \left(\frac{m_2}{\omega_2(\frac{a_i^A}{L})} + 1 \right) \frac{1}{\omega_2(\frac{a_\beta^B}{L})} \left\langle (V_v + V_s) \left(\frac{k-1}{r} - \frac{d}{dr} \right) \right\rangle_{i,l_A;\beta,l_B} \\ &+ \frac{1}{N_i^A N_\beta^B} \frac{1}{\omega_2(\frac{a_i^A}{L})} \left(\frac{-m_2}{\omega_2(\frac{a_\beta^B}{L})} + 1 \right) \left\langle \left(\frac{k+1}{r} + \frac{d}{dr} \right)^\dagger (V_v - V_s) \right\rangle_{i,l_A;\beta,l_B}, \end{aligned}$$

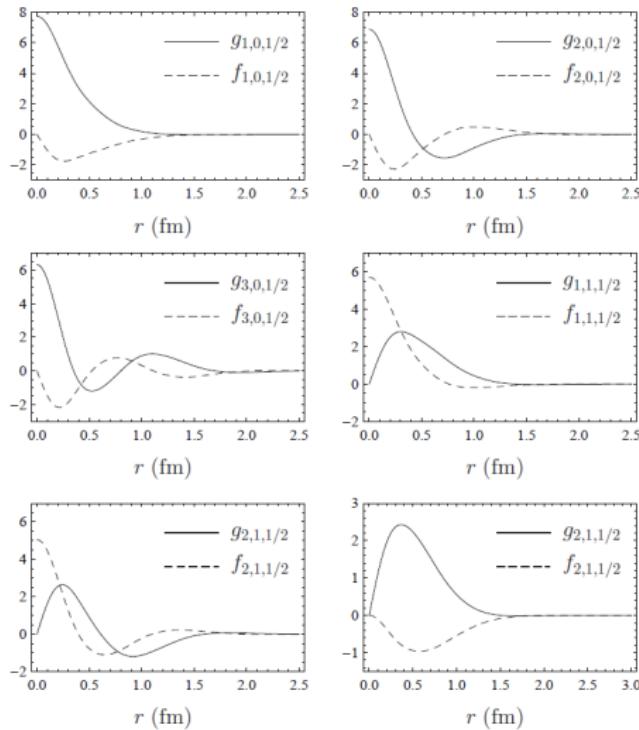
$$< H_c >_{\alpha j} = < H_b >_{j\alpha}^*,$$

$$\begin{aligned} < H_d >_{\alpha\beta} &= \frac{1}{N_\alpha^B N_\beta^B} \left(\frac{-m_2}{\omega_2(\frac{a_\alpha^B}{L})} + 1 \right) \left(\frac{-m_2}{\omega_2(\frac{a_\beta^B}{L})} + 1 \right) < V_v - V_s >_{\alpha,l_B;\beta,l_B} \\ &+ \frac{1}{N_\alpha^B N_\beta^B} \frac{1}{\omega_2(\frac{a_\alpha^B}{L})} \frac{1}{\omega_2(\frac{a_\beta^B}{L})} \left\langle \left(\frac{k-1}{r} - \frac{d}{dr} \right)^\dagger (V_v + V_s) \left(\frac{k-1}{r} - \frac{d}{dr} \right) \right\rangle_{\alpha,l_B;\beta,l_B} \end{aligned}$$

Diagonalizing the Hermitian matrix of H_0 , we can get the eigenenergy of H_0 and the coefficients g_i, f_α , then the eigenequation associated with H_0 is solved and the eigenfunction is obtained.

- ① Motivation
- ② The Relativistic Model
- ③ Solving the Wave Equation
- ④ Numerical Results and Discussion
- ⑤ Summary

Wave Functions



The String Tension b

$$H_0^{\text{Schr}} = \omega_1 + \omega_2 + V_v(r) + V_s(r),$$

$$H_0^{\text{Dirac}} = \omega_1 + H_2(\mathbf{p}) + V_v(r) + \beta^{(2)} V_s(r),$$

$$H_0^{B-S} = \omega_1 + \omega_2 + \frac{1}{2} (1 + h_2) (V_v(r) + \beta^{(2)} V_s(r)) \frac{1}{2} (1 + h_2).$$

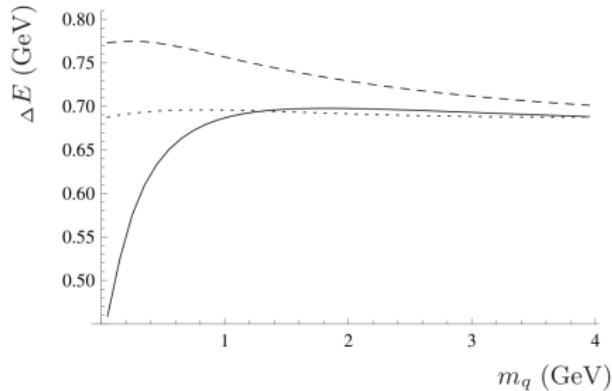


Figure: The energy gap ΔE between the first radial excitation and the ground state as a function of m_q . The dashed, dotted and solid lines stand for the Schrödinger, Dirac and Bethe-Salpeter schemes, respectively.

The String Tension b

- When m_q is taken large enough, the three schemes tend to give the same value for the energy gap ΔE . It indicates the equivalence of the three schemes when dealing with double-heavy mesons.
- In the region $m_q < 1\text{GeV}$, which is the case for heavy-light mesons, the three schemes give quite different values for the energy gap. It has the pattern: $\Delta E^{Schr} > \Delta E^{Dirac} > \Delta E^{B-S}$. In order to give the same energy gap for a specific meson, the confinement parameter should be chosen as: $b^{Schr} < b^{Dirac} < b^{B-S}$.
- In the Schrödinger and Dirac schemes, the energy gap changes slowly over m_q , this is especially true when m_q is less than 1 GeV, ΔE^{Schr} and ΔE^{Dirac} can be viewed as constants. While in the Bethe-Salpeter scheme, ΔE^{B-S} changes drastically over m_q . Experimentally, the ΔE 's are not sensitive to their light quark masses. Thus b^{Schr} and b^{Dirac} can be taken as a constant, while b^{B-S} varies with the quark mass.

Spectrum of D mesons

| $n^j L_J$ | Meson | $E_{\text{expt.}} [2, 9]$ | this work | previous work [27] | Ref. [39] | Ref. [34] |
|---------------|-----------------|---------------------------|-----------|--------------------|-----------|-----------|
| $1^{1/2} S_0$ | D | 1869.62 ± 0.15 | 1871 | 1859 | 1871 | 1868 |
| $1^{1/2} S_1$ | D^* | 2010.28 ± 0.13 | 2008 | 2026 | 2010 | 2005 |
| $1^{1/2} P_0$ | $D_0^*(2400)^0$ | 2318 ± 29 | 2364 | 2357 | 2406 | 2377 |
| $1^{1/2} P_1$ | | | 2507 | 2529 | 2469 | 2490 |
| $1^{3/2} P_1$ | $D_1(2420)$ | 2421.3 ± 0.6 | 2415 | 2434 | 2426 | 2417 |
| $1^{3/2} P_2$ | $D_2^*(2460)$ | 2464.4 ± 1.9 | 2460 | 2482 | 2460 | 2460 |
| $1^{3/2} D_1$ | | | 2836 | 2852 | 2788 | 2795 |
| $1^{3/2} D_2$ | | | 2881 | 2900 | 2850 | 2833 |
| $1^{5/2} D_2$ | $D_J(2740)^0$ | $2737.0 \pm 3.5 \pm 11.2$ | 2737 | 2728 | 2806 | 2775 |
| $1^{5/2} D_3$ | $D_J^*(2760)^0$ | $2760.1 \pm 1.1 \pm 3.7$ | 2753 | 2753 | 2863 | 2799 |
| $1^{5/2} F_2$ | | | 3122 | 3107 | 3090 | 3101 |
| $1^{5/2} F_3$ | | | 3139 | 3134 | 3145 | 3123 |
| $1^{7/2} F_3$ | $D_J^*(3000)^0$ | 3008.1 ± 4.0 | 2980 | 2942 | 3129 | 3074 |
| $2^{1/2} S_0$ | $D_J(2580)^0$ | $2579.5 \pm 3.4 \pm 5.5$ | 2594 | 2575 | 2581 | 2589 |
| $2^{1/2} S_1$ | $D_J^*(2650)^0$ | $2649.2 \pm 3.5 \pm 3.5$ | 2672 | 2686 | 2632 | 2692 |
| $2^{1/2} P_0$ | | | 2895 | 2902 | 2919 | 2949 |
| $2^{1/2} P_1$ | | | 2983 | 2999 | 3021 | 3045 |
| $2^{3/2} P_1$ | | | 2926 | 2932 | 2932 | 2995 |
| $2^{3/2} P_2$ | $D_J(3000)^0$ | 2971.8 ± 8.7 | 2965 | 2969 | 3012 | 3035 |
| $2^{3/2} D_1$ | | | 3230 | 3228 | 3228 | |
| $2^{3/2} D_2$ | | | 3259 | 3260 | 3307 | |
| $2^{5/2} D_2$ | | | 3159 | 3139 | 3259 | |
| $2^{5/2} D_3$ | | | 3176 | 3160 | 3335 | |
| $2^{5/2} F_2$ | | | 3455 | 3425 | | |
| $2^{5/2} F_3$ | | | 3465 | 3444 | 3551 | |
| $2^{7/2} F_3$ | | | 3346 | 3301 | | |

Spectrum of D_s mesons

| $n^j L_J$ | Meson | $E_{\text{expt.}} [2, 9]$ | this work | previous work [27] | Ref. [39] | Ref. [34] |
|---------------|--------------------|--------------------------------------|-----------|--------------------|-----------|-----------|
| $1^{1/2} S_0$ | D_s^\pm | 1968.49 ± 0.32 | 1964 | 1949 | 1969 | 1965 |
| $1^{1/2} S_1$ | $D_s^{*\pm}$ | 2112.3 ± 0.5 | 2107 | 2110 | 2111 | 2113 |
| $1^{1/2} P_0$ | $D_{s0}^*(2317)$ | 2317.8 ± 0.6 | 2437 | 2412 | 2509 | 2487 |
| $1^{1/2} P_1$ | $D_{s1}(2536)$ | 2535.12 ± 0.13 | 2558 | 2562 | 2574 | 2605 |
| $1^{3/2} P_1$ | $D_{s1}(2460)$ | 2459.6 ± 0.6 | 2524 | 2528 | 2536 | 2535 |
| $1^{3/2} P_2$ | $D_{s2}^*(2573)$ | 2571.9 ± 0.8 | 2570 | 2575 | 2571 | 2581 |
| $1^{3/2} D_1$ | $D_{s1}(2860)^-$ | $2859 \pm 12 \pm 6 \pm 23$ [7] | 2885 | 2873 | 2913 | 2913 |
| $1^{3/2} D_2$ | | | 2923 | 2916 | 2961 | 2953 |
| $1^{5/2} D_2$ | | | 2857 | 2829 | 2931 | 2900 |
| $1^{5/2} D_3$ | $D_{s3}^*(2860)^-$ | $2860.5 \pm 2.6 \pm 2.5 \pm 6.0$ [7] | 2871 | 2852 | 2971 | 2925 |
| $1^{5/2} F_2$ | | | 3172 | 3128 | 3230 | 3224 |
| $1^{5/2} F_3$ | | | 3184 | 3152 | 3266 | 3247 |
| $1^{7/2} F_3$ | | | 3107 | 3049 | 3254 | 3203 |
| $2^{1/2} S_0$ | $D_{sJ}(2632)$ | 2632.5 ± 1.7 [3] | 2647 | 2624 | 2688 | 2700 |
| $2^{1/2} S_1$ | $D_{s1}^*(2710)$ | $2708 \pm 9^{+11}_{-10}$ [5] | 2734 | 2729 | 2731 | 2806 |
| $2^{1/2} P_0$ | | | 2945 | 2918 | 3054 | 3067 |
| $2^{1/2} P_1$ | $D_{sJ}(3040)$ | $3044 \pm 8^{+30}_{-5}$ [6] | 3028 | 3017 | 3154 | 3165 |
| $2^{3/2} P_1$ | | | 3009 | 2994 | 3067 | 3114 |
| $2^{3/2} P_2$ | | | 3047 | 3031 | 3142 | 3157 |
| $2^{3/2} D_1$ | | | 3277 | 3247 | 3383 | |
| $2^{3/2} D_2$ | | | 3305 | 3278 | 3456 | |
| $2^{5/2} D_2$ | | | 3260 | 3217 | 3403 | |
| $2^{5/2} D_3$ | | | 3274 | 3237 | 3469 | |
| $2^{5/2} F_2$ | | | 3508 | 3449 | | |
| $2^{5/2} F_3$ | | | 3517 | 3468 | 3710 | |
| $2^{7/2} F_3$ | | | 3459 | 3390 | | |

Spectrum of B mesons

| $n^j L_J$ | Meson | $E_{\text{expt.}} [9]$ | this work | previous work [27] | Ref. [39] | Ref. [34] |
|---------------|---------------|------------------------|-----------|--------------------|-----------|-----------|
| $1^{1/2} S_0$ | B | 5279.25 ± 0.17 | 5273 | 5262 | 5280 | 5279 |
| $1^{1/2} S_1$ | B^* | 5325.2 ± 0.4 | 5329 | 5330 | 5326 | 5324 |
| $1^{1/2} P_0$ | | | 5776 | 5740 | 5749 | 5706 |
| $1^{1/2} P_1$ | | | 5837 | 5812 | 5774 | 5742 |
| $1^{3/2} P_1$ | $B_1(5721)$ | 5723.5 ± 2.0 | 5719 | 5736 | 5723 | 5700 |
| $1^{3/2} P_2$ | $B_2^*(5747)$ | 5743 ± 5 | 5739 | 5754 | 5741 | 5714 |
| $1^{3/2} D_1$ | | | 6143 | 6128 | 6119 | 6025 |
| $1^{3/2} D_2$ | | | 6165 | 6147 | 6121 | 6037 |
| $1^{5/2} D_2$ | | | 5993 | 5989 | 6103 | 5985 |
| $1^{5/2} D_3$ | | | 6004 | 5998 | 6091 | 5993 |
| $1^{5/2} F_2$ | | | 6379 | 6344 | 6412 | 6264 |
| $1^{5/2} F_3$ | | | 6391 | 6354 | 6420 | 6271 |
| $1^{7/2} F_3$ | | | 6202 | 6175 | 6391 | 6220 |
| $2^{1/2} S_0$ | | | 5957 | 5915 | 5890 | 5886 |
| $2^{1/2} S_1$ | | | 5997 | 5959 | 5906 | 5920 |
| $2^{1/2} P_0$ | | | 6270 | 6211 | 6221 | 6163 |
| $2^{1/2} P_1$ | | | 6301 | 6249 | 6281 | 6194 |
| $2^{3/2} P_1$ | | | 6216 | 6189 | 6209 | 6175 |
| $2^{3/2} P_2$ | | | 6232 | 6200 | 6260 | 6188 |
| $2^{3/2} D_1$ | | | 6514 | 6458 | 6534 | |
| $2^{3/2} D_2$ | | | 6527 | 6471 | 6554 | |
| $2^{5/2} D_2$ | | | 6401 | 6357 | 6528 | |
| $2^{5/2} D_3$ | | | 6411 | 6365 | 6542 | |
| $2^{5/2} F_2$ | | | 6692 | 6621 | | |
| $2^{5/2} F_3$ | | | 6700 | 6629 | 6786 | |
| $2^{7/2} F_3$ | | | 6553 | 6493 | | |

Spectrum of B_s mesons

| $n^j L_J$ | Meson | $E_{\text{expt.}} [9]$ | this work | previous work [27] | Ref. [39] | Ref. [34] |
|--------------|------------------|------------------------|-----------|--------------------|-----------|-----------|
| $1^{1/2}S_0$ | B_s | 5366.77 ± 0.24 | 5363 | 5337 | 5372 | 5373 |
| $1^{1/2}S_1$ | B_s^* | $5415.4^{+2.4}_{-2.1}$ | 5419 | 5405 | 5414 | 5421 |
| $1^{1/2}P_0$ | | | 5811 | 5776 | 5833 | 5804 |
| $1^{1/2}P_1$ | | | 5864 | 5841 | 5865 | 5842 |
| $1^{3/2}P_1$ | $B_{s1}(5830)$ | 5829.4 ± 0.7 | 5819 | 5824 | 5831 | 5805 |
| $1^{3/2}P_2$ | $B_{s2}^*(5840)$ | 5839.7 ± 0.6 | 5838 | 5843 | 5842 | 5820 |
| $1^{3/2}D_1$ | | | 6167 | 6146 | 6209 | 6127 |
| $1^{3/2}D_2$ | | | 6186 | 6163 | 6218 | 6140 |
| $1^{5/2}D_2$ | | | 6098 | 6085 | 6189 | 6095 |
| $1^{5/2}D_3$ | | | 6109 | 6094 | 6191 | 6103 |
| $1^{5/2}F_2$ | | | 6405 | 6363 | 6501 | 6369 |
| $1^{5/2}F_3$ | | | 6416 | 6373 | 6515 | 6376 |
| $1^{7/2}F_3$ | | | 6313 | 6276 | 6468 | 6332 |
| $2^{1/2}S_0$ | | | 6010 | 5961 | 5976 | 5985 |
| $2^{1/2}S_1$ | | | 6048 | 6003 | 5992 | 6019 |
| $2^{1/2}P_0$ | | | 6291 | 6227 | 6318 | 6264 |
| $2^{1/2}P_1$ | | | 6323 | 6266 | 6345 | 6296 |
| $2^{3/2}P_1$ | | | 6288 | 6249 | 6321 | 6278 |
| $2^{3/2}P_2$ | | | 6304 | 6263 | 6359 | 6292 |
| $2^{3/2}D_1$ | | | 6540 | 6478 | 6629 | |
| $2^{3/2}D_2$ | | | 6553 | 6491 | 6651 | |
| $2^{5/2}D_2$ | | | 6487 | 6434 | 6625 | |
| $2^{5/2}D_3$ | | | 6496 | 6441 | 6637 | |
| $2^{5/2}F_2$ | | | 6723 | 6647 | | |
| $2^{5/2}F_3$ | | | 6731 | 6654 | 6880 | |
| $2^{7/2}F_3$ | | | 6650 | 6580 | | |

- ① Motivation
- ② The Relativistic Model
- ③ Solving the Wave Equation
- ④ Numerical Results and Discussion
- ⑤ Summary

Summary

- A equation that is equivalent to the originally relativistic Bethe-Salpeter equation is derived.
- The relativistic model is obtained through a heavy-quark expansion with the Foldy-Wouthuysen transformation.
- the string tension b depend on the mass of the constituent quark, especially of the light quark.
- Theoretical results are in good agreement with available experimental data except for the anomalous $D_{s0}^*(2317)$ and $D_{s1}(2460)$ states.
- Heavy-light meson decays can be studied in the relativistic model in further researches.

Thank you!