

# Three-body bound state in a box

— Preliminary results, work in progress

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CRC110 general meeting of 2017, Beijing,  
08/31/2017

# Introduction

## Spectrum of two-body bound state in finite volume

- Zero angular momentum [M.Lüscher, Commun. Math. Phys. 104, 177-206 (1986)]

$$\Delta E_0 = -3|\gamma|^2 \frac{e^{-\kappa L}}{\mu L} + \mathcal{O}(e^{-\sqrt{2}\kappa L})$$

- Non-zero angular momentum [S.König, D. Lee, and H.W.Hammer, PRL 107, 112001 (2011)]

$$\Delta E_l = c_l |\gamma|^2 \frac{e^{-\kappa L}}{\mu L} + \mathcal{O}(e^{-\sqrt{2}\kappa L})$$

## Spectrum of three-body bound state in finite volume

- Identical particles and zero angular momentum [U.Meißner, G.Ríos, A.Rusetsky, PRL 114, 091602(2015).]

$$\Delta E_0 = c_0 \frac{\kappa^2}{m} (\kappa L)^{-\frac{3}{2}} |A|^2 \exp\left(-\frac{2\kappa L}{\sqrt{3}}\right) + \dots$$

- The result is strictly valid in the unitary limit.

# Motivation:

- For spectrum of three-body bound state in finite volume
  - ▶ Non-identical three particles ?
  - ▶ Non-zero angular momentum ?
- For quantization condition of three-body in finite volume
  - ▶ Akaki's talk on three particles quantization condition.

# General description of three-body system

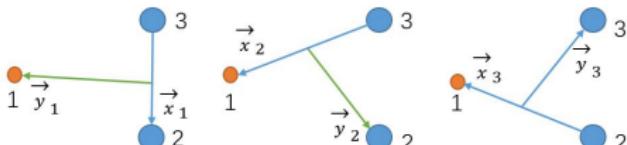
## Schrödinger equation in finite volume

$$\left[ \sum_{i=1}^3 \left( -\frac{1}{2m_i} \Delta_i^2 + V_i^L(\mathbf{x}_i) \right) + E_L \right] \psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = 0$$

- Jacobi coordinates set {i}

$$\mathbf{x}_i = \mu_{jk} (\mathbf{r}_j - \mathbf{r}_k)$$

$$\mathbf{y}_i = \mu_{i(jk)} \left( \frac{m_j \mathbf{r}_j + m_k \mathbf{r}_k}{m_j + m_k} - \mathbf{r}_i \right)$$



- Mass coefficient

$$\mu_{jk} = \sqrt{m_j m_k / (m(m_j + m_k))}, \quad \mu_{i(jk)} = \sqrt{m_i (m_j + m_k) / (m(m_i + m_j + m_k))}$$

Note:  $m$  is a normalization mass.

- Periodic boundary condition:

$$V_i^L(\mathbf{x}_i) = \sum_{\mathbf{n} \in \mathbb{Z}^3} V_i(\mathbf{x}_i + \mu_{jk} \mathbf{n} L)$$

# Energy shift formula

Considering the three-body systems:  $(m_1, m_2, m_3) = (z, 1, 1)$

- For  $z > 2/\sqrt{3} - 1 \approx 0.2$

$$\begin{aligned}\Delta E = & 6 \sum_{i=1}^3 \int d^3 \mathbf{x}_i d^3 \mathbf{y}_i \psi^*(\mathbf{x}_i, \mathbf{y}_i) V_i(\mathbf{x}_i) \psi(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i + \frac{m_k \mu_{i(jk)}}{m_j + m_k} eL) \\ & + 6 \sum_{i=1}^3 \int d^3 \mathbf{x}_i d^3 \mathbf{y}_i \psi^*(\mathbf{x}_i, \mathbf{y}_i) V_i(\mathbf{x}_i) \psi(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i + \frac{m_j \mu_{i(jk)}}{m_j + m_k} eL)\end{aligned}$$

- For  $0 < z < 2/\sqrt{3} - 1 \approx 0.2$

$$\begin{aligned}\Delta E = & 6 \sum_{i=1}^3 \int d^3 \mathbf{x}_i d^3 \mathbf{y}_i \psi^*(\mathbf{x}_i, \mathbf{y}_i) V_i(\mathbf{x}_i) \psi(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i + \frac{m_k \mu_{i(jk)}}{m_j + m_k} eL) \\ & + 6 \sum_{i=1}^3 \int d^3 \mathbf{x}_i d^3 \mathbf{y}_i \psi^*(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i) V_i(\mathbf{x}_i) \psi(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i)\end{aligned}$$

# Wave function of bound state

Wave function of non-identical three-body bound state with angular momentum ( $L, M$ ) under Hyperspherical coordinates ( $R, \alpha_i$ ) is given by [E. Nielsen et al, Physics Reports, 347, 373-459(2001)]

$$\begin{aligned} \psi^{LM}(x_i, y_i) = & \mathcal{N}_{L,M} R^{-\frac{5}{2}} \sum_{j=1}^3 \sum_{l_x l_y} f_0(R) A_j^{(l_x l_y)} \sin^{l_x} \alpha_j \cos^{l_y} \alpha_j \\ & \times P_{\nu_L}^{1/2 + l_x, 1/2 + l_y}(-\cos 2\alpha_j) \sum_{m_x + m_y = M} C_{l_x m_x, l_y m_y}^{LM} Y_{l_x m_x}(\Omega_{x_j}) Y_{l_y m_y}(\Omega_{y_j}) \end{aligned}$$

- Hyperspherical coordinates :  $x_i = R \sin \alpha_i, \quad y_i = R \cos \alpha_i$  .
- Radial asymptotic solution:  $f_0(R) = R^{1/2} K_i \xi_L(\kappa R), \quad \kappa \equiv (-2mE_T)^{1/2}$  .
- Jacobi polynomial:  $P_{\nu}^{a,b}(-z) = (-1)^{\nu} P_{\nu}^{a,b}(z)$  .
- Normalization coefficient  $\mathcal{N}_{L,M} : \int d^3 x_i d^3 y_i |\psi^{LM}(x_i, y_i)|^2 = A_i^{(l_x l_y)} (A_i^{(l_x l_y)})^*$  .
- $\nu_L$  is defines by eigenvalue  $\lambda_L$ :  $\nu_L \equiv -(2 + l_x + l_y)/2 + (4 + \lambda_L)^{1/2}/2$  .
  - ▶ For Efimov bound state:  $\lambda_L < -4$ , it is usually introduce:  $\lambda_L = -4 - \xi_L^2$ .

## Considering

- $l_x \geq 1$  Eigenfunctions are suppressed, thus choosing  $(l_x, l_y) = (0, L)$ .
- Unitary limit: any two-body scattering length  $a \gg L$ .

Under above conditions,  $\nu_L$  is determined by the equation [E. Nielsen et al, Physics Reports, 347, 373-459(2001)]

$$\begin{bmatrix} p & q_{12} & q_{13} \\ q_{21} & p & q_{23} \\ q_{31} & q_{32} & p \end{bmatrix} \begin{bmatrix} A_1^{(0,L)} \\ A_2^{(0,L)} \\ A_3^{(0,L)} \end{bmatrix} = 0$$

- where

$$p = \frac{\sin(\pi(\nu_L + d/2))}{\sin(\pi d/2)}$$

$$q_{ij} = \frac{\Gamma(d/2)\Gamma(\nu_L + d/2 + L)}{\Gamma(d/2 + L)\Gamma(\nu_L + d/2)} F(-\nu_L, \nu_L + L + d - 1; d/2 + L; \cos \gamma_{ij})(-\cos \gamma_{ij})^L$$

- Relation of  $A_i^{(0,L)}$ :

$$\frac{A_2^{(0,L)}}{A_3^{(0,L)}} = \frac{pq_{23} - q_{21}q_{13}}{q_{21}q_{12} - p^2} \equiv A_{23}^L, \quad \frac{A_1^{(0,L)}}{A_3^{(0,L)}} = \frac{pq_{12} - q_{32}q_{13}}{q_{32}p - q_{12}q_{31}} \equiv A_{13}^L$$

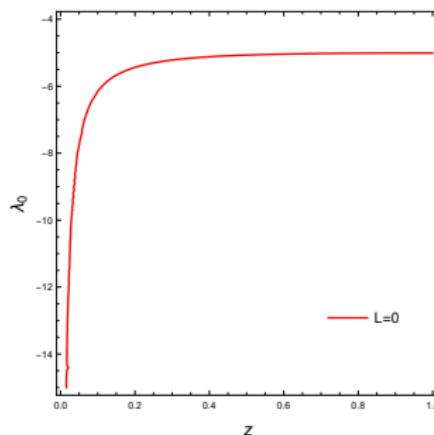
# Angular eigenvalue

With above considering:

$$\psi^{LM}(x_i, y_i) = (-1)^{\nu_L} \frac{\mathcal{N}_{L,M}}{\sqrt{4\pi}} R^{-\frac{5}{2}} f_0^{(L)}(R) \sum_{j=1}^3 A_j^{(0,L)} (\cos^L \alpha_j) P_{\nu_L}^{1/2, 1/2+L}(\cos 2\alpha_j) Y_{LM}(\Omega_{y_j})$$

Eigenvalue is obtained as:

- The lowest angular eigenvalue  $\lambda_L$ .
- $z = 1, \xi_0 = 1.00624$ .
- Only  $L = 0$  considered at present,  
so we omit the superscript of  $A_i^{(0,L)}$ .
- Results for  $L = 0, 1$  with the assumption  
 $A_1 = 0$  is obtained by [E. Nielsen  
et al, Physics Reports, 347, 373-459(2001)].



# Transformation

## Denoting

- $\psi^{LM}(x_i, y_i) \doteq \sum_{j=1}^3 \phi^{LM}(R, \alpha_j)$
- $\psi^{LM}(x_i - \mu_{jk} eL, y_i + \frac{m_k \mu_{i(jk)}}{m_j + m_k} eL) = \sum_{j=1}^3 \phi(R', \alpha'_j)$

## Coordinates transformation

- $(x_i, y_i) \rightarrow (x_i - \mu_{jk} eL, y_i + \frac{m_k \mu_{i(jk)}}{m_j + m_k} eL) ; \quad \tan \alpha'_i \xrightarrow{L \rightarrow \infty} \frac{\mu_{jk} (m_j + m_k)}{\mu_{i(jk)} m_k}$
- $(x_j, y_j) \rightarrow (x_j + \mu_{ki} eL, y_j + \frac{m_k \mu_{j(ki)}}{m_j + m_k} eL) ; \quad \tan \alpha'_j \xrightarrow{L \rightarrow \infty} \frac{\mu_{ki} (m_j + m_k)}{\mu_{j(ki)} m_k}$
- $(x_k, y_k) \rightarrow (x_k, y_k - \mu_{k(ij)} eL) ; \quad \tan \alpha'_k \xrightarrow{L \rightarrow \infty} \frac{R \sin \alpha'_k}{\mu_{k(ij)} L}$
- $R' \xrightarrow{L \rightarrow \infty} \mu_{k(ij)} L - \frac{\mu_{jk}}{\mu_{k(ij)}} e \cdot x_i + \frac{m_k \mu_{i(jk)}}{(m_j + m_k) \mu_{k(ij)}} e \cdot y_i$

## Hyperradial wavefunction

- $f_0(R') \rightarrow \sqrt{\frac{\pi}{2}} e^{-\kappa R'} \frac{1}{\sqrt{\kappa}}$

# Results

- Identity

$$V_i(\mathbf{x}_i)\psi(\mathbf{x}_i, \mathbf{y}_i) = [\frac{1}{m_i}(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}) - \kappa^2]\phi(R, \alpha_i)$$

- For system ( $z > 2/\sqrt{3} - 1$ ), it arrives at

$$\begin{aligned} \Delta E_0 = & 6 \sum_{i=1}^3 \int d^3x_i d^3y_i [\psi^*(\mathbf{x}_i, \mathbf{y}_i) V_i(\mathbf{x}_i) \psi(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i + \frac{m_k \mu_{i(jk)}}{m_j + m_k} eL) \\ & + \psi^*(\mathbf{x}_i, \mathbf{y}_i) V_i(\mathbf{x}_i) \psi(\mathbf{x}_i - \mu_{jk} eL, \mathbf{y}_i + \frac{m_j \mu_{i(jk)}}{m_j + m_k} eL)] \\ = & - 12 \times 2^{-1/2} \pi^{5/2} |C_0|^2 |\mathcal{N}_{0,0}|^2 (\kappa L)^{-3/2} |A_3|^2 \\ & \times [\frac{J_1 A_{13}}{\sin \gamma_{13}} \frac{[\mu_{3(12)} + \mu'_{3(12)}] \mu_{3(12)}^{-5/2}}{m_1} \exp(-\mu_{3(12)} \kappa L) + \frac{J_2 A_{23} A_{13}}{\sin \gamma_{21}} \frac{\mu_{1(23)}^{-3/2}}{m_2} \exp(-\mu_{1(23)} \kappa L) \\ & + \frac{J_3 A_{23}}{\sin \gamma_{32}} \frac{\mu_{2(31)}^{-3/2}}{m_3} \exp(-\mu_{2(31)} \kappa L)] + \dots \end{aligned}$$

►  $C_0 = -\frac{1}{4\pi} \frac{\Gamma(\nu_0+1/2)}{\Gamma(\nu_0+2)\Gamma(3/2)} (\nu_0 + 1) \cosh(\pi\xi_0/4), \mu'_{2(31)} = (1-z)[(2+z)(1+z)]^{-1/2}.$

►  $J_i = \int dz d\theta \cos \theta_{y_i} K_{i\xi_0} \exp(-a_i z \cos \theta_{y_i}).$

►  $a_i = [m_i m_k / ((m_i + m_j)(m_j + m_k))]^{1/2}, \gamma_{ij} = \arctan(\sigma\{i, j, k\} \sqrt{\frac{m_k(m_i+m_j+m_k)}{m_i m_j}}).$

# Results

- If denoting

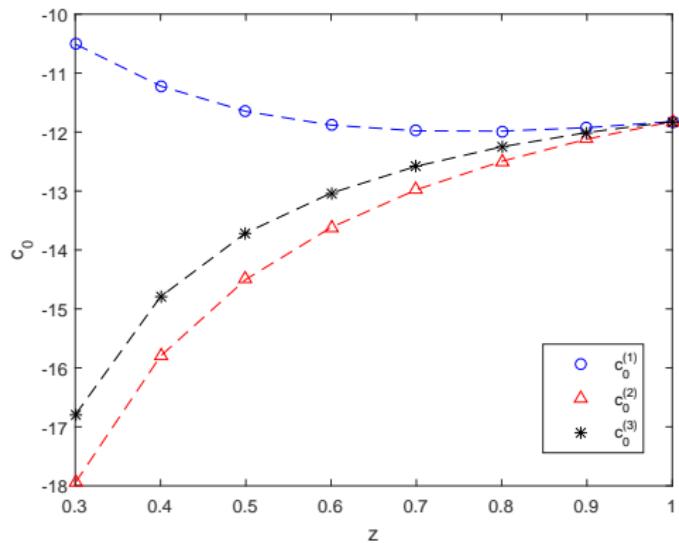
$$\Delta E_0 = (\kappa L)^{-3/2} \kappa^2 |A_3|^2 \left( \frac{c_0^{(1)}}{m_1} \exp(-\mu_{3(12)} \kappa L) + \frac{c_0^{(2)}}{m_2} \exp(-\mu_{1(23)} \kappa L) \right. \\ \left. + \frac{c_0^{(3)}}{m_3} \exp(-\mu_{2(31)} \kappa L) \right) + \dots$$

- ▶  $c_0^{(1)} = -12 \times 2^{-1/2} \pi^{5/2} |C_0|^2 |\mathcal{N}_{0,0}|^2 \frac{J_1 A_{13}}{\sin \gamma_{13}} [\mu_{3(12)} + \mu'_{3(12)}] \mu_{3(12)}^{-5/2}$ .
- ▶  $c_0^{(2)} = -12 \times 2^{-1/2} \pi^{5/2} |C_0|^2 |\mathcal{N}_{0,0}|^2 \frac{J_2 A_{23} A_{13}}{\sin \gamma_{21}} \mu_{1(23)}^{-3/2}$ .
- ▶  $c_0^{(3)} = -12 \times 2^{-1/2} \pi^{5/2} |C_0|^2 |\mathcal{N}_{0,0}|^2 \frac{J_3 A_{23}}{\sin \gamma_{32}} \mu_{2(31)}^{-3/2}$ .

- The dependence of  $(c_0^{(1)}, c_0^{(2)}, c_0^{(3)})$  on  $z$  arrives at

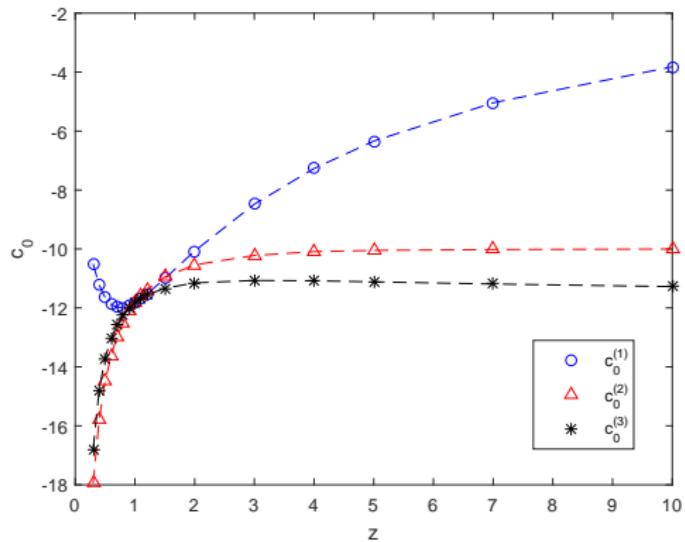
- ▶  $(m_1, m_2, m_3) = (z, 1, 1), L = 0$

# Results



- Especially, the point for  $z = 1$  is just the equal-mass case, which is studied by [U.Meißner,G.Ríos, A.Rusetsky, PRL 114,091602(2015)].

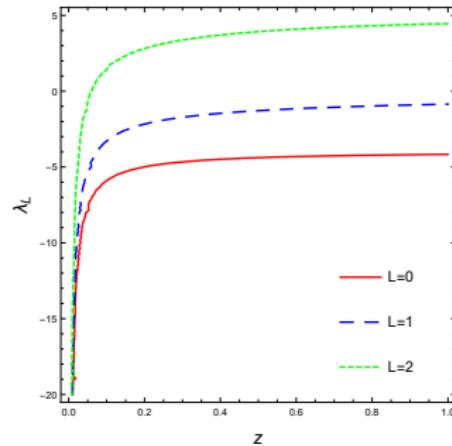
# More results



- $(c_0^{(1)}, c_0^{(2)}, c_0^{(3)}) = (-0.47; -10.15; -11.59)$  for  $z = 100$ .
- $(c_0^{(1)}, c_0^{(2)}, c_0^{(3)}) = (-0.048; -10.16; -11.63)$  for  $z = 1000$ .

Question: why not consider  $L \geq 1$  for system  $z > 2/\sqrt{3} - 1 \approx 0.2$  ?

- Choosing  $A_1^L = 0$  for simplicity.



- $z < 0.067$  for  $\lambda_1 < -4$ ;  $z < 0.025$  for  $\lambda_2 < -4$ .

- Conclusion:

No Efimov bound state exists with  $z > 2/\sqrt{3} - 1$  for  $L \geq 1$ .

# Summary and outlook

## • Summary

- ▶ Spectrum of identical three-body bound state in finite volume is generalized to non-identical case.
- ▶ Three-body system with  $(z, 1, 1)$  for  $z > 2/\sqrt{3} - 1$  is studied for  $L = 0$ , and the exact energy shift formula in this case is obtained.
- ▶ Our method here can be applied for three-body system with any masses and any angular momentum  $L$ .

## • Outlook

- ▶ Considering the three-body system with  $z < 2/\sqrt{3} - 1$  for  $L = 0$ .
- ▶ Considering  $L \neq 0$ .

Work in progress ...