Calculate PDFs using “lattice cross sections”

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Based on YQM, Qiu, 1404.6860, and work *in preparation*

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Outline

I. Introduction to PDFs

II. Lattice cross sections

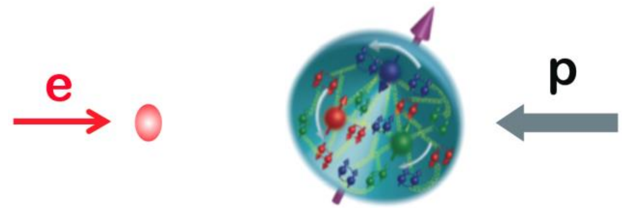
III. Relation to other methods

IV. Summary

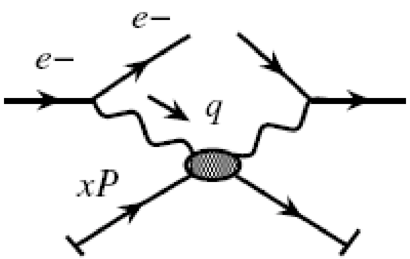
QCD factorization

- The key and a first principle method to relate experimental data to QCD theory

- Electron-hadron:



$\sigma_{\text{tot}}^{\text{DIS}}$ =



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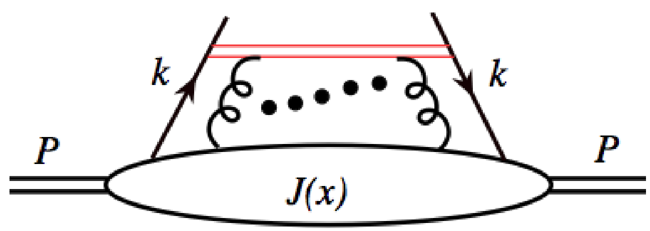


$+\,O\left(\frac{1}{QR}\right)$

Hard-part Probe	Parton-distribution Structure	Correction Approximation

Operator definition of PDFs

- Spin-averaged quark distribution

$$f_{q/p}(x,\mu^2) = \int \frac{d\xi_-}{4\pi} e^{-ix\xi_- P_+} \langle P|\overline{\psi}(\xi_-)\,\gamma_+ \exp\left\{-ig\int_0^{\xi_-} d\eta_- A_+(\eta_-)\right\}\psi(0)|P\rangle$$

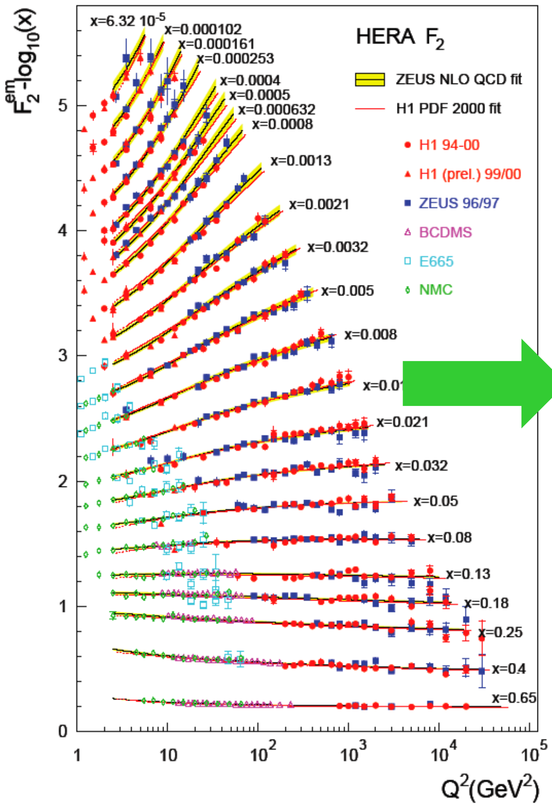
  - Simplest of all parton correlation functions
  - Unlike cross section, not direct physical observable; but well defined in QCD

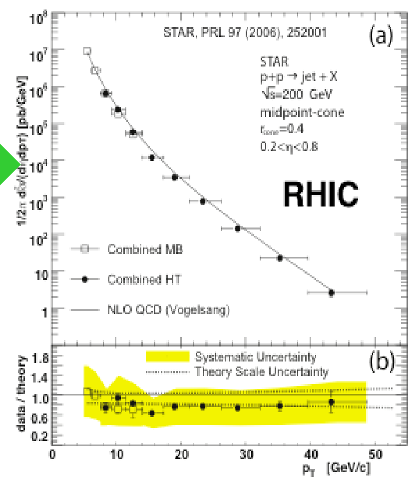
- Boost invariant along “+” direction
- Parton interpretation emerges in $A_+ = 0$ gauge
- Logarithmic UV divergent, renormalizable
- Time dependent!

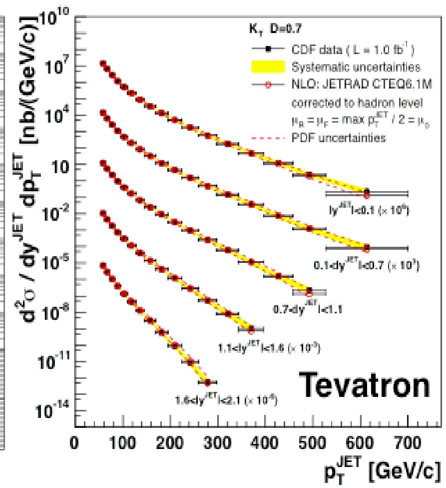
Extract PDFs by fitting data

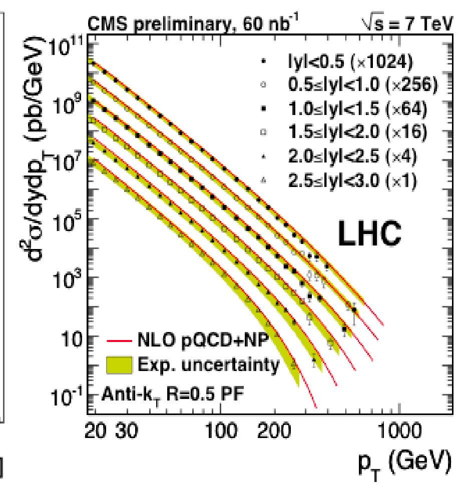
- **Successful**

Measure e-p at 0.3 TeV (HERA)
Predict p-p at 0.2, 1.96, and 7 TeV

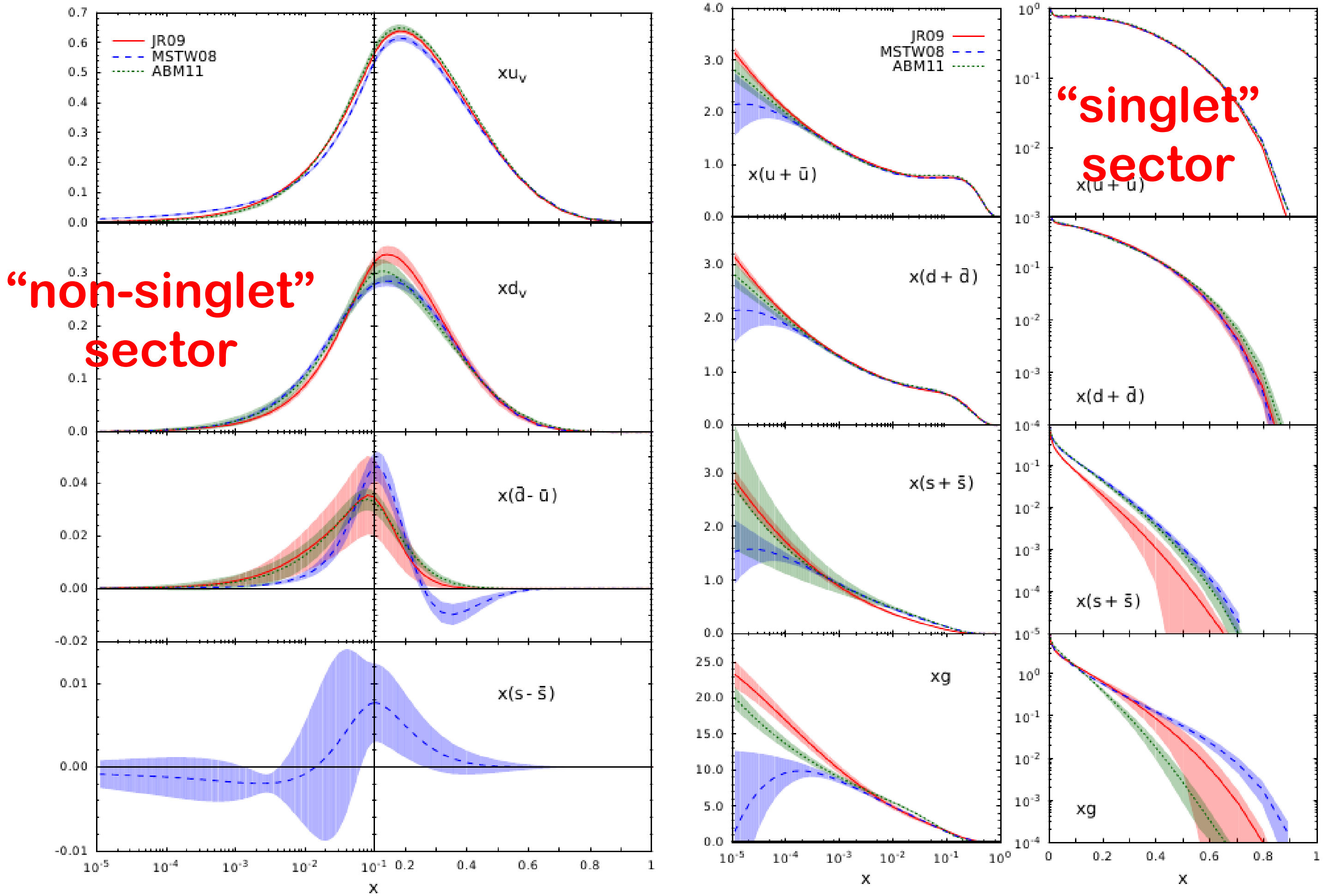








Uncertainty of PDFs

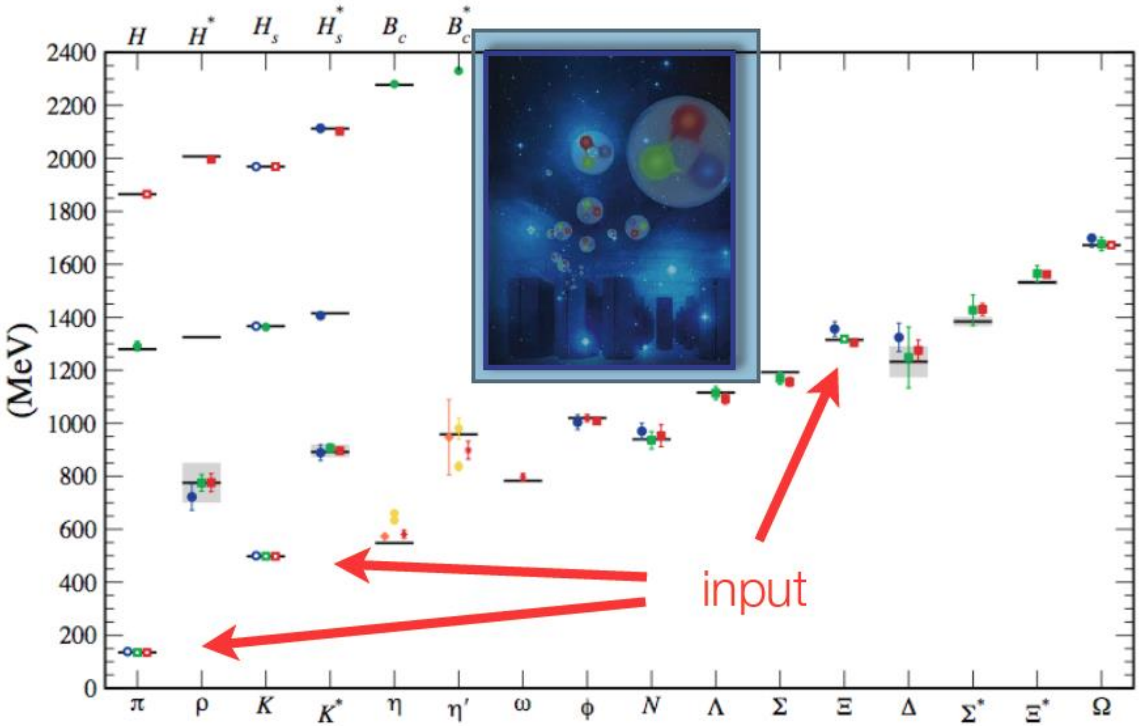


Question

How to determine PDFs nonperturbatively from first principle?

Lattice QCD

- The main nonperturbative approach to solve QCD
- Predict the hadron mass



(Kronfeld)

- An intrinsically Euclidean time: $\tau = i\, t$

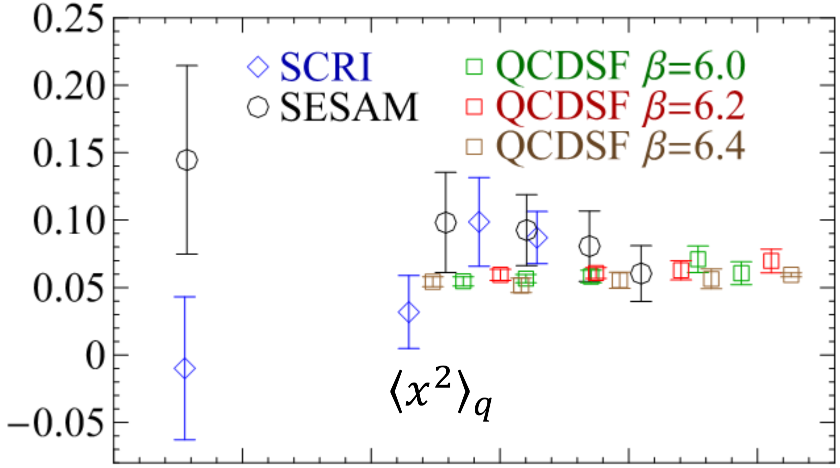
Cannot calculate PDFs directly

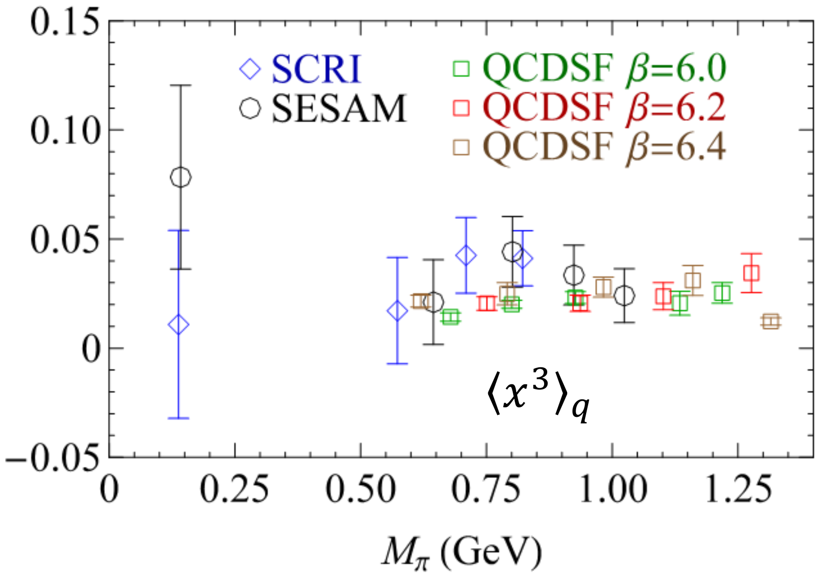
Traditional method

- ➢ **Moments: matrix elements of local operators**

$$\langle x^n(\mu^2)\rangle_q \equiv \int_0^1 dx\, x^n f_{q/p}(x,\mu^2)$$

- ➢ **Works, but only for limited moments**





Dolgov et al., hep-lat/0201021 Gockeler et al., hep-ph/0410187

Preview of new approaches

- **Quasi PDFs** Ji, 1305.1539

goes to PDFs as $P_z \to \infty$

$$\tilde{f}_{q/p}(x,\mu^2,P_z) = \int \frac{d\xi_z}{4\pi} e^{-ix\xi_z P_z} \langle P|\overline{\psi}(\xi_z)\,\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle$$

- **OPE without OPE** Chambers, *et. al.* , 1703.01153

$$T_{\mu\nu}(p,q) = \rho_{\lambda\lambda'} \int d^4x e^{iq\cdot x} \langle p,\lambda'|TJ_\mu(x)J_\nu(0)|p,\lambda\rangle$$

- **Lattice cross sections** YQM, Qiu, 1404.6860, and work *in preparation*

$$\sigma_n(\xi^2,\omega,P^2) = \langle P|T\{\mathcal{O}_n(\xi)\}|P\rangle$$

$$\widetilde{\sigma}_n(q^2,\widetilde{\omega},P^2) = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} \sigma_n(\xi^2, P\cdot\xi, P^2)$$

Both “Quasi PDFS” and “OPE /o OPE” are special cases of “Lattice cross sections”

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A "no-go theorem"

- **One can never accurately calculate a time-dependent quantity on lattice QCD**
  - Lattice QCD has imaginary time
  - One will encounter difficulty whenever approaching its exact value
  - Quasi PDFs goes to PDFs as $P_z \to \infty$, but one cannot take $P_z \to \infty$ on lattice QCD

- **Comparison**
  - One can never exactly determine the position of a particle in quantum mechanism

Generalized uncertainty principle

- **A possible way to determine a time-dependent quantity on lattice:**
  - Summing over time with a weighting function

    $P(u) = \int dt\ C(t,u) O(t) \rightarrow$ calculate $P(u)$
  - If $C(t,u)$ is peaked around t_0, $P(u)$ can be used to determine $O(t_0)$
  - If $C(t,u)$ is very sharp, uncertainty of calculated $P(u)$ will be large
  - A "generalized uncertainty principle": determine $O(t)$ within a tolerant uncertainty
- **Comparison**
  - One can determine the position of a particle within the uncertainty:

$$\Delta x\ \Delta p \geq \hbar/2$$

"Lattice cross sections"

$$P(u) = \int dt\, C(t,u) O(t)$$

➢ Conditions for $P(u)$ to be useful

1. Calculable on lattice QCD using an Euclidean time: has an operator definition in QCD, no time dependence
2. With known $P(u)$, $C(t,u)$, which relating $P(u)$ to $O(t)$, can be calculated using other methods

➢ Benefit from asymptotic freedom

- Occasionally, $C(t,u)$ is perturbatively calculable
- $P(u)$ can be factorized to $O(t)$

Then $P(u)$ is a "lattice cross section" (LCS) to determine $O(t)$

LCSs to determine PDFs

- **LCSs in coordinate space** $\sigma_n(\xi^2,\omega,P^2)$ $\quad \omega = P\cdot\xi$

  and in momentum space $\widetilde{\sigma}_n(q^2,\widetilde{\omega},P^2)$ $\quad \widetilde{\omega} = \frac{2P\cdot q}{q^2}$
  - Basically hadronic matrix elements
  - $1/\xi^2$ and q^2: hard scales to enable factorization
  - ω and $\widetilde{\omega}$: parameters
- **Conditions for a good LCS**

  ① Calculable on Euclidean lattice QCD

  ② Renormalizable for UV divergences

  ③ Factorizable for CO divergence with IR safe coefficients

$$\sigma_n(\xi^2,\omega,P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x,\mu^2) K_n^a(\xi^2, x\omega, x^2P^2, \mu^2) + O(\xi^2\Lambda_{\rm QCD}^2)$$

  - The last condition relates LCSs to PDFs

Hadronic matrix elements

➢ Coordinate space or momentum space

$$\sigma_n(\xi^2, \omega, P^2) = \langle P | T\{\mathcal{O}_n(\xi)\} | P \rangle \qquad \widetilde{\sigma}_n(q^2, \widetilde{\omega}, P^2) = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} \sigma_n(\xi^2, P\cdot\xi, P^2)$$

➢ Possible choices of the nonlocal operator

$$\begin{aligned}
\mathcal{O}_S(\xi) &= \xi^4 Z_S^{-2} [\overline{\psi}_q \psi_q](\xi)\, [\overline{\psi}_q \psi_q](0)\,, \\
\mathcal{O}_{V_1}(\xi) &= -\frac{\xi^4}{2} Z_V^{-2} [\overline{\psi}_q \gamma_\nu \psi_q](\xi)\, [\overline{\psi}_q \gamma^\nu \psi_q](0)\,, \\
\mathcal{O}_{V_2}(\xi) &= \xi^2 Z_V^{-2} [\overline{\psi}_q \not{\xi} \psi_q](\xi)\, [\overline{\psi}_q \not{\xi} \psi_q](0)\,, \\
\mathcal{O}_{V_2'}(\xi) &= \xi^2 Z_{V'}^{-2} [\overline{\psi}_q \not{\xi} \psi_{q'}](\xi)\, [\overline{\psi}_{q'} \not{\xi} \psi_q](0)\,,
\end{aligned}$$

- Gauge invariant locally
- Renormalization is very simple

$$\mathcal{O}_q(\xi) = Z^{-1}(\xi^2) \overline{\psi}_q(\xi)\, \not{\xi}\, \Phi(\xi, 0)\, \psi_q(0)\,,$$

$$\Phi(\xi, 0) = \mathcal{P} e^{-ig\int_0^1 \xi\cdot A(\lambda\xi)\, d\lambda}$$

- Gauge dependent locally, path ordered gauge link is needed
- Renormalization is complicated but known

Ishikawa YQM, Qiu, Yoshida, 1707.03107

Straight forward to construct much more operators with both quark fields and gluon fields

Factorization

➢ **All of them are factorizable to all orders in perturbation theory**

YQM, Qiu, work *in preparation*

$$\sigma_n(\xi^2,\omega,P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x}\, f_a(x,\mu^2)\, K_n^a(\xi^2, x\omega, x^2P^2,\mu^2) + O(\xi^2\Lambda_{\rm QCD}^2)$$

$$\widetilde{\sigma}_n = \sum_a f_a \otimes \widetilde{K}_n^a + O(\Lambda_{\rm QCD}^2/q^2)$$

$$f_{\bar{a}/h}(x,\mu^2) = -f_{a/h}(-x,\mu^2)$$

where

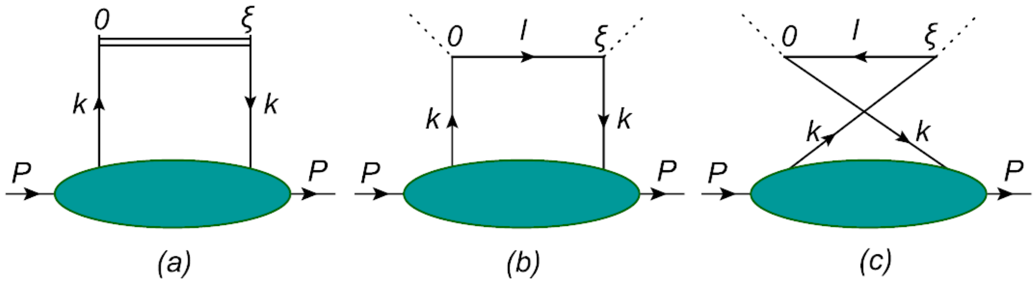
$$K_n^a = \sum_J 2W_n^{(J,a)}(\xi^2,\mu^2)\,\Sigma_J(x\omega, x^2P^2\xi^2)$$

$$\widetilde{K}_n^a = \int \frac{d^4\xi}{\xi^4}\, e^{iq\cdot\xi} K_n^a(\xi^2, xP\cdot\xi, x^2P^2,\mu)$$

- **Note:** $\widetilde{K}_n^a$ **is unambiguous only for** $\widetilde{\omega}^2 < 1$

Matching coefficients

- **Obtained by calculating Feynman diagrams**



$$K_q^{q(0)}(Q^2, x\omega, 0, \mu) = \frac{1}{2}\mathrm{Tr}[\not{k}\not{\xi}]e^{-i\xi\cdot k} = 2x\omega e^{-ix\omega}$$

$$K_S^{q(0)}(Q^2, x\omega, 0, \mu) = ix\omega\left(e^{ix\omega} - e^{-ix\omega}\right)$$

$$\widetilde{K}_S^{q(0)}(Q^2, x\widetilde{\omega}, 0, \mu) = \frac{x^2\widetilde{\omega}^2}{1 - x^2\widetilde{\omega}^2 - i\varepsilon}$$

Good LCSs

➢ Conditions satisfied up to now

- **Operators, and thus matrix elements, are renormalizable**
- **Matrix elements are factorizable**

➢ Final condition: calculable on Euclidean lattice

$$\sigma_n(\xi^2, \omega, P^2) = \langle P|T\{\mathcal{O}_n(\xi)\}|P\rangle$$ **with** $\xi_0 = 0$

$$\widetilde{\sigma}_n(q^2, \widetilde{\omega}, P^2) = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} \sigma_n(\xi^2, P\cdot\xi, P^2)$$ **with** $q_0 = 0$

With these conditions, σ_n and $\tilde{\sigma}_n$ are good LCSs to extract PDFs

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Relation to quasi PDFs

➤ Quasi PDFs

Ji, 1305.1539

$$\tilde{f}_{q/p}(x,\mu^2,P_z) = \int \frac{d\xi_z}{4\pi} e^{-ix\xi_z P_z} \langle P|\overline{\psi}(\xi_z)\,\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle$$

➤ Linear combination of LCSs

$$\sigma_n^{\mathrm{II}}(|\vec{P}|^2\cos^2\theta, x, P^2) = \int \frac{d\omega}{\omega}\frac{e^{-ix\omega}}{4\pi}\sigma_n(\xi^2,\omega,P^2)$$

with fixed $|\vec{P}|$ $\quad \xi^2 = -\frac{\omega^2}{|\vec{P}|^2\cos^2\theta}$

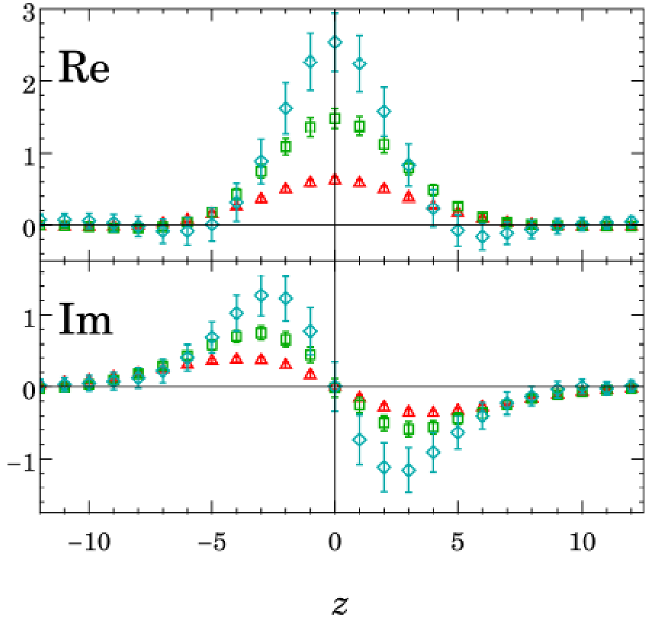
σ_q^{II} **reproduces quasi PDF if**

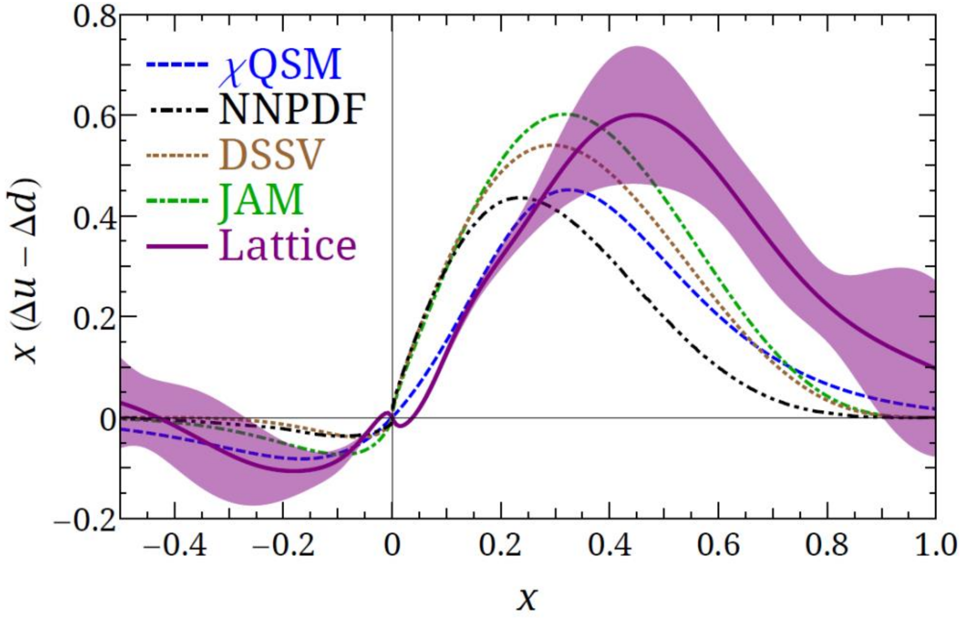
both $\vec{P}$ and $\vec{\xi}$ are along "z" direction

Lattice results of quasi PDFs

➢ Exploratory studies

Lin et al. 1402.1462
Alexandrou et al. 1504.07455
Chen et al. 1603.06664





- Works, convergence not bad
- Shape similar to experimental data
- Renormalization is needed, but complicated

Relation to OPE /o OPE

➢ OPE without OPE

Chambers, *et. al.* , 1703.01153

$$T_{\mu\nu}(p,q) = \rho_{\lambda\lambda'} \int d^4x e^{iq\cdot x} \langle p,\lambda' | T J_\mu(x) J_\nu(0) | p,\lambda \rangle$$

With $\mu = \nu = 3$ and $p_3 = q_3 = q_4 = 0$

$$T_{33}(p,q) = 4\omega \int_0^1 dx \frac{\omega x}{1-(\omega x)^2} F_1(x,q^2)$$

➢ $\tilde{\sigma}_n$ with $n = S$

$$\widetilde{\sigma}_S \approx \int_{-1}^{1} dx \, \frac{x\widetilde{\omega}^2}{1 - x^2\widetilde{\omega}^2 - i\varepsilon} f_q(x,\mu^2) + O(\alpha_s)$$

- Reproduces the T_{33} in "OPE /o OPE"
- Can be used to extract PDFs if $\widetilde{\omega}^2 < 1$

Lattice results of OPE /o OPE

➢ **Exploratory studies**

Chambers, *et. al.* , 1703.01153

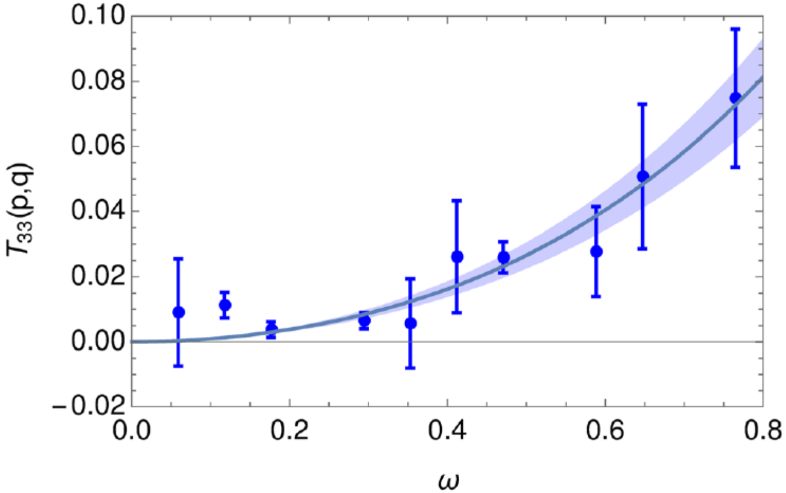


FIG. 6. The proton Compton amplitude $T_{33}(p,q)$ for momenta $\vec{p} = (2,-1,0)$, $(-1,1,0)$, (1,0,0), (0,1,0), (2,0,0), $(-1,2,0)$, (1,1,0), (0,2,0), (2,1,0), (1,2,0), from left to right, and $\vec{q} = (3,5,0)$, in lattice units. The current has been attached to the d quark, leading to the 'handbag' diagram in Fig. 1. Z_V has been taken from [17]. The solid line shows a sixth order polynomial fit (giving $\chi^2/\mathrm{dof} = 0.9$), and the shaded area shows the error.

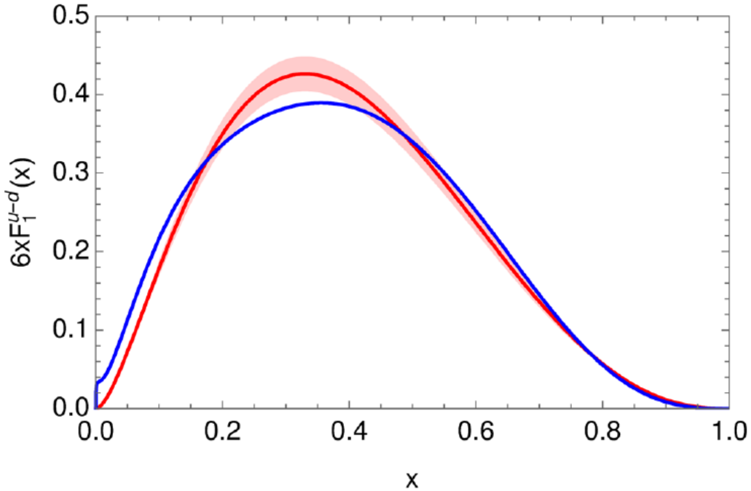


FIG. 5. The structure function $F_1^{u-d}(x)$ obtained from the Mellin transform of Eq. (22) fitted to the moments (−), compared with the target structure function $6xF_1(x) = x[u(x) - d(x)]$ (−). For the error estimate see the text.

- **Shape similar to target structure function**
- **Constrained by $\tilde{\omega}^2 < 1$, may not have enough information to determine PDFs**

Summary

- “Lattice cross section” = hadronic matrix elements that are calculable + renormalizable + factorizable
- Quasi-PDFs and “OPE /o OPE” are special cases of good LCSs
- Much more good LCSs are available to determine PDFs
- Construction of good LCSs for GPDs, TMDs, …, are straight forward

Thank you!

Back up

OPE

➢ Factorization

$$\sigma_n(\xi^2,\omega,P^2) = \sum_{J,a} W_n^{(J,a)}(\xi^2,\mu^2)\xi^{\nu_1}\cdots\xi^{\nu_J}\langle P|\mathcal{O}^{(J,a)}_{\nu_1\cdots\nu_J}(\mu^2)|P\rangle$$

$$\langle P|\mathcal{O}^{(J,a)}_{\nu_1\cdots\nu_J}(\mu^2)|P\rangle = 2A^{(J,a)}(\mu^2)\left(P_{\nu_1}\cdots P_{\nu_J} - \text{traces}\right)$$

$$\begin{aligned}\Sigma_J(\omega, P^2\xi^2) &\equiv \xi^{\nu_1}\cdots\xi^{\nu_J}\left(P_{\nu_1}\cdots P_{\nu_J} - \text{traces}\right)\\ &= \sum_{i=0}^{i_{\max}} C^i_{J-i}(\omega)^{J-2i}\left(P^2\xi^2/4\right)^i ,\end{aligned}$$

$$A^{(J,a)}(\mu^2) = \int_{-1}^{1} dx x^{J-1} f_{a/h}(x,\mu^2)$$

$$K_n^a = \sum_J 2W_n^{(J,a)}(\xi^2,\mu^2)\,\Sigma_J(x\omega, x^2P^2\xi^2) \qquad |\omega| \ll 1 \text{ and } |P^2\xi^2| \ll 1$$

$$\widetilde{K}_n^a = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} K_n^a(\xi^2, xP\cdot\xi, x^2P^2, \mu)$$

Momentum space

➢ Condition for factorization

$$\widetilde{\sigma}_n(q^2, \widetilde{\omega}, P^2) = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} \sigma_n(\xi^2, P\cdot\xi, P^2) \qquad \widetilde{\omega} = \frac{2P\cdot q}{q^2}$$

$$\int \frac{d^4\xi}{\xi^4} \xi^\nu \, e^{i(q+xP)\cdot\xi}$$

$$\widetilde{\sigma}_n = \sum_a f_a \otimes \widetilde{K}_n^a + O(\Lambda_{\rm QCD}^2/q^2)$$

$$\widetilde{K}_n^a = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} K_n^a(\xi^2, xP\cdot\xi, x^2P^2, \mu) \qquad \widetilde{\omega}^2 < 1$$

Factorization

YQM, Qiu, 1404.6860, 1412.2688

- Factorize the last kernel, and then recursively:

$\widehat{\mathcal{P}}$: pick up the singular part of integration

$$\tilde{f}_{q/p} = \lim_{m\to\infty} C_0 \sum_{i=0}^{m} K_0^i + \text{UVCT} = \lim_{m\to\infty} C_0 \sum_{i=0}^{m} K_0^i$$

$$= \lim_{m\to\infty} C_0 \left[1 + \sum_{i=0}^{m-1} K^i (1-\widehat{\mathcal{P}})K\right]_{\text{ren}} + \tilde{f}_{q/p}\,\widehat{\mathcal{P}}\,K$$

$$= \lim_{m\to\infty} C_0 \left[1 + \sum_{i=1}^{m} \left[(1-\widehat{\mathcal{P}})K\right]^i\right]_{\text{ren}} + \tilde{f}_{q/p}\,\widehat{\mathcal{P}}\,K\,,$$

Normal PDFs

All CO divergences of quasi PDF

$$\tilde{f}_{q/p} = \left[C_0 \frac{1}{1-(1-\widehat{\mathcal{P}})K}\right]_{\text{ren}} \left[\frac{1}{1-\widehat{\mathcal{P}}K}\right]$$

Finite

$$\tilde{\sigma}_{\text{M}}(\tilde{x}, \tilde{\mu}^2, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_{i/h}(x, \mu^2)\, \mathcal{C}_i\!\left(\frac{\tilde{x}}{x}, \tilde{\mu}^2, \mu^2, P_z\right) + O\!\left(\tilde{\mu}^{-2} + (\tilde{x}P_z)^{-2}\right)$$

- Factorizable as far as quasi-PDFs are multiplicatively renormalized

How about renormalization?

➢ Coordinate space definition

$$\tilde{F}_{q/p}(\xi_z, \tilde{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z} \langle h(p)|\overline{\psi}_q(\xi_z)\, \frac{\gamma_z}{2} \Phi_{n_z}^{(f)}(\{\xi_z, 0\})\, \psi_q(0)|h(p)\rangle$$

➢ Conjecture of all-orders renormalization

$$\tilde{F}^R_{i/p}(\xi_z, \tilde{\mu}^2, p_z) = e^{-C_i|\xi_z|} Z_{wi}^{-1} Z_{vi}^{-1} \tilde{F}^b_{i/p}(\xi_z, \tilde{\mu}^2, p_z)$$

Ishikawa, YQM, Qiu, Yoshida, 1609.02018
Chen, Ji, Zhang, 1609.08102
Constantinou, H. Panagopoulos, 1705.11193
…

- Rigorous proof is needed!

Proof: Importance and difficulty

➢ Why proof is important?

- All-order proof of factorization needs multiplicative renormalization YQM, Qiu, 1404.6860, 1412.2688
- Whether mixing with other operators under renormalization? A close set of operators are needed

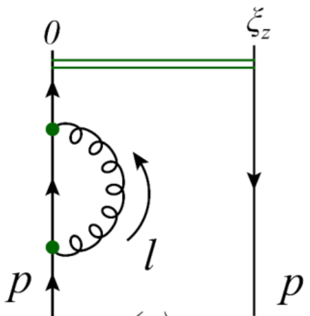
➢ Why proof is difficult

- Because of z-direction dependence, Lorentz symmetry is broken, hard to exhaust all possible UV divergences
- Renormalization of composite operator is needed

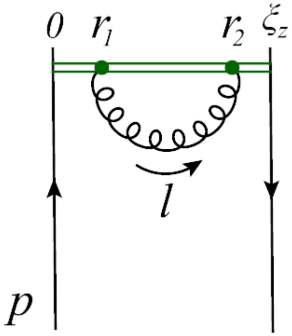
Broken of Lorentz symmetry

➢ Identifying UV divergences

- Renormalization of QCD in covariant gauge: only from 4-dimensional loop integration, all components become large
- Quasi-PDFs: 3-dimensional integration as while as 4-dimensional integration can generate UV divergences



(c)



(a)

$$l^\mu = \bar{l}^\mu + l_z n_z^\mu$$

UV: 4-D integration

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 (p-l)^2}$$

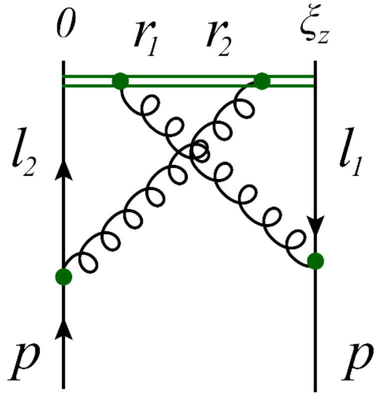
UV: 3-D integration

$$\int \frac{d^3 \bar{l}}{l^2} = \int \frac{d^3 \bar{l}}{\bar{l}^2 - l_z^2}$$

Broken of Lorentz symmetry con.

➢ Hard to identify all UV regions

- Need to consider 3-D and 4-D integrations for each loop



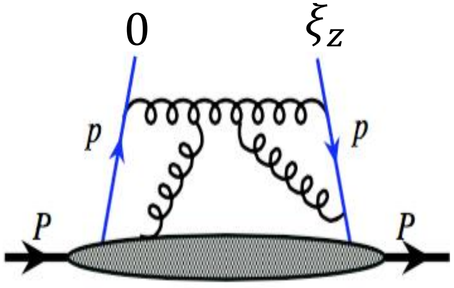
① l_1: 3-D, l_2: 3-D
② l_1: 3-D, l_2: 4-D
③ l_1: 4-D, l_2: 3-D
④ l_1: 4-D, l_2: 4-D

- A n-loop diagram, to identify all possible UV divergences, needs consider 2^n different cases!

Composition operator renormalization

➢ Quasi-quark PDF in $A_z = 0$ gauge: no gauge link

$$\tilde{F}_{q/p}(\xi_z, \tilde{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z} \langle h(p)|\overline{\psi}_q(\xi_z)\, \frac{\gamma_z}{2}\, \psi_q(0)|h(p)\rangle$$



- Renormalization of quark field $\bar{\psi}_q$ and ψ_q: taking care by renormalized QCD Lagrangian
- Renormalization of the bi-local operator as a whole: still needs to study

➢ Comparison: Quark PDF in $A_+ = 0$ gauge

- Similar for quark field renormalization
- Renormalization of the bi-local operator as a whole: needed!
- It is this renormalization that mixes quark PDF with gluon PDF

Keys for a rigorous proof

Ishikawa YQM, Qiu, Yoshida, 1707.03107

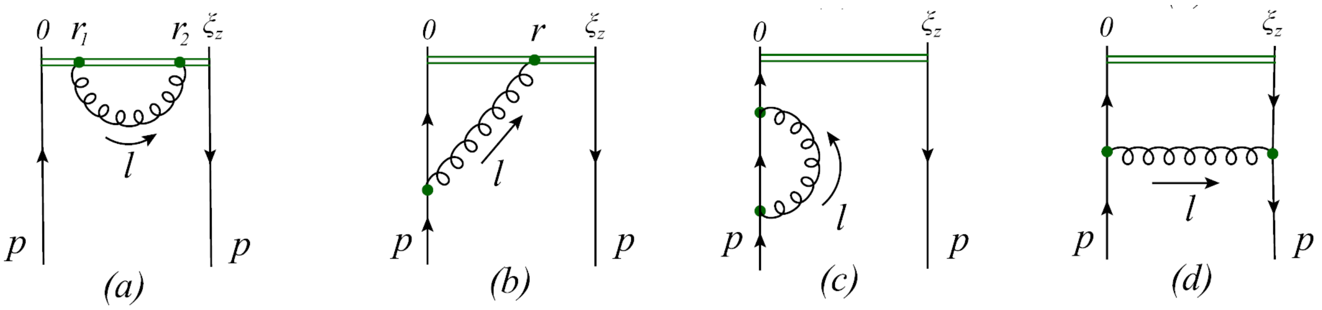
- **Working in Feynman gauge**
  - Because renormalization of QCD Lagrangian in Feynman gauge is well known

- **Key to prove the renormalization: show that UV divergences are local in space-time**
  - Nontrivial conclusion! E.g. UV divergences for normal PDFs are non-local in "−" direction
  - The most difficult part in our proof
  - One can guess this, but a rigorous proof is badly needed

One-loop calculation

One-loop diagrams: quark in a quark

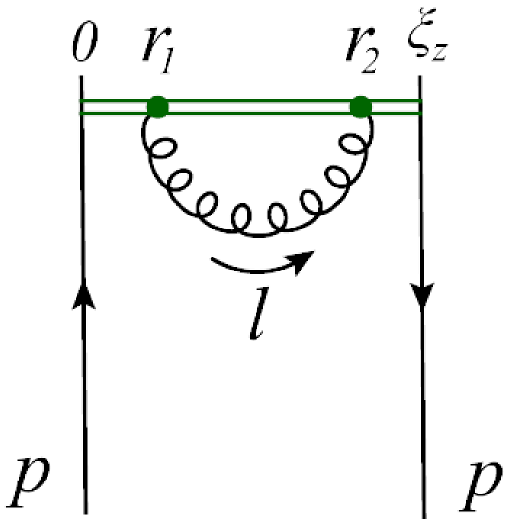
➢ Quasi quark PDFs at one loop level



- **Will demonstrate that UV divergences are local in space-time, which is significantly different from normal PDFs**

- **Note: normal PDFs, UV divergences from the regio$(l_+, l_-, l_\perp) \sim (1, \lambda^2, \lambda)$ with $\lambda \to \infty$, nonlocal in '-' direction in coordinate space.**
- **Thus, renormalization of normal PDFs is a convolution, while renormalization of quasi-PDFs is multiplicative factor**

Fig.1 (a)



(a)

$$M_{1a} = \frac{e^{ip_z\xi_z}}{p_z}\frac{1}{N_c}\mathrm{Tr}_c[T^aT^a]\int_a^{\xi_z-2a}dr_1\int_{r_1+a}^{\xi_z-a}dr_2$$
$$\times\int\frac{d^4l}{(2\pi)^4}\,e^{-ip_z\xi_z}\,e^{il_z(r_2-r_1)}\left(\frac{-ig_{\mu\nu}}{l^2}\right)$$
$$\times(-ig_sn_z^\mu)(-ig_sn_z^\nu)\mathrm{Tr}\left[\frac{1}{2}\not{p}\,\frac{1}{2}\gamma_z\right]$$
$$=\frac{\alpha_sC_F}{4i\pi^3}\int_a^{\xi_z-2a}dr_1\int_{r_1+a}^{\xi_z-a}dr_2\int d^4l\frac{e^{il_z(r_2-r_1)}}{l^2}$$

- **Cutoff “a” between fields along gaugelink**
- **Conclusion independent of regulators**

$$\int\frac{d^3\bar{l}}{l^2} = \int\frac{d^3\bar{l}}{\bar{l}^2-l_z^2}$$
$$= \int d^3\bar{l}\left(\frac{1}{\bar{l}^2}+\frac{l_z^2}{(\bar{l}^2-l_z^2)\bar{l}^2}\right)$$

$$d^4l = d^3\bar{l}\,dl_z \qquad l^2 = \bar{l}^2 - l_z^2$$
$$\int dl_z e^{il_z(r_2-r_1)} = 2\pi\delta(r_2-r_1)$$

- **First term vanishes because $r_1 \neq r_2$, thus 3D integration is finite**

Fig. 1(a) cont.

- **Fix 3D, l_z integration is finite**
- **UV divergent only if all 4 components of l^μ go to infinity**

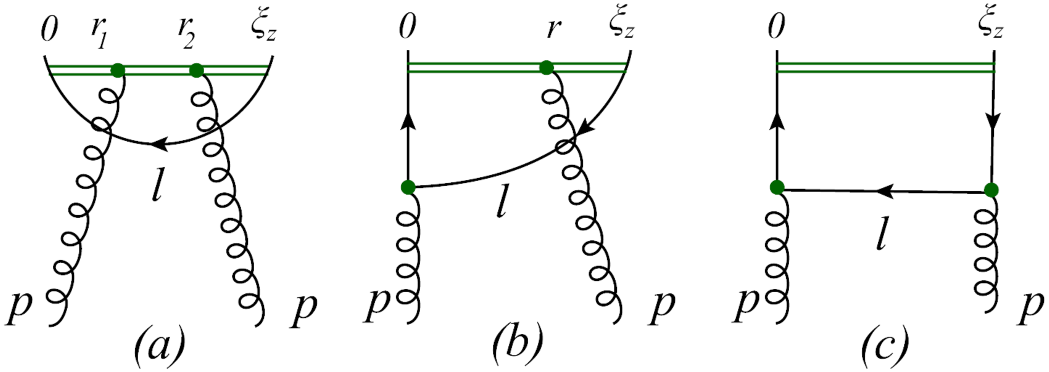
$$M_{1a} \overset{\text{div}}{=} -\frac{\alpha_s C_F}{\pi}\frac{|\xi_z|}{a} + \frac{\alpha_s C_F}{\pi}\ln\frac{|\xi_z|}{a}$$

- **At this order, UV divergences only come from the region where all loop momenta go to infinity, thus localized in coordinate space.**
- **Will show next: this behavior remains true up to all order in perturbation theory.**

$$\begin{aligned} M^{(1)} &\overset{\text{div}}{=} M_{1a} + 2\times M_{1b} + 2\times\frac{1}{2}M_{1c} + M_{1d} \\ &= \frac{\alpha_s C_F}{\pi}\left(-\frac{|\xi_z|}{a} + 2\ln\frac{|\xi_z|}{a} - \frac{1}{4\epsilon}\right). \end{aligned}$$

One-loop diagrams: quark in a gluon

➢ Gluon to quark



$$M_{2a} \propto \int_0^{\xi_z} dr_1 \int_{r_1}^{\xi_z} dr_2 \int d^4 l\, e^{-il_z\xi_z} \frac{l_z}{l^2}$$
$$= \frac{\xi_z^2}{2} \int dl_z\, e^{-il_z\xi_z}\, l_z \int d^3\bar{l} \left(\frac{1}{\bar{l}^2} + \frac{l_z^2}{(\bar{l}^2 - l_z^2)\bar{l}^2} \right)$$

- **UV divergence from 3-D $\propto \delta'(\xi_z)$, vanishes for finite ξ_z**

One-loop diagrams: quark in a gluon con.

➢ Finite term

$$
\begin{aligned}
&\frac{\xi_z^2}{2}\int dl_z\, e^{-il_z\xi_z}\, l_z \int d^3\bar{l}\frac{l_z^2}{(\bar{l}^2-l_z^2)\bar{l}^2}\\
\propto&\frac{\xi_z^2}{2}\int dl_z\, e^{-il_z\xi_z}\,\frac{l_z^3}{|l_z|}\\
=&\frac{2i}{\xi_z},
\end{aligned}
$$

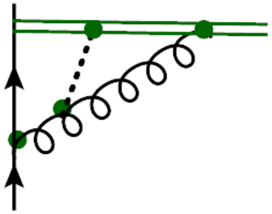
- **Divergent as $\xi_z \to 0$**
- **Result in bad large $\tilde{x}$ behavior in momentum space**

Outline

Power counting

Divergence index

➢ UV divergence at higher loops

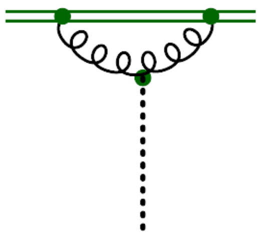


- Construct higher-loop diagrams from lower-loop diagrams by adding gluons to it
- Define divergence index ω_3 (ω_4) for 3D (4D) integration
- Using $\Delta\omega_3$ ($\Delta\omega_4$) to denote divergence index changes for 3D (4D) integration

➢ Condition for renormalizability

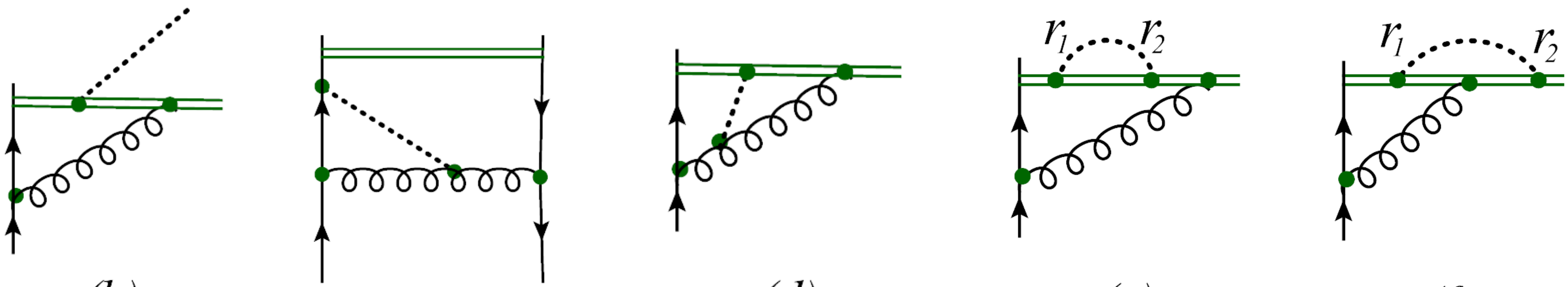
- Finite number of divergent topologies
- Sufficient condition: $\Delta\omega_3 \leq 0$ and $\Delta\omega_4 \leq 0$ for all cases, but not a necessary condition

Cases I-V



(a)

Case I: Only one end of the gluon is attached to parton lines of a loop, like Fig. 3(a). In this case we have one more propagator and one more QCD vertex in the loop, which results in $\Delta\omega_3 = -1$ and $\Delta\omega_4 = -1$. Thus a linear divergence can be changed to a logarithmic divergence, and a logarithmic divergent diagram is changed to be finite.



(b) *(c)* *(d)* *(e)* *(f)*

II. $\Delta\omega_3 = 0$, $\Delta\omega_4 = -1$

III. $\Delta\omega_3 = -1$, $\Delta\omega_4 = 0$

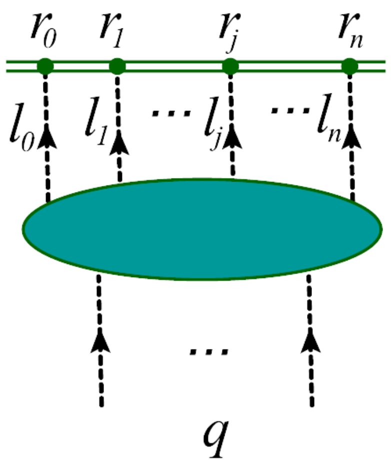
IV. $\Delta\omega_3 = 0$, $\Delta\omega_4 = 0$

V. $\Delta\omega_3 = 1$, $\Delta\omega_4 = 0$

- $\Delta\omega_3 > 0$ **for case V, may result in infinite topologies of UV div.**
- **Dangerous for the renormalizability**

GLI diagram

➢ Gauge-link-irreducible (GLI) diagram



- Diagram is connected no matter how many cuts are applied on the gauge link, or remove it
- Similar as the terminology 1PI

$$l_0 = q - l_1 - \cdots - l_n$$

- Can be generated from one-loop diagrams combined with insertions in Cases I, III, IV, all of which has $\Delta\omega_3 \leq 0$ and $\Delta\omega_4 \leq 0$
- Thus superficial UV divergence index $\omega \leq 1$

Finiteness of 3-D integration for GLI

➢ Dependence on l_j

$$e^{iq_z r_0}\prod_{j=1}^{n}\int_{r_{j-1}+a}^{r_{max}-a} dr_j \int \frac{d^4 l_j}{(2\pi)^4} e^{il_{jz}(r_j-r_0)}\mathcal{M}(q,l_1,\cdots,l_n)$$

- Numerator in M: decompose to $\bar{l}_j$ and l_{jz}
- Denominator in M:

$$\frac{1}{(l_j+k)^2}=\frac{1}{\Delta-2k_z l_{jz}-l_{jz}^2}$$
$$=\frac{1}{\Delta}+\frac{2k_z l_{jz}}{\Delta^2}+\frac{(\Delta+4k_z^2+2k_z l_{jz})l_{jz}^2}{(\Delta-2k_z l_{jz}-l_{jz}^2)\Delta^2}$$

$$\Delta=(\bar{l}_j+\bar{k})^2-k_z^2$$

- Last term: finite for integration of $\bar{l}_j$

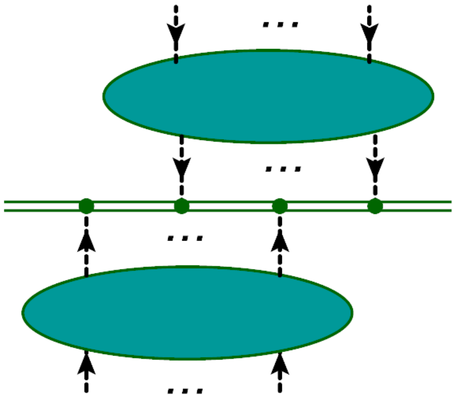
➢ UV divergence from integration of $\bar{l}_j$

- l_{jz} dependence is factorized out, vanish for finite $r_j - r_0$

$$\int dl_{jz} e^{il_{jz}(r_j-r_0)} l_z^m \propto \delta^{(m)}(r_j-r_0)$$

Quasi-PDFs: UV divergences local

➢ A non-GLI diagram made up by 2 GLI dia.



- Superficial UV divergence index $\omega \leq 2$
- For each GLI sub-diagram, similar argue for GLI diagram. UV finite if any 3-D integration is applied

➢ Easily generate to any non-GLI diagram:

- Overall UV divergence, obtained by fixing "z" component of any loop momentum, eventually vanishes after the integration of this "z" component
- UV divergences of quasi-PDFs: from the region whether all loop momenta become large → local in space-time
- As $\Delta\omega_4 \leq 0$ for all cases: finite div. topology, renormalizable

PDFs: UV divergences non-local

➢ 3-D' (l_- and $l_\perp$) integration of PDFs

$$\frac{1}{(l+k)^2} = \frac{1}{\hat{\Delta} + 2l_+(l_- + k_-)}$$

$$= \frac{1}{\hat{\Delta}} - \frac{2(l_- + k_-)l_+}{\hat{\Delta}^2} + \frac{4(l_- + k_-)^2 l_+^2}{(\hat{\Delta} + 2l_+(l_- + k_-))\hat{\Delta}^2}$$

$$\hat{\Delta} = 2k_+(l_- + k_-) - (\vec{l}_\perp + \vec{k}_\perp)^2$$

- Similar argue as quasi-PDFs: l_+ is factorized in the first two terms ,vanish under 3-D' integration
- But the last term is still UV divergent under 3-D' integration

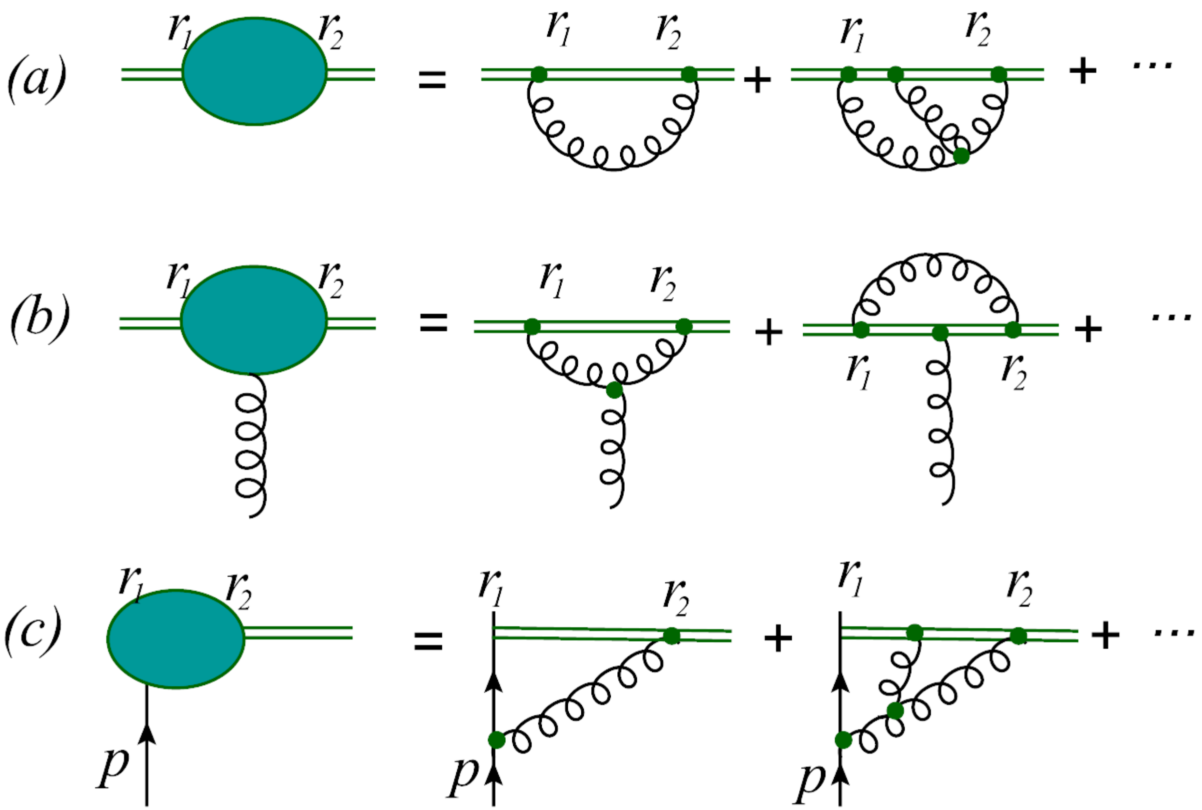
➢ UV divergent region and non-locality

$$(l_+, l_-, \vec{l}_\perp) \sim (1, \lambda^2, \lambda) \text{ as } \lambda \to \infty$$

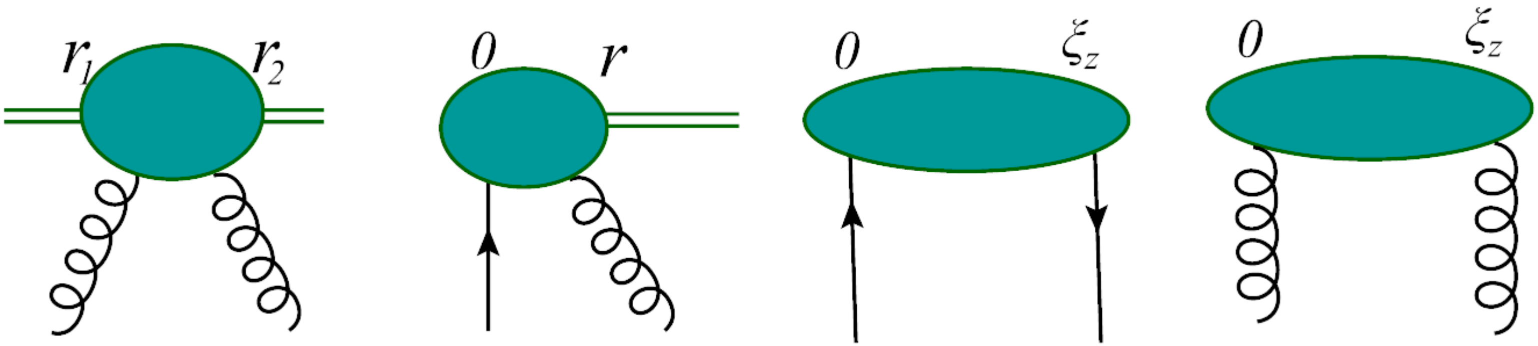
$$l_- l_+ \sim l_\perp^2 \sim \lambda^2$$

- Non-local in "-" direction in space-time

UV divergent topologies



UV finite topologies



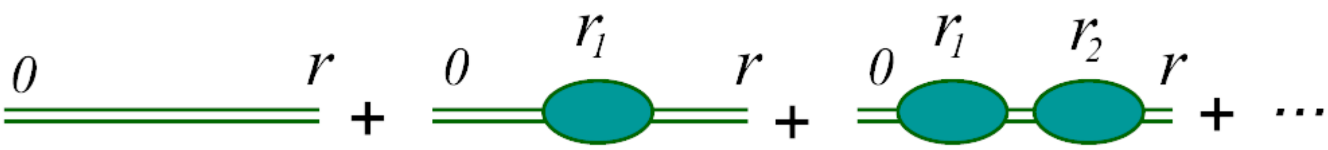
- The last diagram: no mixing between quasi-quark PDF and quasi-gluon PDF

Outline

Renormalization

Power divergence

➢ Renormalization



$$1 + c \int_0^r dr_1 + c^2 \int_0^r dr_1 \int_{r_1}^r dr_2 + \cdots$$
$$= \mathcal{P} e^{\, c \int_0^r dr'} = e^{\, c\, r} \, ,$$

- It is allowed to introduce an overall factor $e^{-c|\xi_z|}$ to remove all power UV divergences

➢ Interpretation

- Mass renormalization of test particle

Dotsenko, Vergeles, NPB (1980)

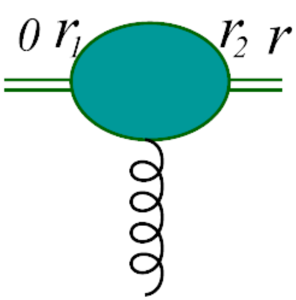
Log divergence related to gaugelink

Dotsenko, Vergeles, NPB (1980)

➢ Log div. from gaugelink self energy

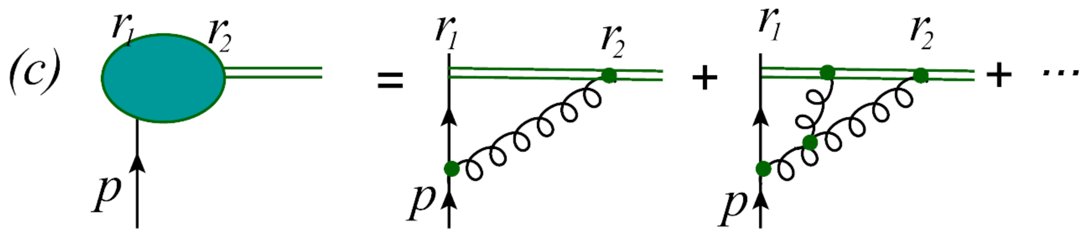
- Besides power divergence, there are also logarithmic UV divergences
- It is known that these divergences can be removed by a "wave function" renormalization of the test particle, Z_{wq}^{-1}.

➢ Log div. from gluon-gaugelink vertex



- Logarithmic UV: can be absorbed by the coupling constant renormalization of QCD.

UV from vertex correction



- Remove UV div. at fixed order
  - The most dangerous UV diagram, may mix with other operators
  - Locality of UV divergence: no dependence on $r_2 - r_1$ or p
  - UV divergence is proportional to quark-gaugelink vertex at lowest order, with a constant coefficient
  - A constant counter term is able to remove this UV divergence.
- Renormalization to all-orders
  - Using bookkeeping forests subtraction method, the net effect is to introduce a constant multiplicative renormalizaton factor Z_{vq}^{-1} for the quark-gaugelink vertex.

Renormalization

Ishikawa YQM, Qiu, Yoshida, 1707.03107

- Using renormalized QCD Lagrangian:

  - All UV divergences (too all orders) can be removed by the following renormalization

$$\tilde{F}^{R}_{i/p}(\xi_z, \tilde{\mu}^2, p_z) = e^{-C_i|\xi_z|} Z_{wi}^{-1} Z_{vi}^{-1} \tilde{F}^{b}_{i/p}(\xi_z, \tilde{\mu}^2, p_z)$$

- Renormalization: multiplicative factor, not mix with other operators

- Significantly different from normal PDFs

- Quasi quark PDF is indeed a "good lattice cross section"