Dispersive analysis of $B \to \pi \ell^+ \ell^-$ and $B \to K \ell^+ \ell^-$ decays at low q^2

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in preparation with Yun-Hua Chen and Feng-Kun Guo

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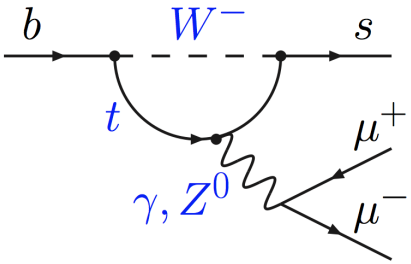
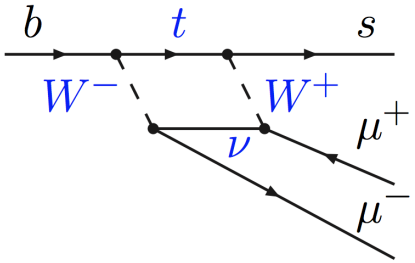
Outline

1. **Introduction**
2. $B \to \pi\gamma^*$ **and** $B \to K\gamma^*$ **form factors in dispersion theory**
   - Pion Vector Form Factor: Omnès equation
   - $B \to 3\pi$ and $B \to 2\pi K$ decay amplitude: Khuri Treiman
3. **Results**
4. **Conclusions**

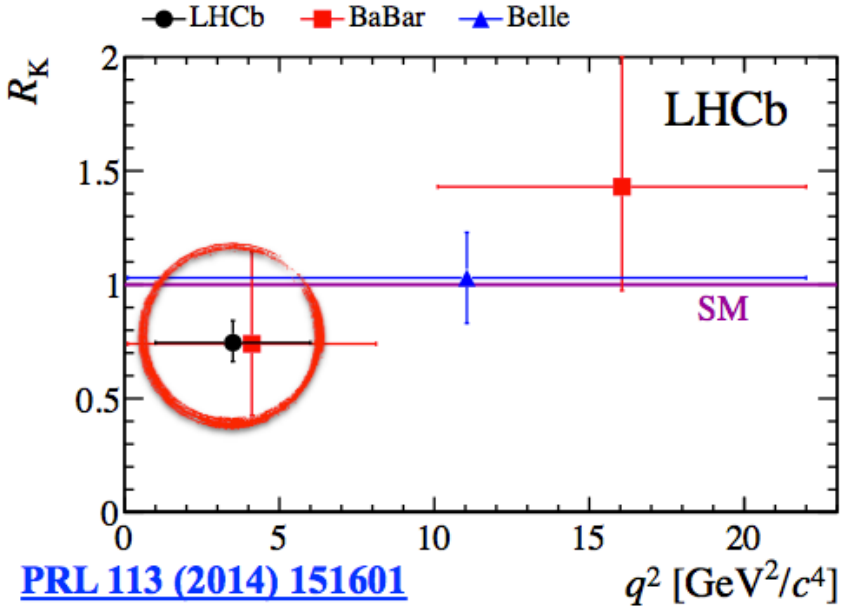
Motivation

- FCNC transitions in the SM: Loop and CKM suppressed





- Lepton Flavour Universality cast in doubt: tensions in $b \to s\ell^+\ell^-$



PRL 113 (2014) 151601

$$R_K = \frac{B\to K\mu^+\mu^-}{B\to Ke^+e^-}\Big|_{\rm exp} = 0.745^{+0.090}_{-0.074} \pm 0.036\,; \qquad q^2 \in [1,6]\ \mathrm{GeV}^2$$

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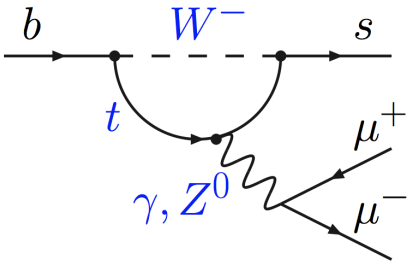

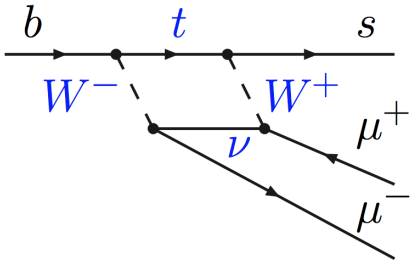

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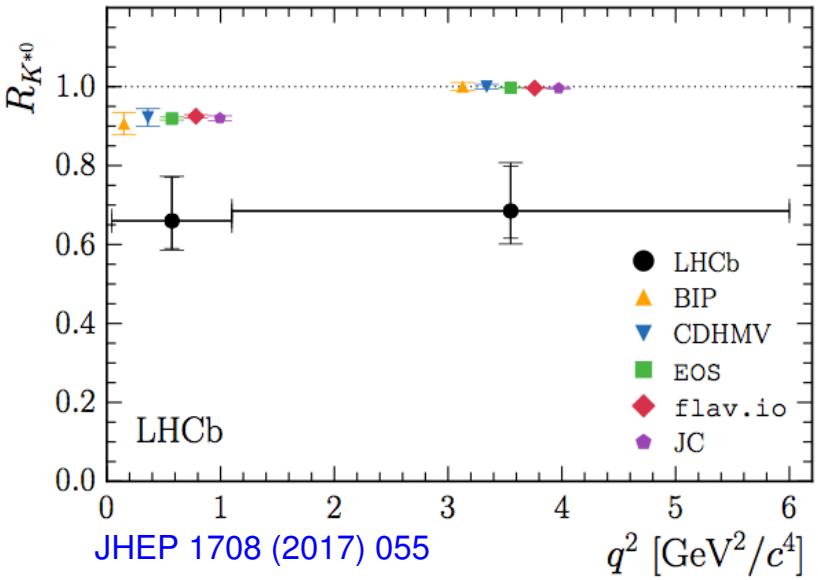
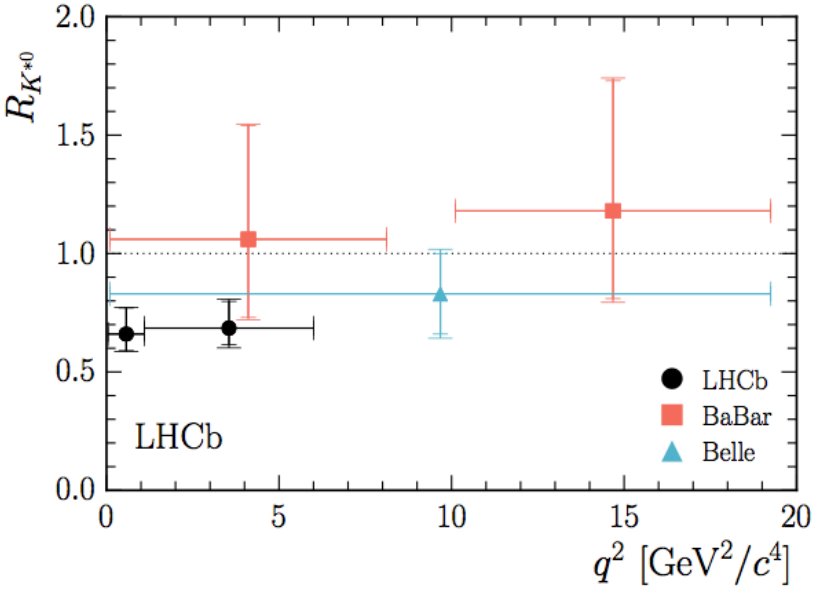


JHEP 1708 (2017) 055



$$R_{K^*} = \frac{B\to K^*\mu^+\mu^-}{B\to K^*e^+e^-}\Big|_{\rm exp} = \begin{cases} 0.66^{+0.11}_{-0.07}({\rm stat}) \pm 0.03({\rm stat})\,; 0.045 < q^2 < 1.1\,{\rm GeV}^2 \\ 0.69^{+0.11}_{-0.07}({\rm stat}) \pm 0.05({\rm syst})\,; 1.1 < q^2 < 6.0\,{\rm GeV}^2 \end{cases}$$

Motivation

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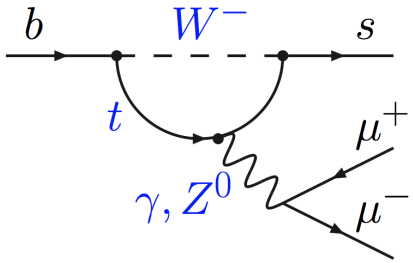

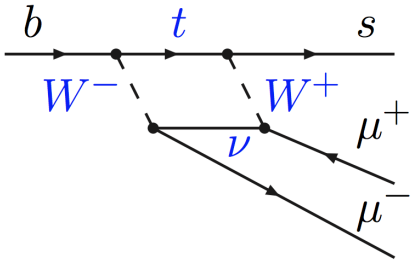

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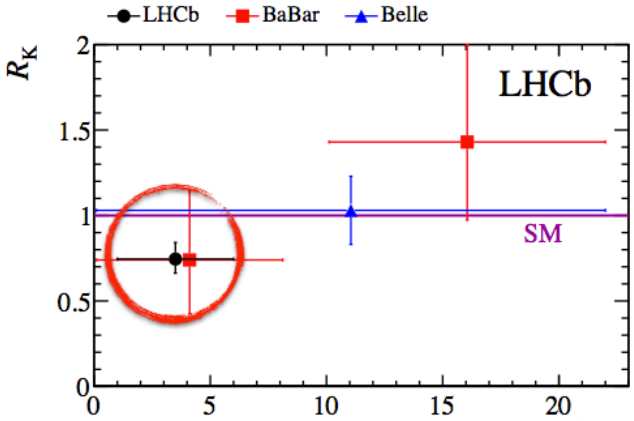
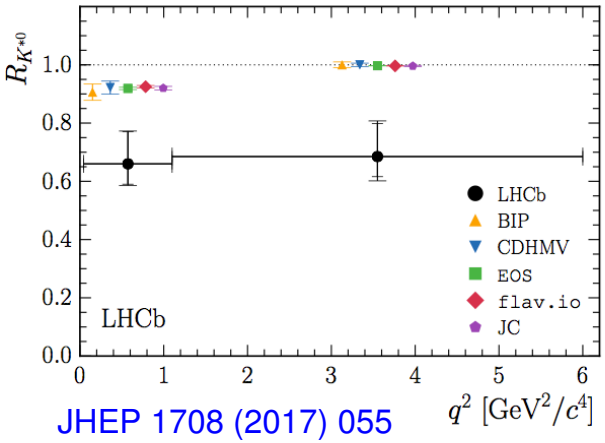


PRL 113 (2014) 151601



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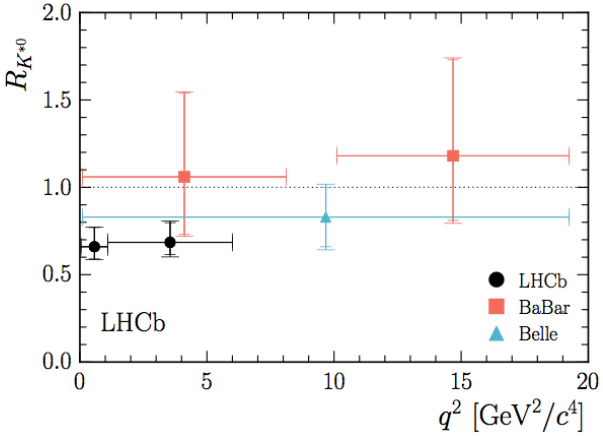
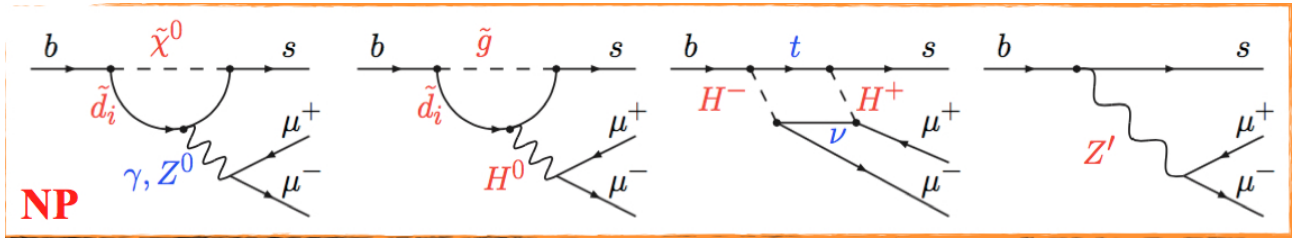
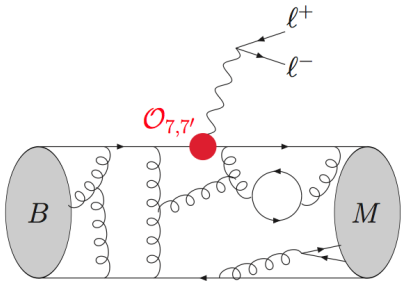


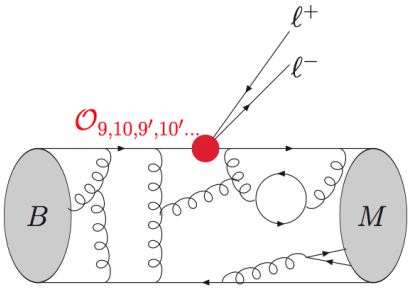
- Indirect searches of NP: new particles with new type interactions?



Anatomy of $b \to d\ell^+\ell^-$ and $b \to s\ell^+\ell^-$ transitions in the SM

- Weak Effective Hamiltonian





$$\mathcal{H}_{\rm eff}^{b\to q} = \frac{4G_F}{\sqrt{2}}\left[V_{ub}V_{uq}^*\sum_{i=1}^{2}C_i\mathcal{O}_i^u + V_{cb}V_{cq}^*\sum_{i=1}^{2}C_i\mathcal{O}_i^c - V_{tb}V_{tq}^*\sum_{i=3}^{10}C_i\mathcal{O}_i\right],$$

$$\mathcal{O}_7 = \frac{e^2}{16\pi^2}m_b\left(\bar{q}\sigma^{\mu\nu}P_R b\right)F^{\mu\nu}, \quad C_7^{\rm eff}(\mu_b) = -0.304,$$

$$\mathcal{O}_9 = \frac{e^2}{16\pi^2}\left(\bar{q}\gamma^\mu P_L b\right)\sum_\ell\left(\bar{\ell}\gamma_\mu\ell\right), \quad C_9^{\rm eff}(\mu_b) = 4.211,$$

$$\mathcal{O}_{10} = \frac{e^2}{16\pi^2}\left(\bar{q}\gamma^\mu P_L b\right)\sum_\ell\left(\bar{\ell}\gamma_\mu\gamma_5\ell\right), \quad C_{10}^{\rm eff}(\mu_b) = -4.103.$$

$$\mathcal{M}(B^+\to\pi^+\ell^+\ell^-) = \frac{G_F\alpha_{\rm em}}{\sqrt{2}\pi}V_{tb}V_{td}^*\Big[C_9^{\rm eff}(q^2)\langle\pi(p_\pi)|\bar{d}\gamma^\mu b|B(p_B)\rangle\bar{\ell}\gamma_\mu\ell$$

$$+C_{10}^{\rm eff}(\mu_b)\langle\pi(p_\pi)|\bar{d}\gamma^\mu b|B(p_B)\rangle\bar{\ell}\gamma_\mu\gamma_5\ell - 2C_7^{\rm eff}(q^2)\frac{m_b}{q^2}\langle\pi(p_\pi)|\bar{d}i\sigma^{\mu\nu}q_\nu b|B(p_B)\rangle\bar{\ell}\gamma_\mu\ell\Big].$$

invariant mass distribution

- Hadronic matrix element

$$\langle \pi(p_\pi)|\bar{d}\gamma^\mu b|B(p_B)\rangle = \left[p_B^\mu + p_\pi^\mu - \frac{m_B^2 - m_\pi^2}{q^2}q^\mu\right] f_+(q^2) + \frac{m_B^2 - m_\pi^2}{q^2}q^\mu f_0(q^2),$$

$$\langle \pi(p_\pi)|\bar{d}i\sigma^{\mu\nu}q_\nu b|B(p_B)\rangle = \left[q^2(p_B^\mu + p_\pi^\mu) - (m_B^2 - m_\pi^2)q^\mu\right]\frac{i f_T(q^2)}{m_B + m_\pi},$$

- Differential decay rate distribution

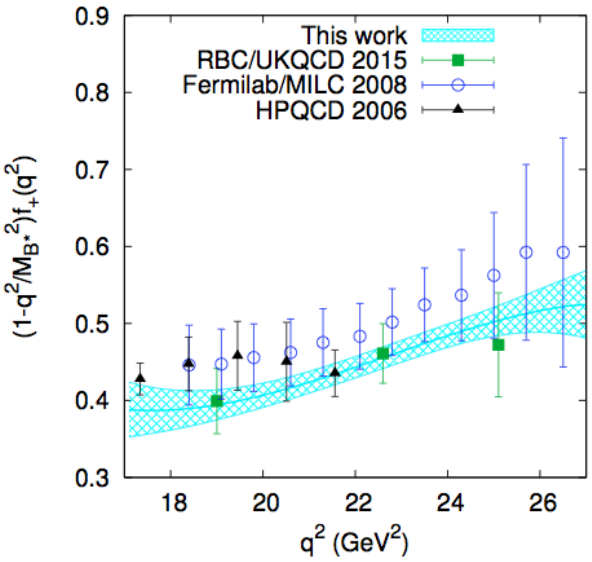
$$\frac{d\Gamma(B^+ \to \pi^+\ell^+\ell^-)}{dq^2} = \frac{G_F^2\alpha_{\rm em}^2}{1024\pi^5 m_B^3}|V_{tb}V_{td}^*|^2\lambda(q^2, m_B^2, m_\pi^2)^{1/2}\sqrt{1 - \frac{4m_\ell^2}{q^2}}$$
$$\times\left\{\frac{2}{3}\lambda(q^2, m_B^2, m_\pi^2)\left[\left(1 + \frac{2m_\ell^2}{q^2}\right)\left|C_9^{\rm eff} f_+(q^2) + \frac{2m_b}{m_B + m_\pi}C_7^{\rm eff} f_T(q^2)\right|^2\right.\right.$$
$$\left.\left. + \left(1 - \frac{4m_\ell^2}{q^2}\right)\left|C_{10}^{\rm eff} f_+(q^2)\right|^2\right] + \frac{4m_\ell^2}{q^2}(m_B^2 - m_\pi^2)^2\left|C_{10}^{\rm eff} f_0(q^2)\right|^2\right\},$$

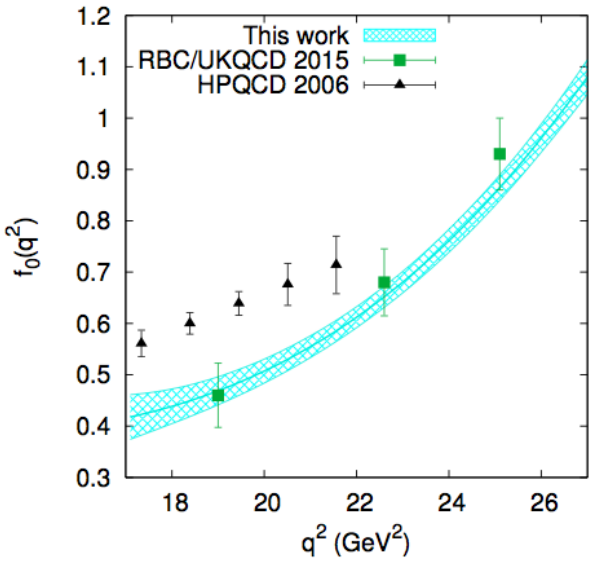
- $f_+(q^2)$ dominates: $f_T(q^2)$ is subdominat $\left(|C_7^{\rm eff}| \ll |C_{9,10}^{\rm eff}|\right)$, while $f_0(q^2)$ is suppressed by m_ℓ^2.

$B \to \pi$ Form Factors

Phys.Rev. D92 (2015) no.1, 014024

- Lattice Form Factors+$B \to \pi \ell \nu_\ell$ data

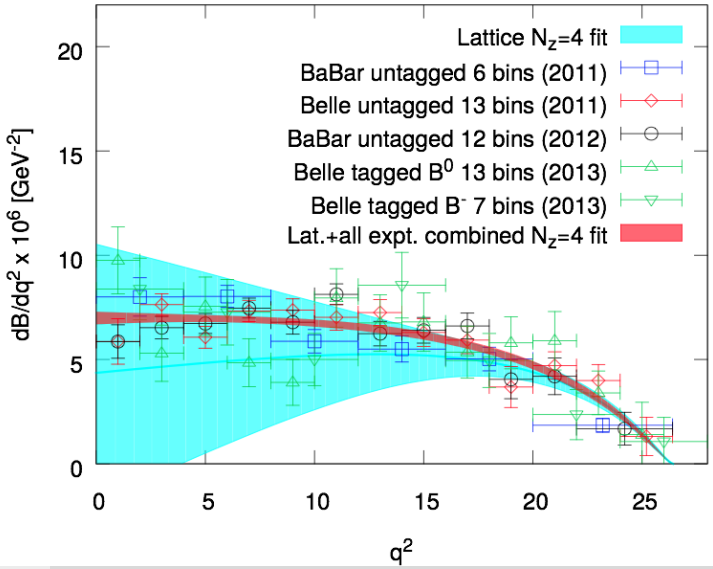




$$\frac{d\Gamma(B\to\pi\ell\nu_\ell)}{dq^2} = \frac{G_F^2|V_{ub}|^2}{192\pi^3 m_B^3}|p_\pi|^3|f_+^{B\pi}(q^2)|^2 ,$$

$$f_+^{B\pi}(q^2) = \frac{1}{1-q^2/m_{B*}^2}\sum_{n=0}^{K} b_k(t_0)[z(q^2,q_0^2)]^k ,$$

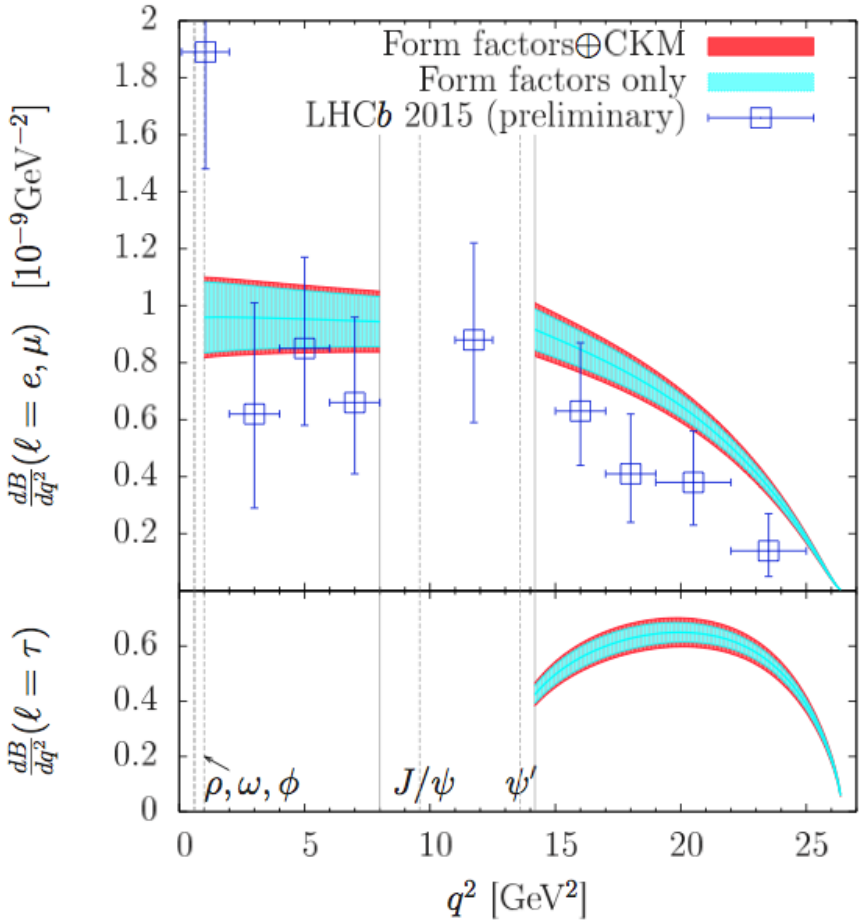
$$z(q^2,q_0^2) = \frac{\sqrt{t_+-q^2}-\sqrt{t_+-q_0^2}}{\sqrt{t_+-q^2}+\sqrt{t_+-q_0^2}}$$



$B \to \pi$ Form Factors

Phys.Rev.Lett. 115 (2015) no.15, 152002

- $B \to \pi \ell^+ \ell^-$ description

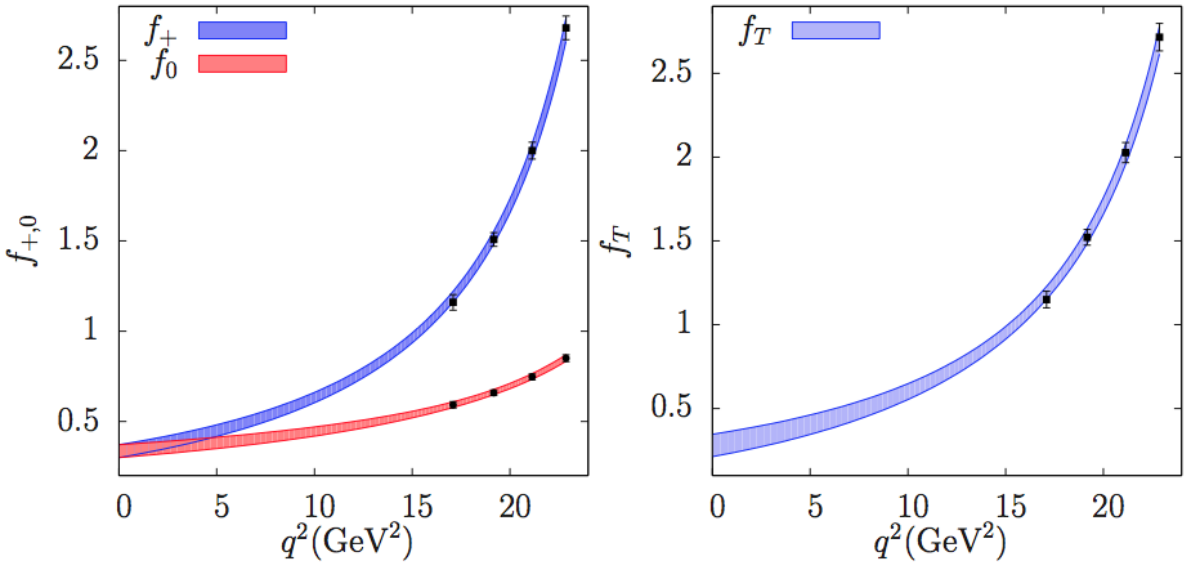


- Our purpose: To describe the region where light vector resonance are produced ($q^2 < 1$ GeV^2)

$B \to K$ Form Factors

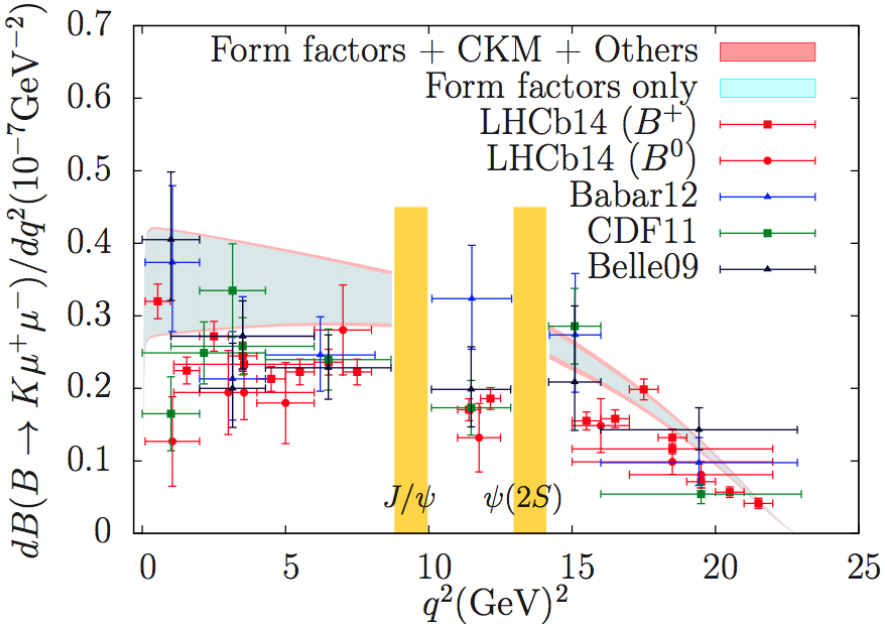
Phys.Rev. D93 (2016) no.2, 025026

- Lattice Form Factors



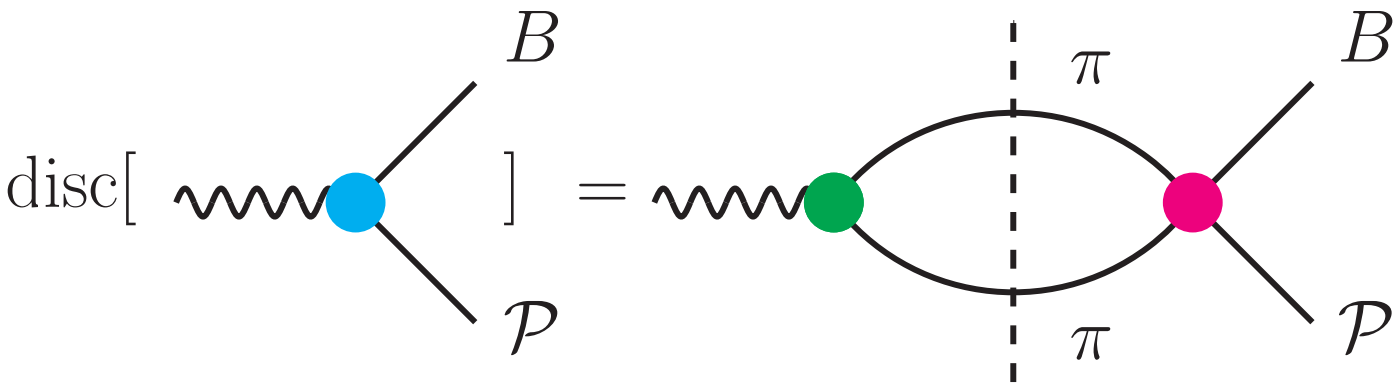
- $B \to K\mu^+\mu^-$ description

Phys.Rev. D93 (2016) no.3, 034005



Transition form factor $B \to \mathcal{P}\ell^+\ell^-$

- Two-pion discontinuity of the $B \to \mathcal{P}\gamma^*$ transition form factor



- *Exempli gratia* : $\mathcal{P} = \pi$

$$\mathrm{disc} f_{B\pi}^{\pi\pi}(s) = 2i\sigma(s) F_\pi^{V*}(s) f_1^{B\to 3\pi}(s)\theta(s-4m_\pi^2)\,,$$

$$f_{B\pi}(s) = \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{disc} f_{B\pi}^{\pi\pi}(s')}{s'-s}\,,$$

Transition form factor $B \to \mathcal{P}\ell^+\ell^-$

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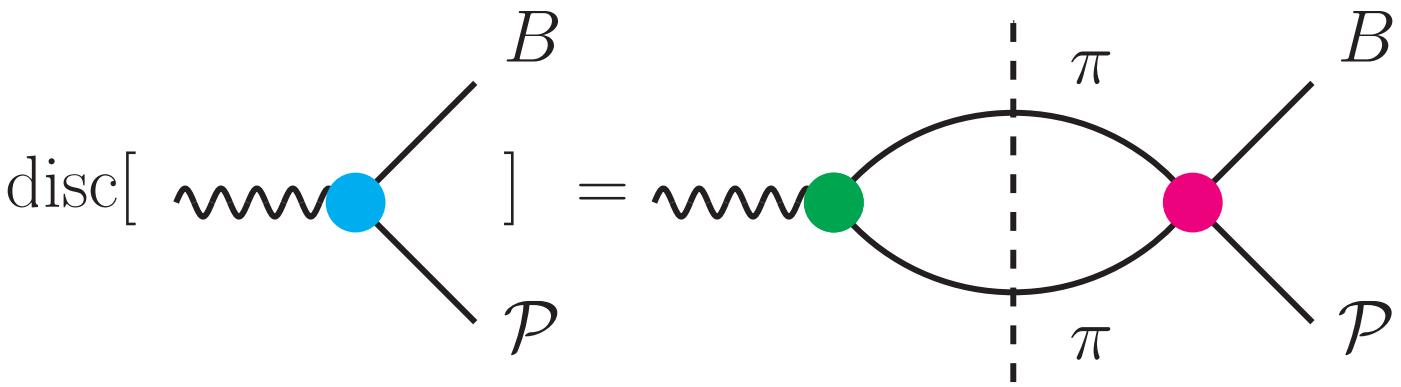

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$$f_{B\pi}(s) = f_{B\pi}(0) + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sigma(s') F_\pi^{V*}(s') f_1^{B\to 3\pi}(s')}{s'(s'-s)}\,,$$

$$f_{B\pi}(0) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\sigma(s') F_\pi^{V*}(s') f_1^{B\to 3\pi}(s')}{s'}\,,$$

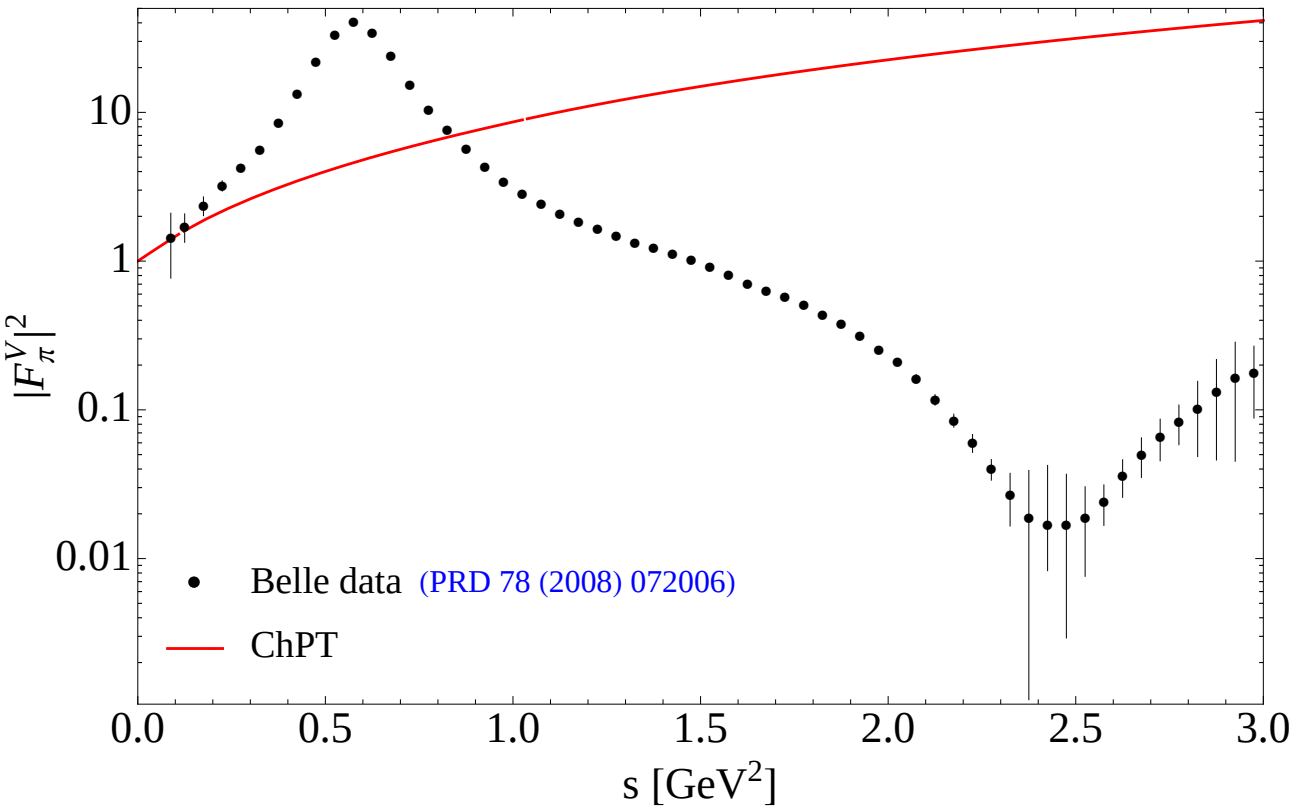
Pion vector Form Factor (Warm-up)

J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 517

- Chiral Perturbation Theory $\mathcal{O}(p^4)$

$$F_\pi^V(s) = 1 + \frac{2L_9^r(\mu)}{F_\pi^2}s - \frac{s}{96\pi^2 F_\pi^2}\left(A_\pi(s,\mu^2) + \frac{1}{2}A_K(s,\mu^2)\right)$$

$$A_P(s,\mu^2) = \log\frac{m_P^2}{\mu^2} + 8\frac{m_P^2}{s} - \frac{5}{3} + \sigma_P^3(s)\log\left(\frac{\sigma_P(s)+1}{\sigma_P(s)-1}\right), \quad \sigma_P(s) = \sqrt{1 - 4\frac{m_P^2}{s}}$$



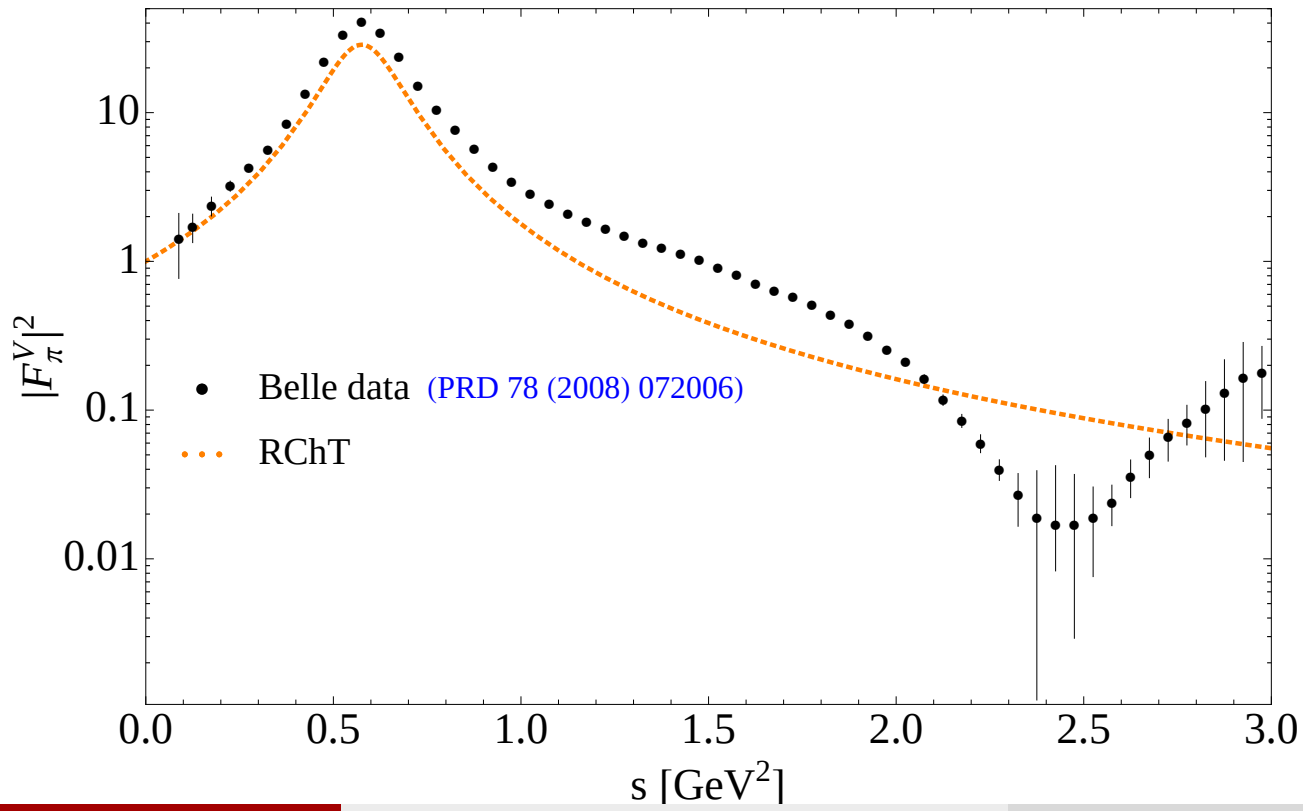
Pion vector Form Factor (Warm-up)

G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B 321 (1989) 311

- Chiral Perturbation Theory with resonances (RχT)

$$F_\pi^V(s) = \frac{m_\rho^2}{m_\rho^2 - s - i m_\rho \Gamma_\rho(s)}$$

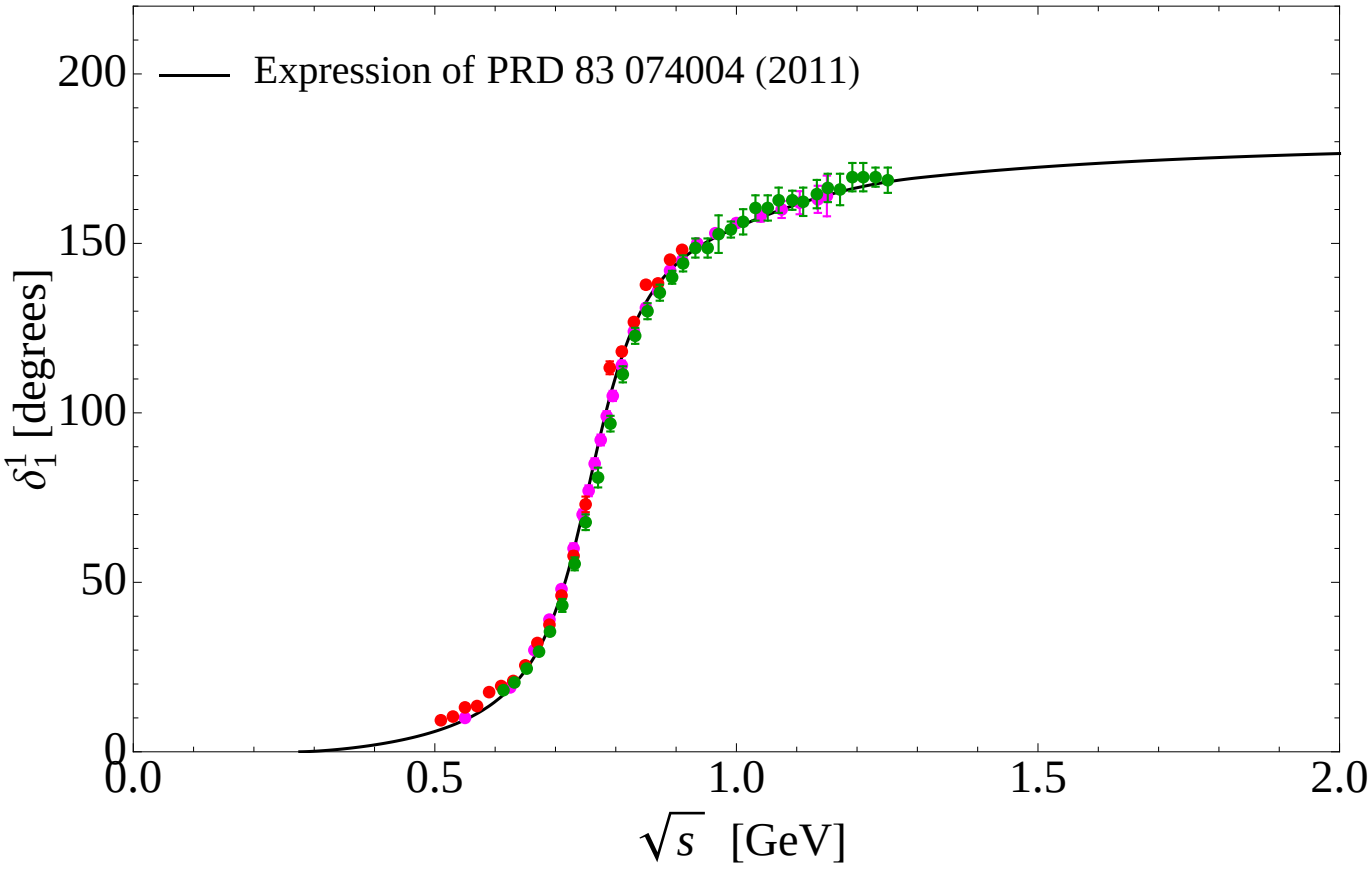
$$\Gamma_\rho(s) = \gamma_\rho \frac{s}{m_\rho^2}\left(\sigma_\pi(s)^3 + \frac{1}{2}\frac{\sigma_K(s)^2}{\sigma_K(m_\rho^2)}\right), \quad \sigma_P(s) = \sqrt{1 - 4\frac{m_P^2}{s}}$$



Pion vector Form Factor

- Omnès

$$F_\pi^V(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1(s)}{s'(s'-s)}\right\}$$



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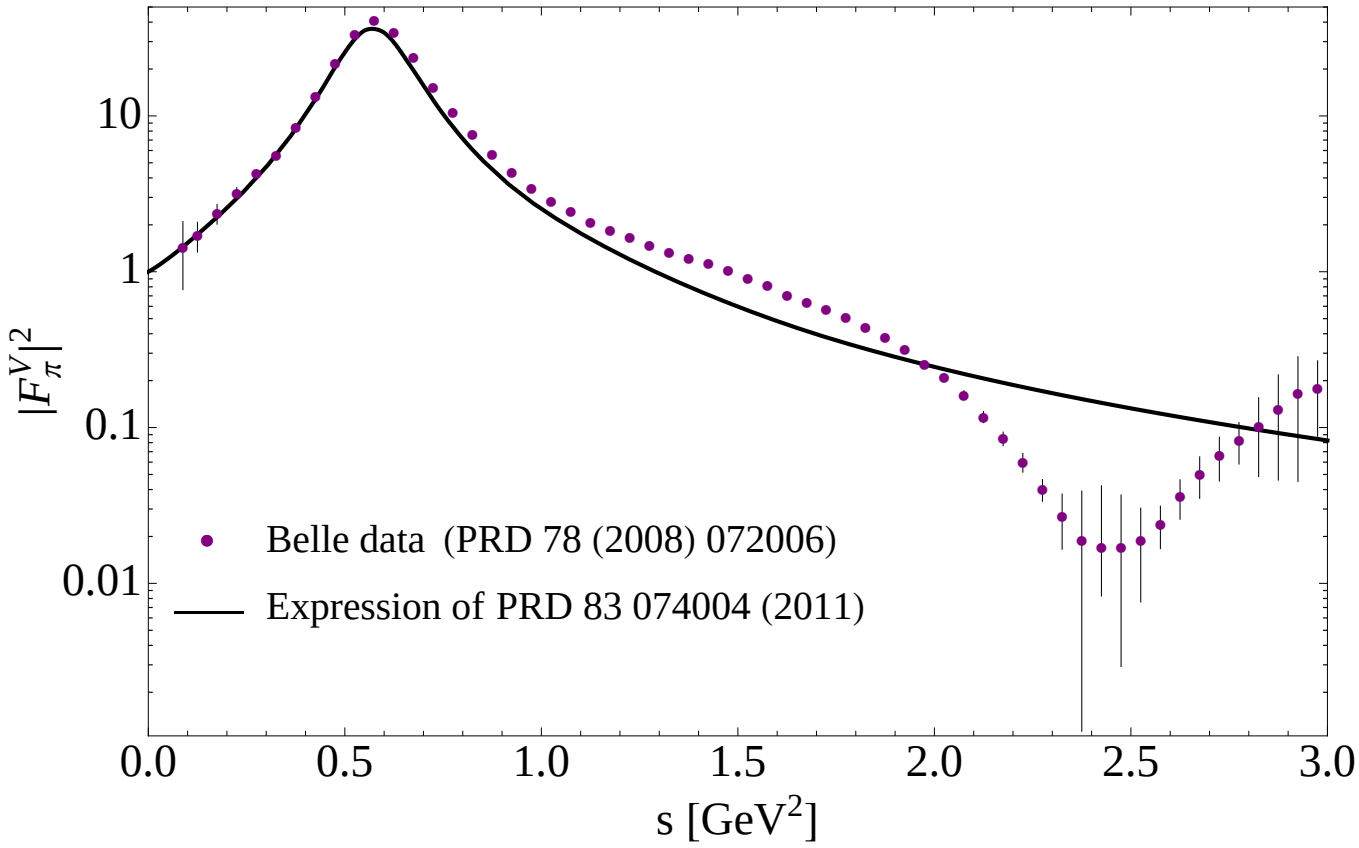

Pion vector Form Factor

P. Roig, Nucl. Phys. B 161 (2012) [ArXiv: 1112.0962]

- Phenomenological model to incorporate $\rho'(1450), \rho''(1700)$

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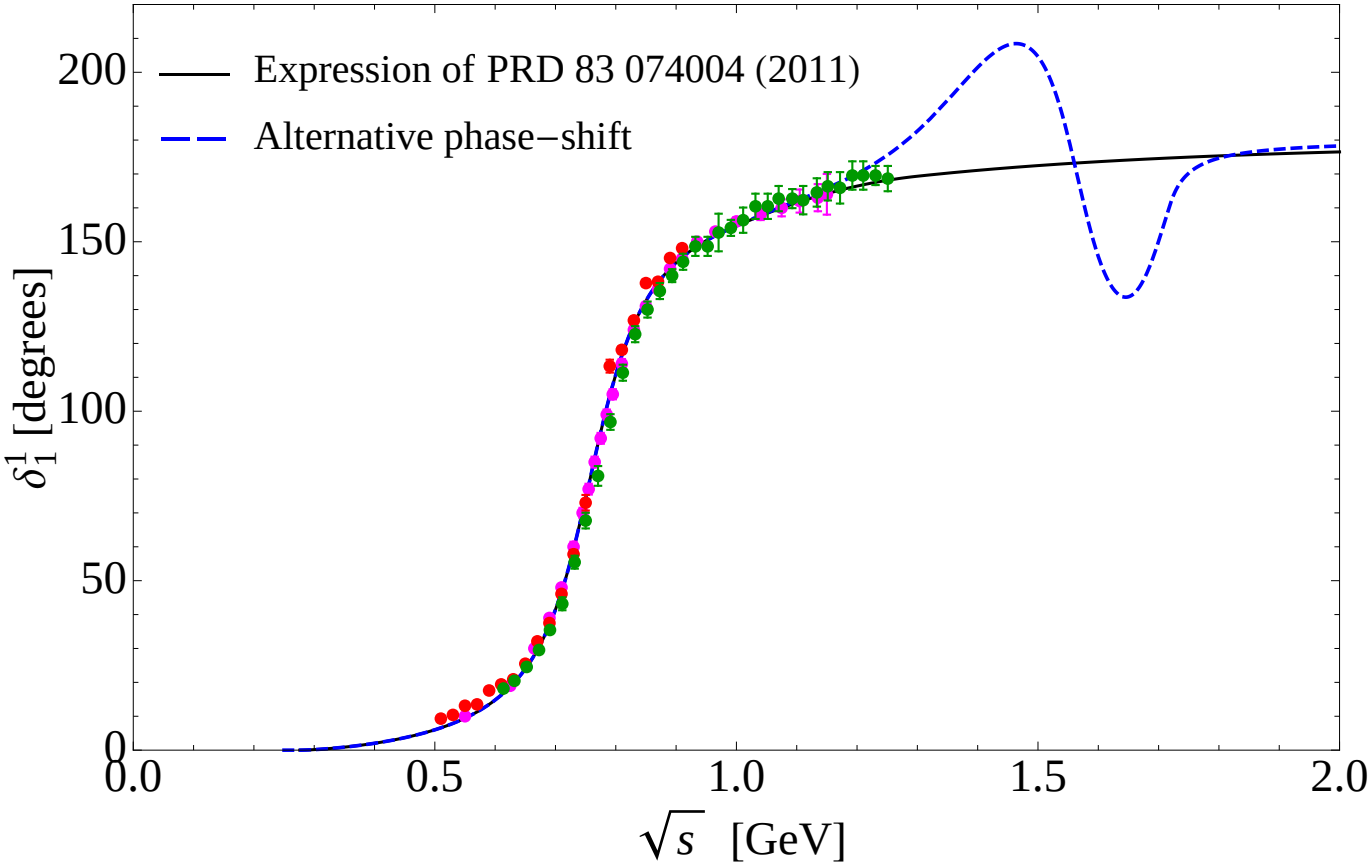

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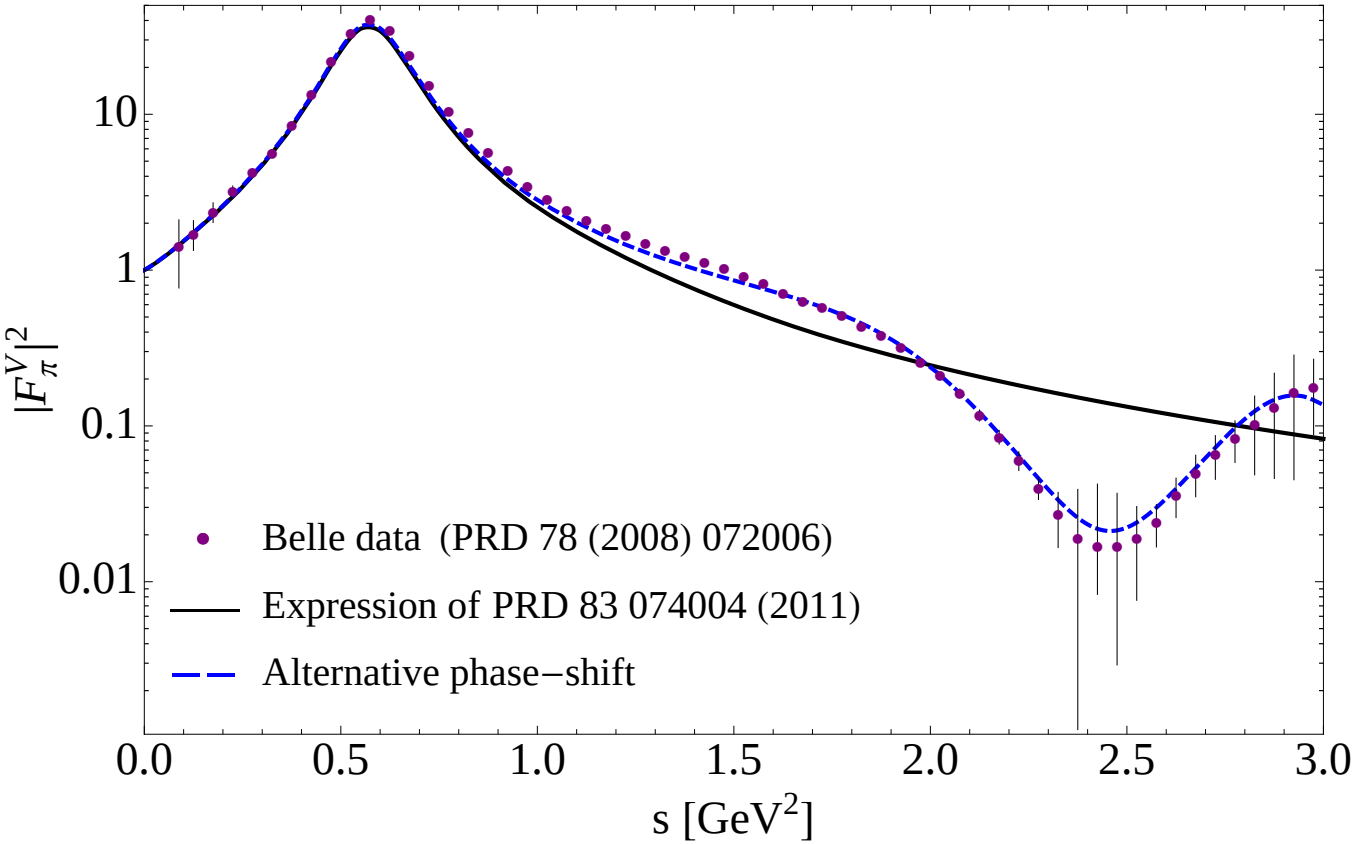
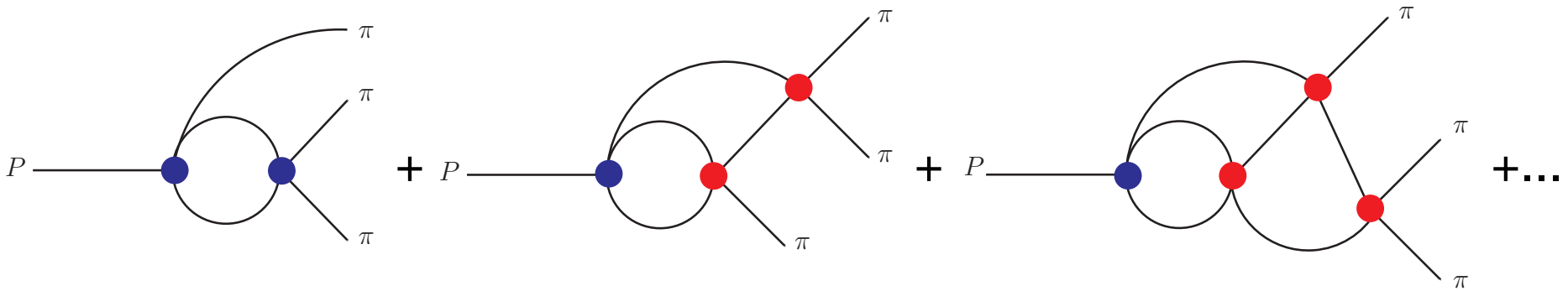


3-body decay: dispersion relations *à la* Khuri Treiman



$$\mathrm{disc} f_1(s) = \mathrm{disc}\mathcal{F}(s) = \left(\mathcal{F}(s) + \hat{\mathcal{F}}(s)\right) \sin\delta_1^1(s) e^{-i\delta_1^1(s)} \theta(s - 4m_\pi^2)(s)\,,$$

$$f_1(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\mathrm{disc} f_1(s')}{s' - s - i\epsilon} = \mathcal{F}(s) + \hat{\mathcal{F}}(s)\,,$$

- $\delta_1^1(s)$: P-wave $\pi\pi$ scattering phase shift
- $\hat{\mathcal{F}}(s)$ is the inhomogeneity: s-channel projection of left-hand cut contributions (t-and-u channels)
- Dispersive integral: accounts for final state interactions between all three final state pions

3-body decay: dispersion relations *à la* Khuri Treiman

- Dispersive integral: accounts crossed-channel rescattering (t-and-u channels)

$$\mathcal{F}(s) = a\Omega(s)\left(1 + \frac{s}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|s'(s'-s)}\right),$$

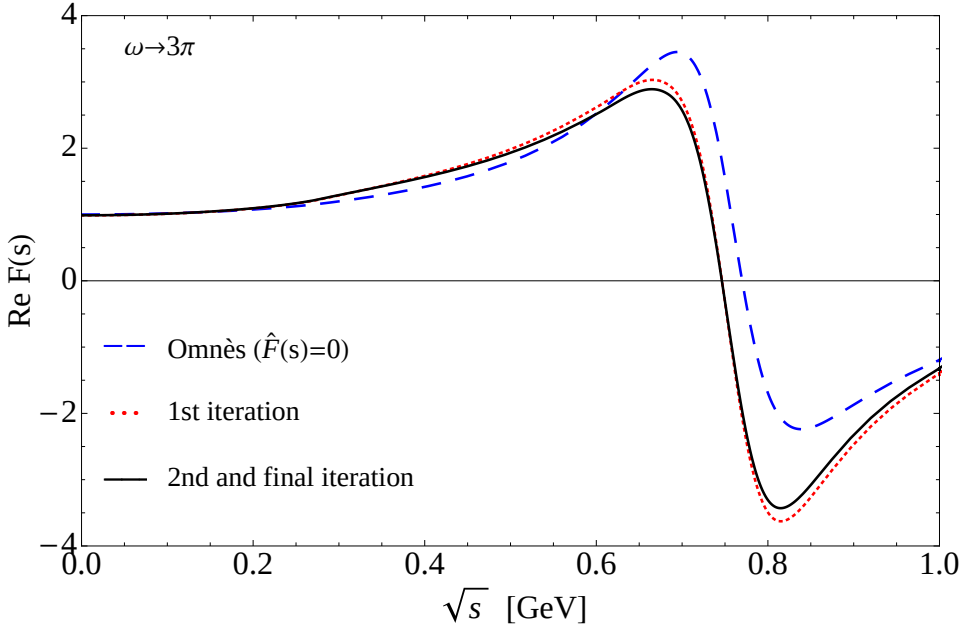
$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1} dz_s(1-z_s^2)\mathcal{F}(t(s,z_s)), \quad z_s = \cos\theta_s = \frac{t-u}{4p(s)q(s)},$$

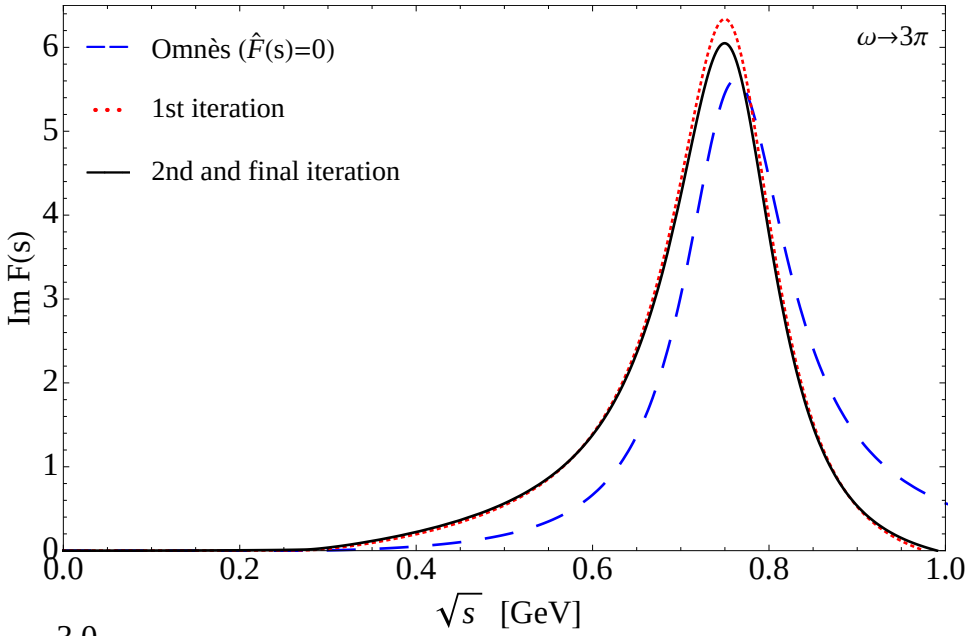
$$p(s) = \frac{1}{2\sqrt{s}}\lambda^{1/2}(s, M_P^2, m_\pi^2), \quad q(s) = \frac{1}{2\sqrt{s}}\lambda^{1/2}(s, m_\pi^2, m_\pi^2),$$

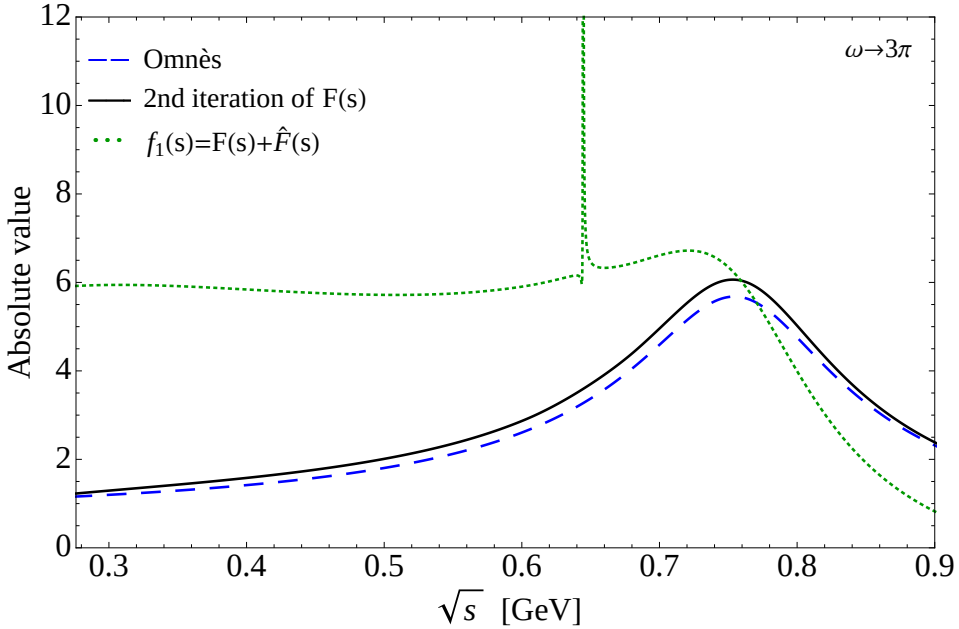
- If $\hat{\mathcal{F}}(s) = 0$ (no left-hand cut contributions)$\Rightarrow$ $\mathcal{F}(s) = a\Omega(s)$ (pion vector form factor)
- a is a subtraction constant: only free parameter of the model

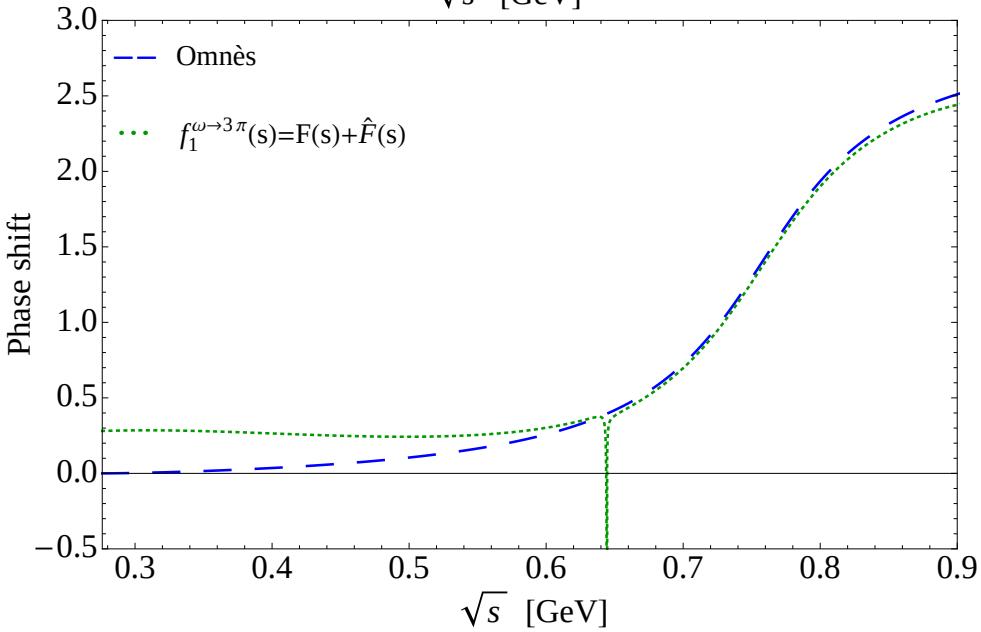
$\omega \to 3\pi$ (warm-up)

B. Kubis et.al., EPJC 72 (2012) 2014
B. Kubis et.al., PRD 86 (2012) 054013
I.V. Danilkin et.al., PRD 91 (2015) 094029



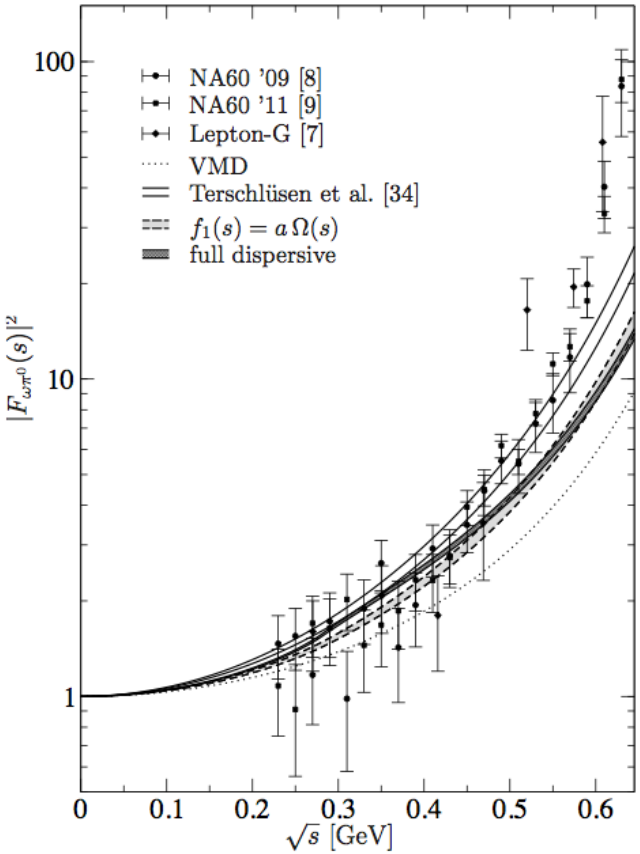




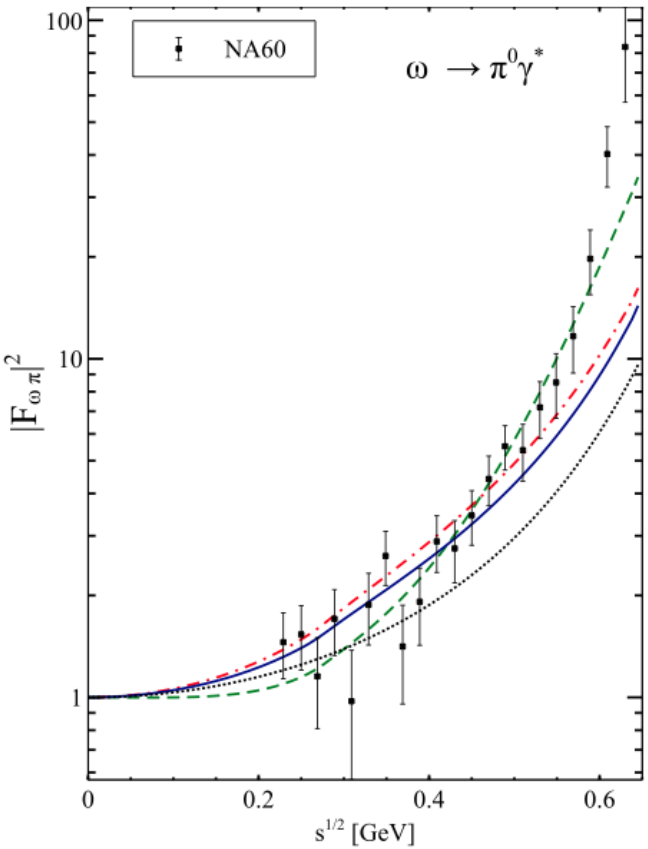


$\omega \to \pi^0\gamma^*$ transition form factor

- Dispersive vs VMD $\quad F(s) = \dfrac{m_\rho^2}{m_\rho^2 - s - i\sqrt{s}\Gamma(s)}$



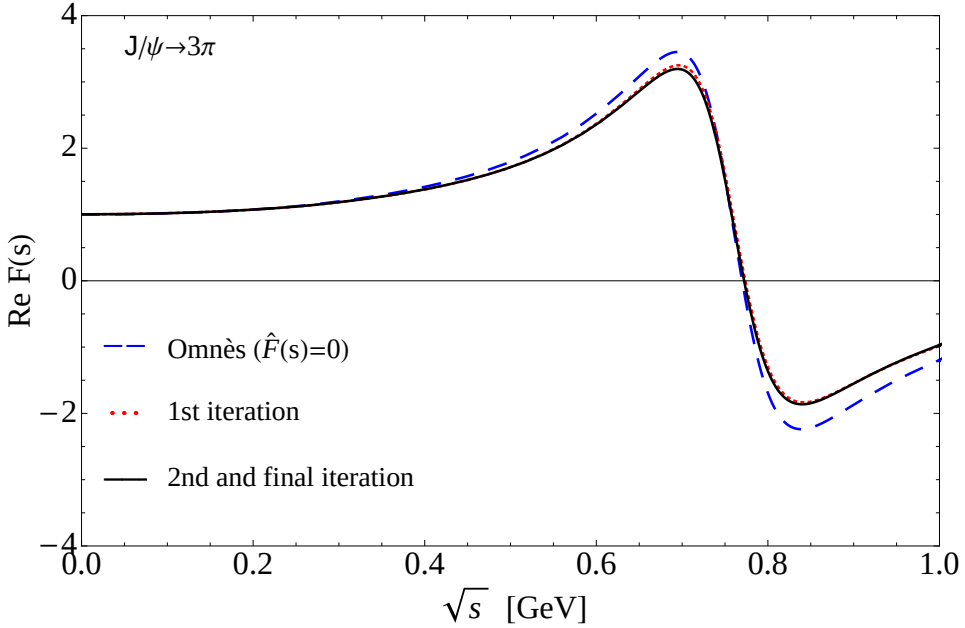
B. Kubis et.al., PRD 86 (2012) 054013

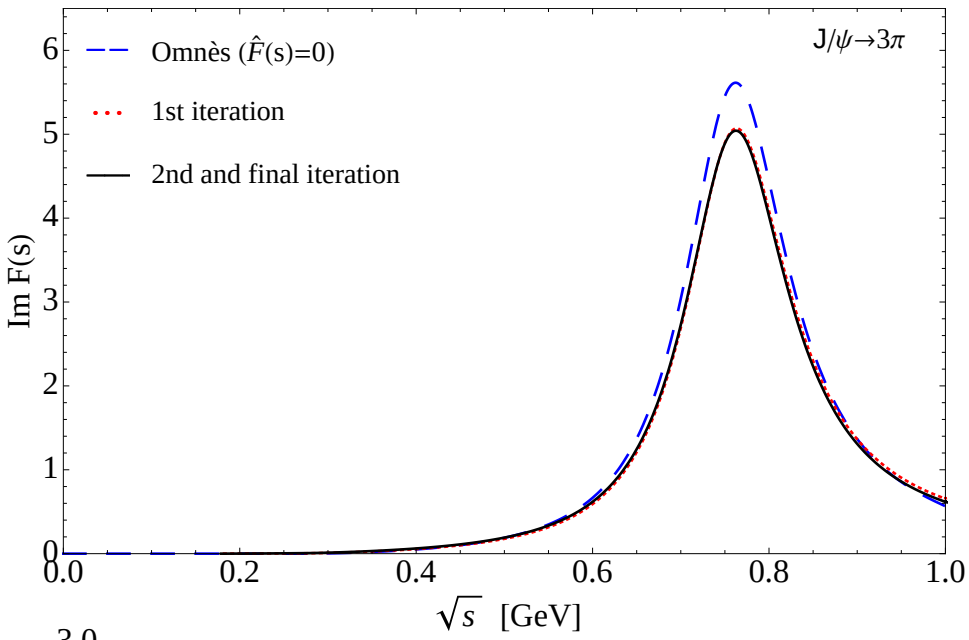


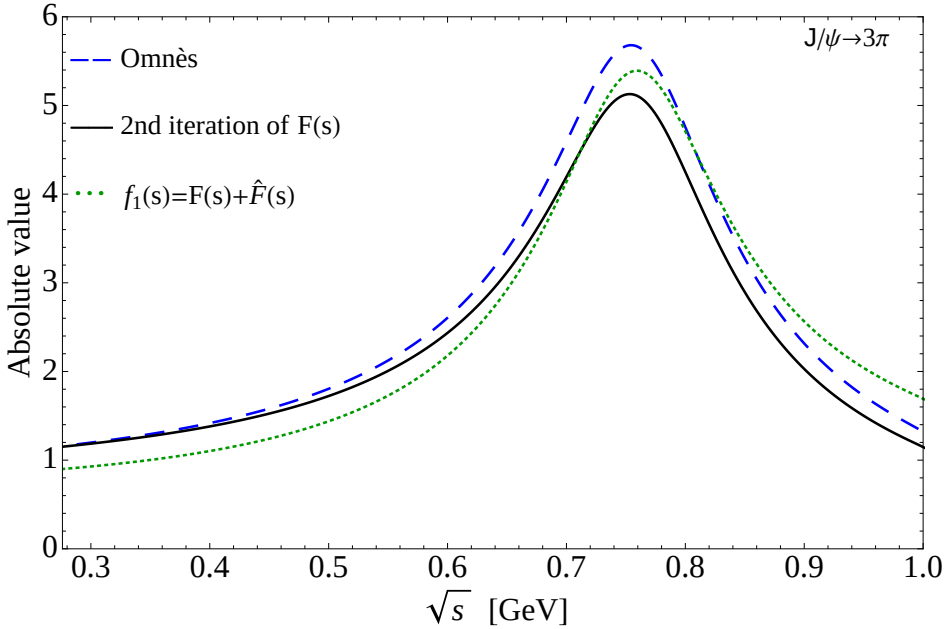
I.V. Danilkin et.al., PRD 91 (2015) 094029

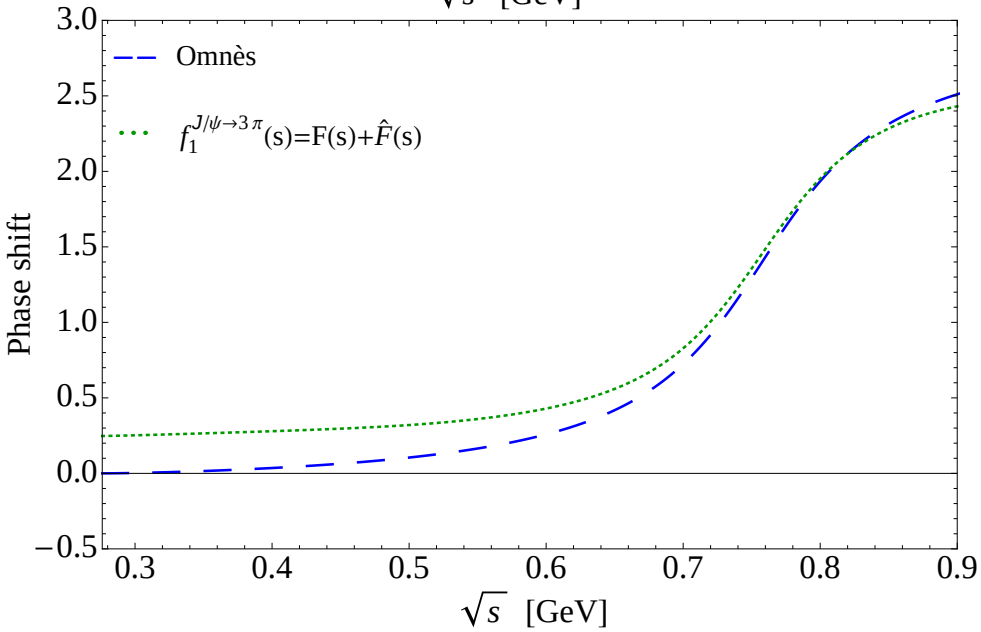
$J/\psi \to 3\pi$ (warm-up)

B. Kubis et.al., PRD 91 (2015) 036004

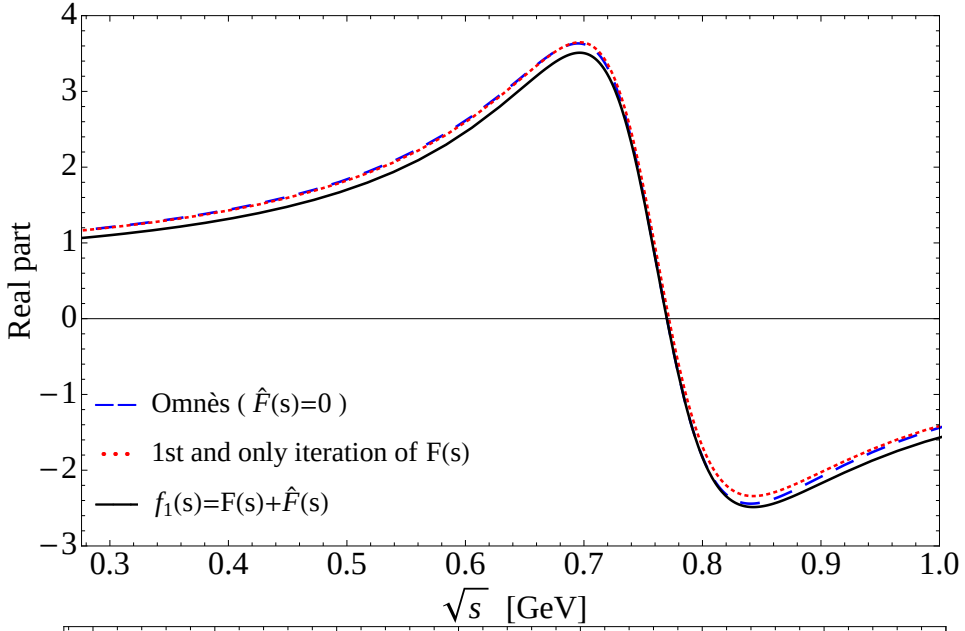


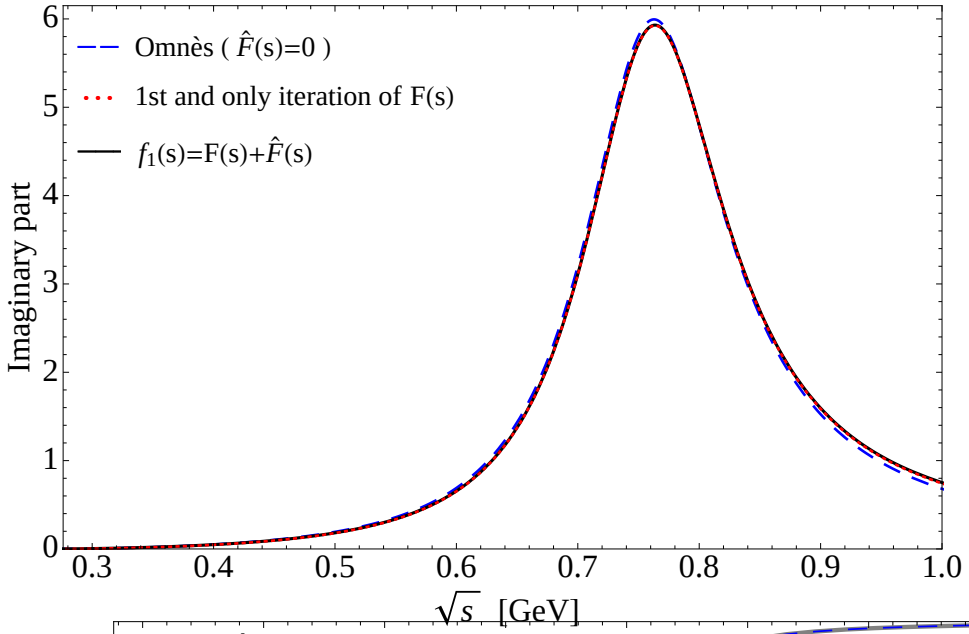


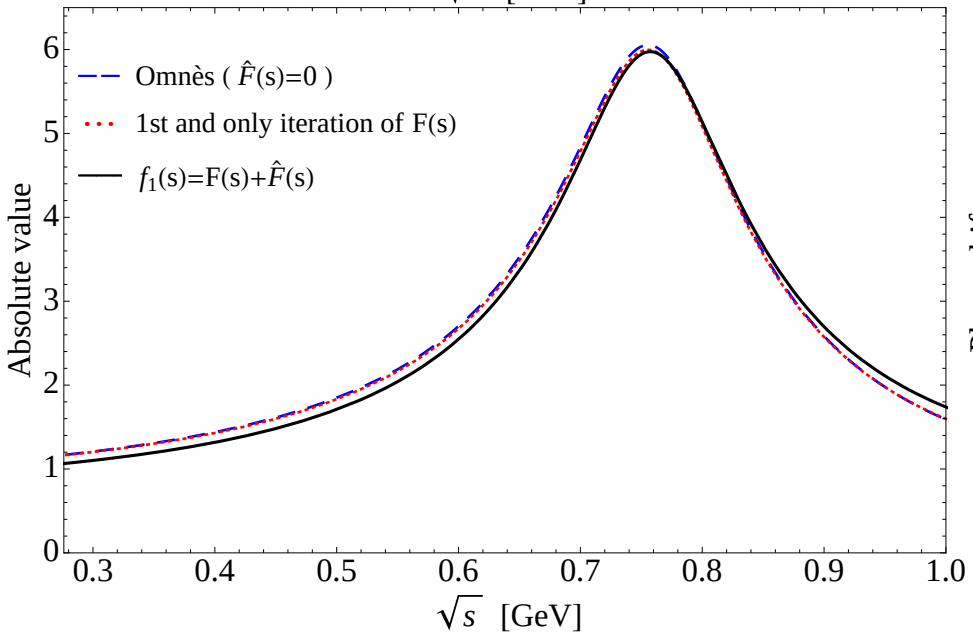


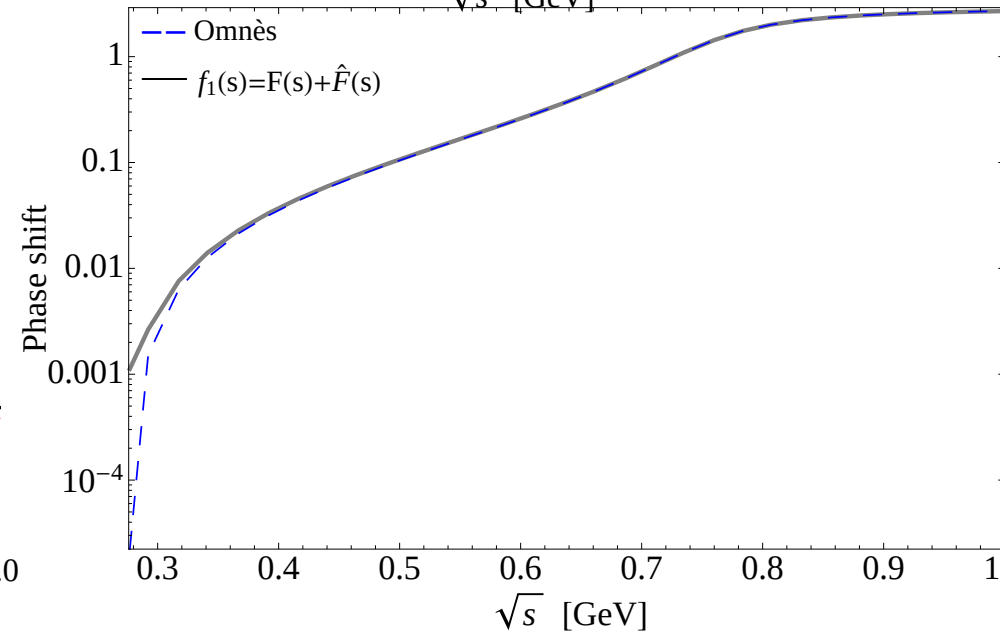


$B \to 3\pi$ (this work)







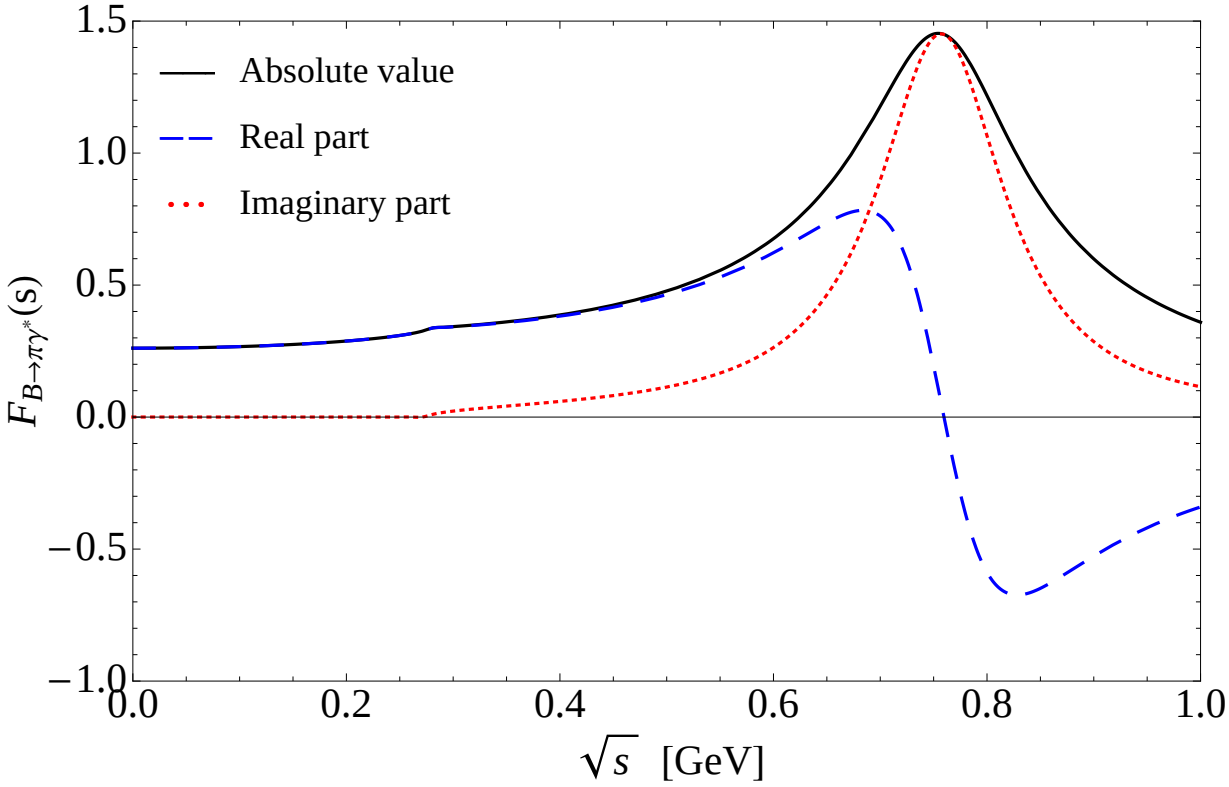


$B \to \pi\gamma^*$ transition form factor

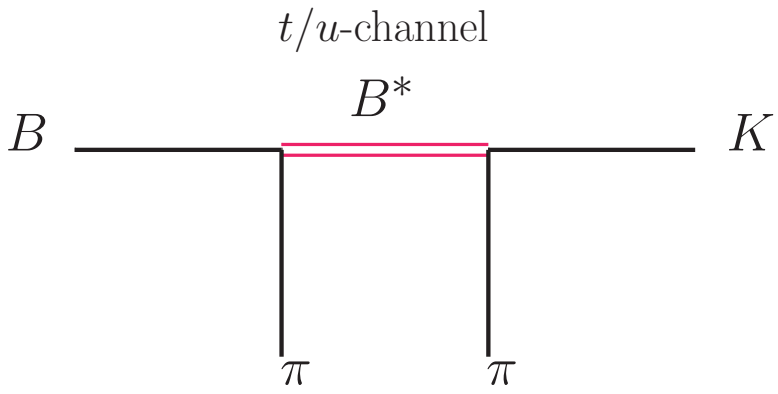
$$f_{B\pi}(s) = f_{B\pi}(0) + \frac{s}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\sigma(s') F_\pi^{V*}(s') f_1^{B\to 3\pi}(s')}{s'(s'-s)},$$

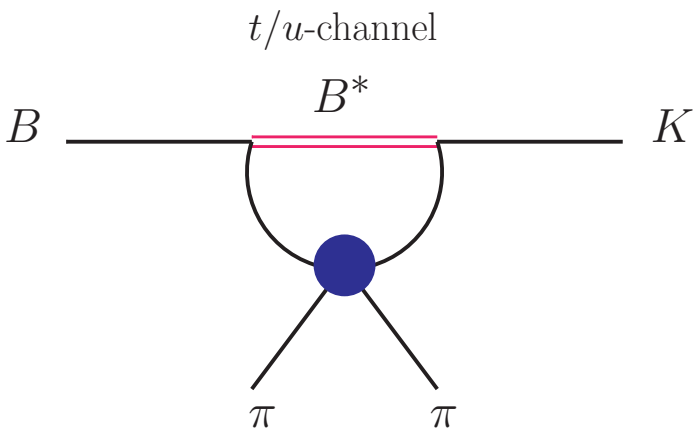
$$f_{B\pi}(0) = \frac{1}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\sigma(s') F_\pi^{V*}(s') f_1^{B\to 3\pi}(s')}{s'} \doteq 0.261^{+0.020}_{-0.023} \Rightarrow a = 0.04,$$

Bharucha
JHEP 1205, 092 (2012)



$B \to 2\pi K$ decay amplitude: B^* resonance exchange





$$\mathcal{T}(t,u) = \frac{g_{BB^*\pi} g_{B^*K\pi}}{2}\left(\frac{m_\pi^2 - m_K^2 - s + u}{t - m_{B^*}^2} + t \leftrightarrow u\right), \quad g_{BB^*\pi} = \frac{2m_B}{f_\pi} g\,,$$

$g = 0.569(76)$ (lattice)

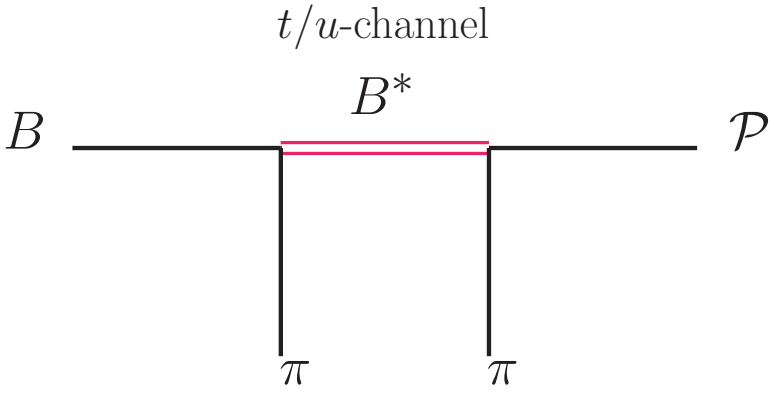
$$\hat{\mathcal{T}}(s) = 3\int_{-1}^{1} \frac{dz_s}{2}(1 - z_s^2)\mathcal{T}(t(s,z_s), u(s,z_s))\,,$$

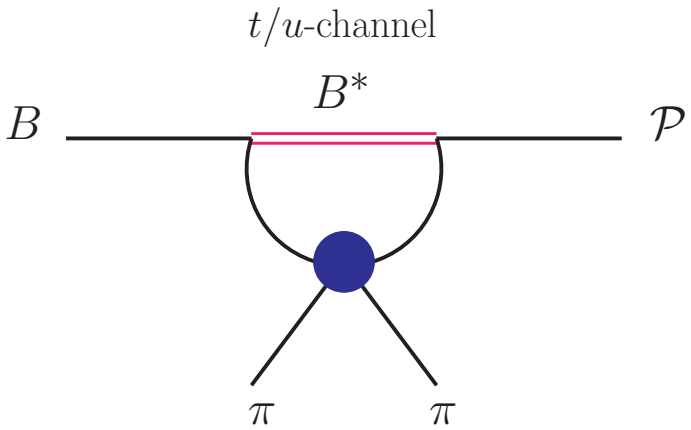
$g_{B^*K\pi} = 5 \cdot 10^{-5}$ (B $\to$ K$^*\pi$)

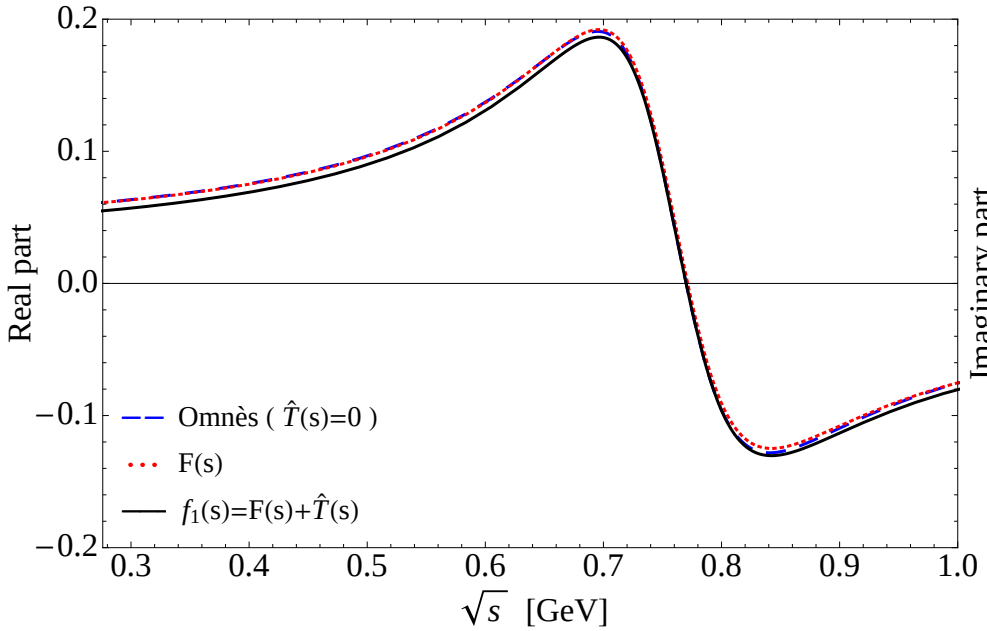
$$\mathcal{F}(s) = \Omega(s)\left(a + \frac{s}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\sin\delta_1^1(s')\hat{\mathcal{T}}(s')}{|\Omega(s')|s'(s'-s)}\right), \quad f_1(s) = \mathcal{F}(s) + \hat{\mathcal{T}}(s)$$

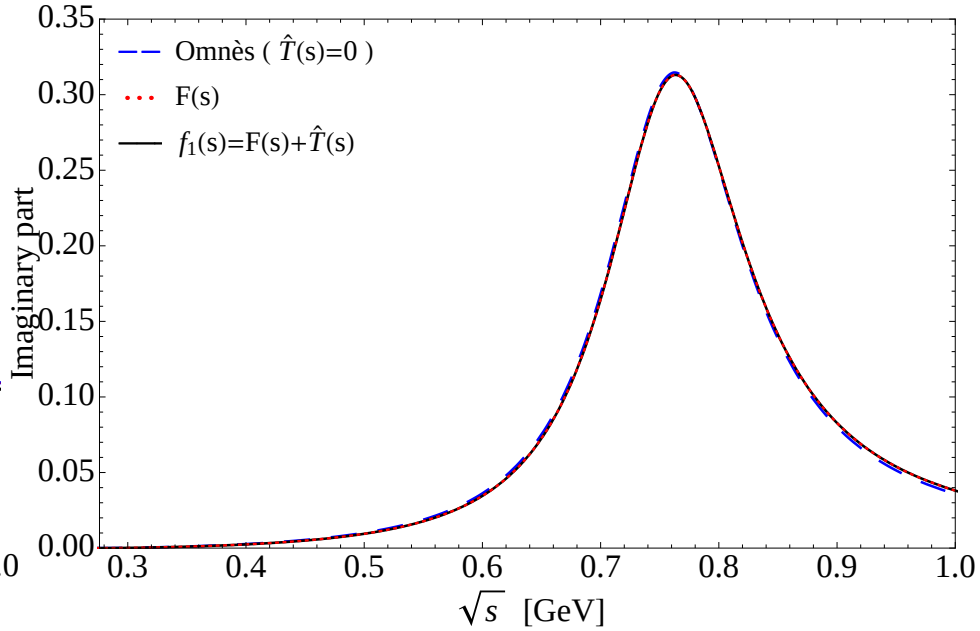
We fix a such that we reproduce $F_{BK}(0) = 0.319$ obtained in the Lattice

$B \to 2\pi K$ decay amplitude: B^* resonance exchange







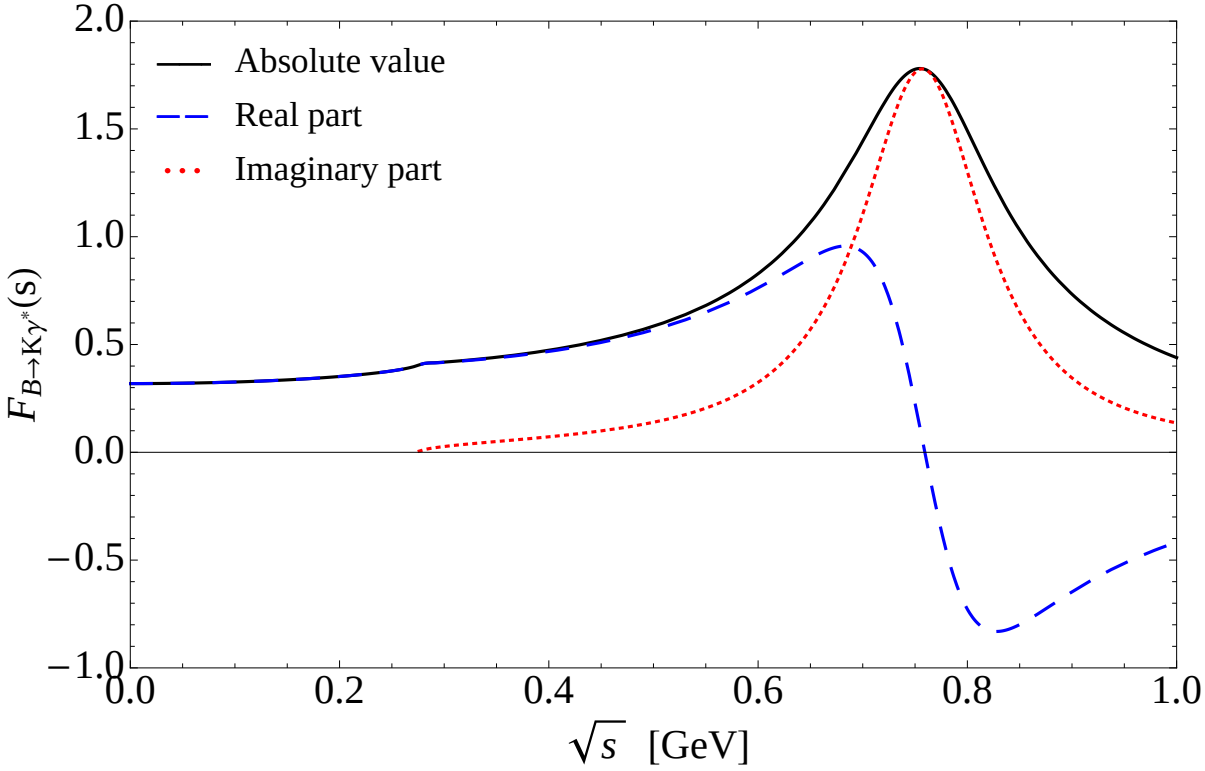


$B \to K\gamma^*$ transition form factor (preliminary)

$$f_{BK}(s) = f_{BK}(0) + \frac{s}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\sigma(s') F_\pi^{V*}(s') f_1^{B\to 2\pi K}(s')}{s'(s'-s)},$$

$$f_{BK}(0) = \frac{1}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\sigma(s') F_\pi^{V*}(s') f_1^{B\to 2\pi K}(s')}{s'} \doteq 0.319(66) \Rightarrow a = 0.05$$

Bouchard
PRD 88, no. 7, 079901 (2013)



Application

- Predictions for R_π and R_K in the range $0.045 < q^2 < 1$ GeV^2

$$R_\pi = \frac{B \to \pi \mu^+ \mu^-}{B \to \pi e^+ e^-}\,, \quad R_K = \frac{B \to K \mu^+ \mu^-}{B \to K e^+ e^-}\,,$$

$\sqrt{s_{\rm cut}}$ (GeV)	1.3	1.8	2.5	5
R_π	0.9300	0.9298	0.9298	0.9297
R_K	0.9300	0.9300	0.9300	0.9300

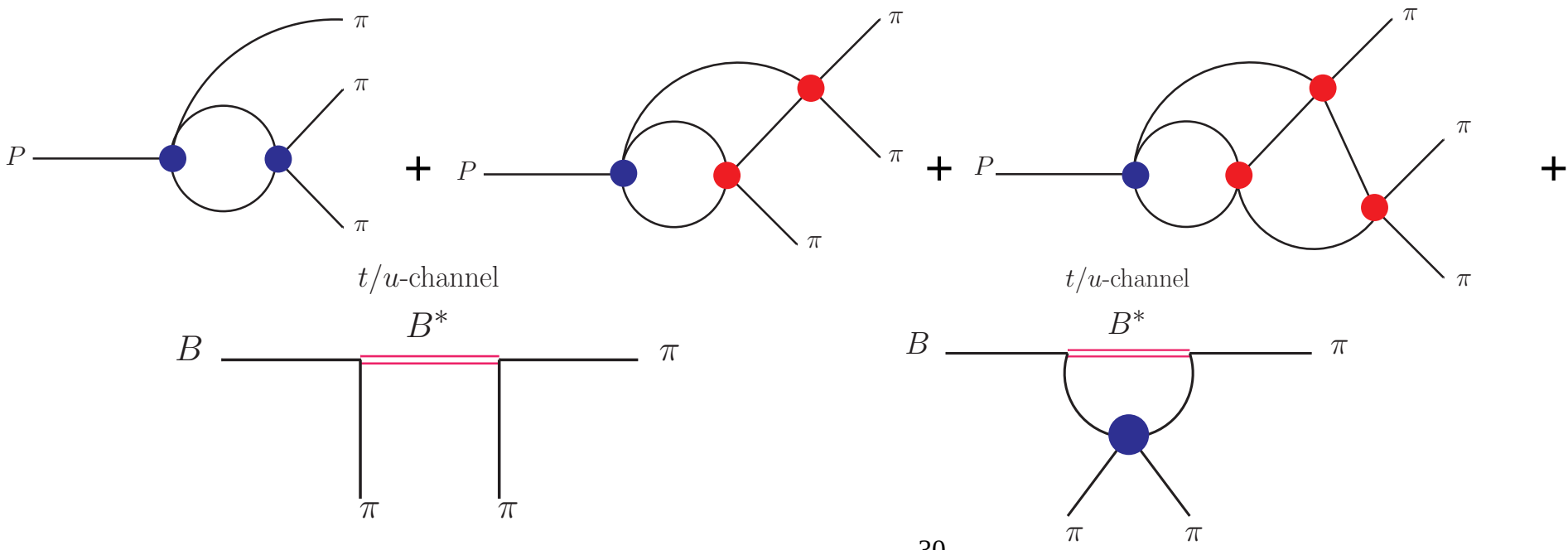
- Results insensitive to the value of the cut-off
- Central results (preliminary): $R_\pi = 0.930(1)$ and $R_K = 0.930(1)$

Conclusions

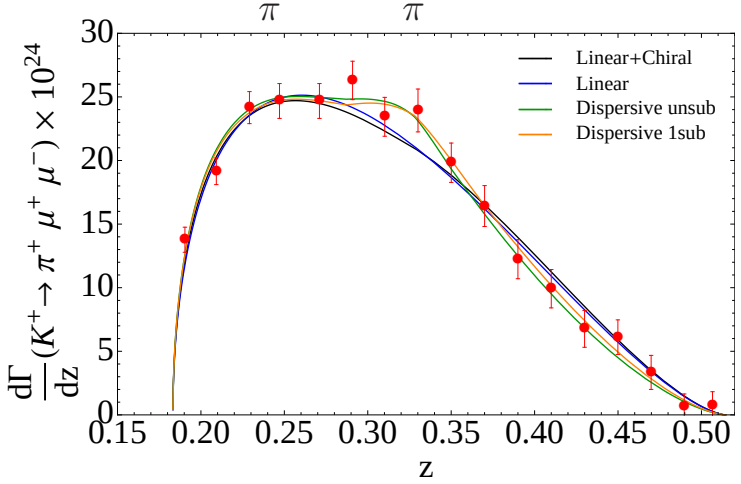
- We describe the $B \to \pi$ and $B \to K$ form factors in dispersion theory in the region where light vector resonances are produced
- Elements required as input:
  - Pion Vector Form Factor: Omnès equation
  - $B \to 3\pi$ and $B \to 2\pi K$ decay amplitudes: Khuri-Treiman
- The $\delta_1^1(s)$ is (almost) all what we need
- Crossed-channels rescattering effects taken into account
- We predict $R_\pi = 0.930(1)$ and $R_K = 0.940(1)$ in the region $0.045 < q^2 < 1$ GeV^2

In progress

- In progress: To add B^* resonance exchange to $B \to 3\pi$



- (Very near) future plan: $K \to \pi \ell^+ \ell^-$



Back-up

The Omnès equation

R. Omnès, Nuovo Cim. 8, 316 (1958)

- Schwartz reflection principle

$$\left.\begin{aligned} f(s+i\varepsilon) &= |f(s)|e^{i\delta(s)} \\ f(s-i\varepsilon) &= |f(s)|e^{-i\delta(s)} \end{aligned}\right\} \quad f(s+i\varepsilon) = e^{2i\delta(s)} f(s-i\varepsilon)\,,$$

- Take the discontinuity

$$\mathrm{disc}\log f(s) = \log f(s+i\varepsilon) - \log f(s-i\varepsilon) = 2i\delta(s)\,.$$

- Write down a dispersion relation for $\log f(s)$

$$\log f(s) = \frac{1}{2i\pi}\int_{s_{\mathrm{th}}}^{\infty} \frac{\mathrm{disc}\log f(s)}{(s'-s-i\varepsilon)} ds'\,,$$

- Omnès solution

$$f(s) = \exp\left[\frac{1}{\pi}\int_{s_{\mathrm{th}}}^{\infty} \frac{\delta(s')}{(s'-s-i\varepsilon)} ds'\right] \equiv \Omega(s)\,.$$

Subtractions

- The discontinuity is not known up to arbitrarily large energies
- Performing subtractions we diminish the importance of the contribution from the high-energy region of the integral
- The subtraction constants absorbs the information encoded in this region

$$f(s) = f(s_0)\Omega(s)\,, \quad \Omega(s) = \exp\left[\frac{s-s_0}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\delta(s')}{(s'-s_0)(s'-s-i\varepsilon)}ds'\right],$$

$$\begin{aligned}
{\rm Re}f(s) &= f(s_0)\exp\left[\mathcal{P}\frac{s-s_0}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\delta(s')}{(s'-s_0)(s'-s)}ds'\right]\cos\delta(s)\,,\\
{\rm Im}f(s) &= \tan\delta(s){\rm Re}f(s)\,,\\
|f(s)| &= f(s_0)\exp\left[\mathcal{P}\frac{s-s_0}{\pi}\int_{s_{\rm th}}^{\infty}\frac{\delta(s')}{(s'-s_0)(s'-s)}ds'\right].
\end{aligned}$$

The Omnès equation

- Most general solution

$$f(s) = P(s)\Omega(s)\,.$$

- To fix $P(s)$ we analyze $s \to \infty$ where usually $\delta_\infty \equiv \lim_{s\to\infty} \delta(s)$ tends to a constant

$$f(s) = |f(s)|e^{i\delta(s)}$$

$$\begin{aligned}
\lim_{s\to\infty} f(s) &= \lim_{s\to\infty} P(s) \exp\left[\frac{s-s_0}{\pi}\frac{\delta_\infty(s)}{s-s_0}\log\left(\frac{s_{\rm th}-s}{s_{\rm th}-s_0}\right)\right] e^{i\delta_\infty} \\
&= \lim_{s\to\infty} P(s)\left(\frac{s_{\rm th}-s_0}{s}\right)^{\delta_\infty/\pi} e^{i\delta_\infty}\,.
\end{aligned}$$

- Assuming $f(s)$ to vanish for $s \to \infty$ implies $P(s)$ to be a constant

$\phi \to 3\pi$

B. Kubis et.al., EPJC 72 (2012) 2014
B. Kubis et.al., PRD 86 (2012) 054013
I.V. Danilkin et.al., PRD 91 (2015) 094029