A novel perspective of Feynman integrals

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Outline

I. Introduction

II. Series representation

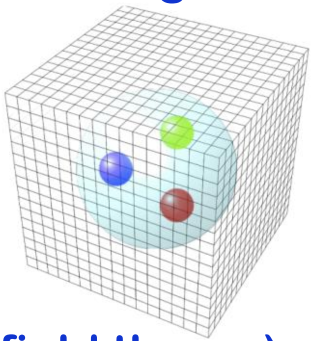
III. Analytical continuation

IV. Outlook: HPET

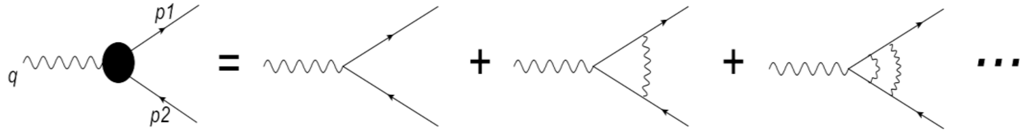
Quantum field theory

- QFT: the underlying theory of modern physics
  - Solving QFT is important for testing the SM and discovering NP

- How to solve QFT:



- **Nonperturbatively** (e.g. lattice field theory): discretize spacetime, numerical simulation complicated, application limited

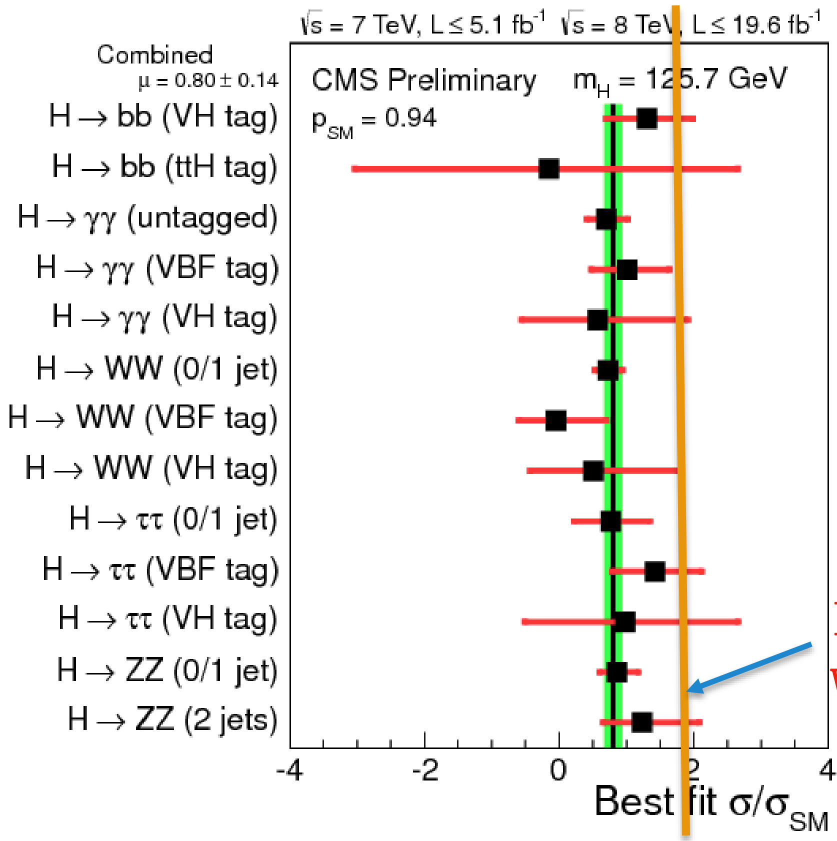


- **Perturbatively** (small coupling constant): generate and calculate Feynman amplitudes, relatively simpler, the primary method

Super computer

Why precision calculation?

- Test of SM



LO @ 8 TeV: 9.6 +/- 25% pb
N3LO @ 8 TeV: 19.47 + 0.32% - 2.99% pb

Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056

If w/o higher order results,
we can NOT interpret the Higgs properly

- No significant new physics signal at LHC up to now, precision is needed for NP discovery

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules
- New developments (参见何颂报告)

2. Calculate Feynman loop integrals

- Direct calculation
- Indirect calculation

This talk

3. Calculate phase-space integrals

- Monte Carlo simulation with IR subtractions
- Relation with loop integrals

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \lim_{\eta\to 0^+} \int \frac{d^D p}{(2\pi)^D}\left(\frac{i}{p^2+i\eta} + \frac{-i}{p^2-i\eta}\right)$$

Feynman loop integrals

- The key in applying pQFT

$$\lim_{\eta\to 0^+}\int \prod_{i=1}^{L}\frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}}\prod_{\alpha=1}^{N}\frac{1}{(q_\alpha^2-m_\alpha^2+\mathrm{i}\eta)^{\nu_\alpha}}$$

q_α: linear combination of loop momenta and external momenta

Multi-loop: a challenge for intelligence

- One-loop calculation: (up to 4 legs) satisfactory approaches existed as early as 1970s

’t Hooft, Veltman, NPB (1979)
Passarino, Veltman, NPB (1979)
Oldenborgh, Vermaseren (1990)

Developments of unitarity-based method in the past decade made the calculation efficient for multi-leg problems

Britto, Cachazo, Feng, 0412103
Ossola, Papadopoulos, Pittau, 0609007
Giele, Kunszt, Melnikov, 0801.2237

- About 40 years later, a satisfactory approach for multi-loop calculation is still missing

Main strategy

1) Reduce loop integrals to basis (Master Integrals)

- Integration-by-parts (IBP) reduction:

  Chetyrkin, Tkachov, NPB (1981)
  Laporta, 0102033

  currently, the only way, main bottleneck

  brute force algorithm, extremely time consuming for complicated problems

  unitarity-based reduction is efficient but cannot give complete reduction

2) Calculate MIs/original integrals

- Differential equations (depends on reduction and BCs) Kotikov, PLB (1991)
- Difference equations (depends on reduction and BCs) Laporta, 0102033
- Sector decomposition (extremely time-consuming) Binoth, Heinrich, 0004013
- Mellin-Barnes representation (nonplanar, time) Usyukina (1975) Smirnov, 9905323

IBP reduction

➢ **A result of dimensional regularization**

Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(v_k^\mu \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}} \right) = 0\,, \quad \forall j, k$$

$$\Downarrow$$

- **Linear equations:** $$\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$
- M_i **scalar integrals,** Q_i **polynomials in** $D, \vec{s}, \eta$

➢ **For each problem, the number of MIs is FINITE**

Smirnov, Petukhov, 1004.4199

- **Feynman integrals form a finite dimensional linear space**
- **Reduce thousands of loop integrals to much less MIs**

Difficulty of IBP reduction

➢ Solve IBP equations

$$\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- Very large scale of linear equations (even millions of)
- Fully coupled
- Hard to do Gaussian elimination for many variables $D, \vec{s}, \eta$

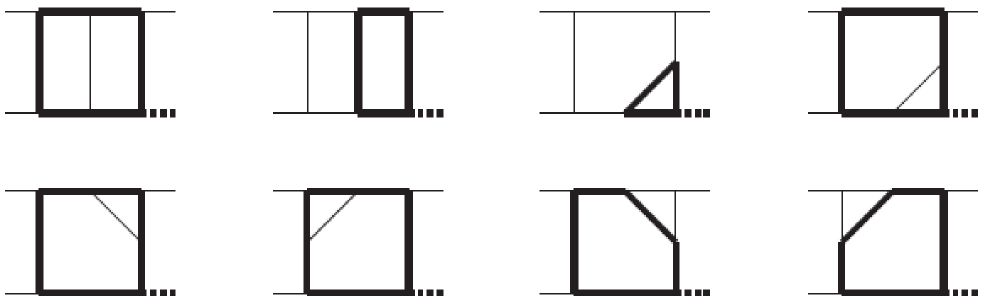
➢ Cutting edge problems

- Hundreds GB RAM
- Months of runtime using super computer

Difficulty of MIs calculation

- Analytical: Higgs → 3 partons (Euclidean Region)

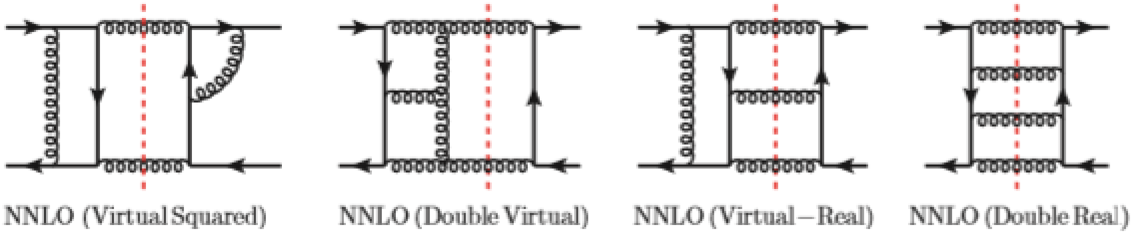
R. Bonciani, et.al 2016



200MB, 10 min

- Numerical: Quarkonium decay at NNLO

Feng, Jia, Sang, 1707.05758



10^5 CPU core-hour

Recent developments

➢ Improvements for IBP reduction

- Finite field method Manteuffel, Schabinger, 1406.4513
- Direct solution Kosower, 1804.00131
- Module-intersection IBP method Böhm, Georgoudis, Larsen, Schönemann, Zhang, 1805.01873
- Obtain one coefficient at each step Chawdhry, Lim, Mitov, 1805.09182

➢ Improvements for evaluating scalar integrals

- Quasi-Monte Carlo method Li, Wang, Yan, Zhao, 1508.02512
- Finite basis Manteuffel, Panzer, Schabinger, 1510.06758
- Uniform-transcendental basis Boels, Huber, Yang, 1705.03444

State-of-the-art computation

(参见朱华星报告)

- 2→2 process with massive particles at two-loop order is already the frontier
  - $g+g\to t+\bar{t},\ g+g\to H+H,\ g+g\to H+g,\dots$
- Very time-consuming
  - Two-loop $g+g\to H+H\ (g)$: complete IBP reduction cannot be achieved within tolerable time
    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - Two-loop decay $Q+\bar{Q}\to g+g$, MIs cost $O(10^5)$ CPU core-hour
    Feng, Jia, Sang, 1707.05758
  - Four-loop nonplanar cusp anomalous dimension, within tolerable computational expense, calculated MIs have 10% uncertainty
    Boels, Huber, Yang, 1705.03444

New ideas are badly needed

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Feynman integrals with an auxiliary variable

Liu, YQM, Wang, 1711.09572
Liu, YQM, 1801.10523

➢ **Dimensional regularized scalar Feynman loop integral with an auxiliary variable η**

$$I(D;\{\nu_\alpha\};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

- Think it as an analytical function of η
- Physical result is defined by

$$I(D;\{\nu_\alpha\};0) \equiv \lim_{\eta\to 0^+} I(D;\{\nu_\alpha\};\eta)$$

Expansion of propagators

- **Expansion of propagators around $\eta \to \infty$**

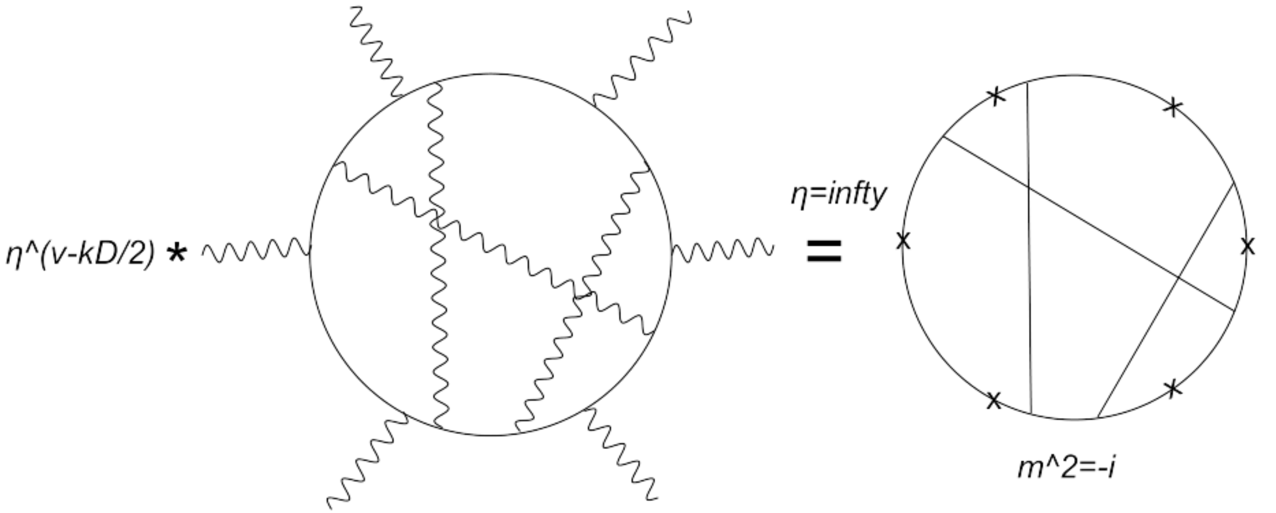
$$\frac{1}{(\ell+p)^2 - m^2 + \mathrm{i}\eta} = \frac{1}{\ell^2 + \mathrm{i}\eta} \sum_{j=0}^{\infty} \left(- \frac{2\ell \cdot p + p^2 - m^2}{\ell^2 + \mathrm{i}\eta} \right)^j$$

  - We proved the validity of the expansion rigorously

- **After expansion: no external momenta in denominator, equal squared masses $-i\,\eta$**

Asymptotic expansion

➢ Leading term



➢ Higher order terms

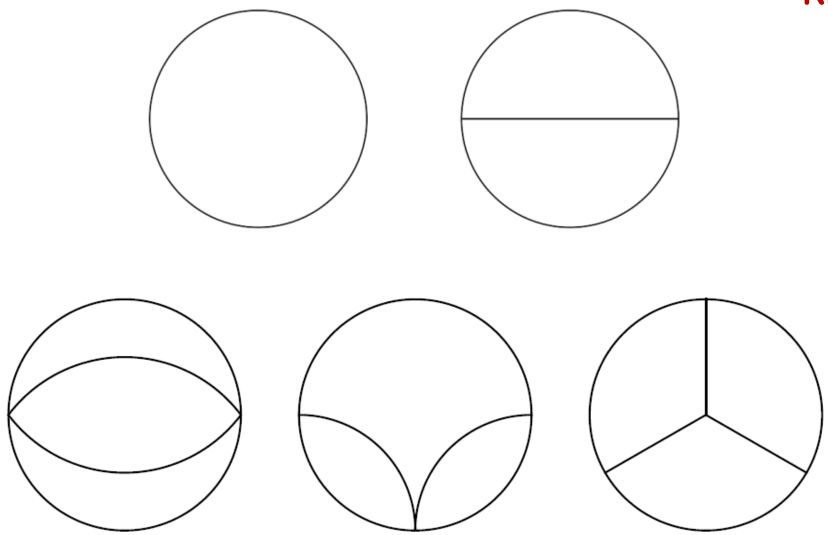
$$I_0^{\mathrm{bub}}(D;\eta) + \frac{1}{\eta}\sum_k c_{1k} I_{1k}^{\mathrm{bub}}(D+2;\eta) + \frac{1}{\eta^2}\sum_k c_{2k} I_{2k}^{\mathrm{bub}}(D+4;\eta) + \cdots\,, \quad \eta \to \infty$$

- All terms are combinations of vacuum bubble integrals

Vacuum bubble integrals

- Vacuum bubble integrals up to 3 loops, analytic results are known

Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136



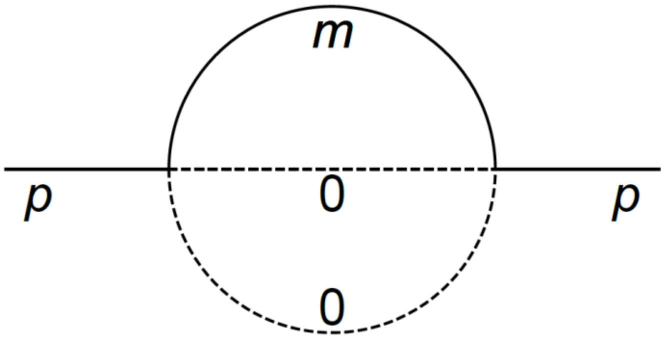
Numerical results known to 5-loop order!!!

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

Example of series expansion

➢ Sunrise integral

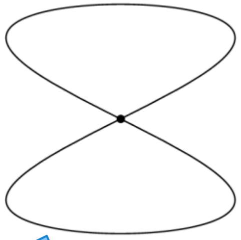
$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$

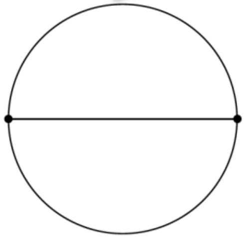


$$\mathcal{D}_1 =(\ell_1+p)^2-m^2,\ \mathcal{D}_2=\ell_2^2,\ \mathcal{D}_3=(\ell_1+\ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1-\frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta}+\frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right]I_{2,2}^{\mathrm{bub}}\right.$$

$$\left.-\mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right]I_{2,1}^{\mathrm{bub}}+\mathcal{O}(\eta^{-2})\right\}$$





Series representation of Feynman integrals

- The infinite series at $\eta \to \infty$ uniquely defines the analytical function $I(\eta)$

$$I(\eta) = \eta^{LD/2-\nu}\left[I_{\rm bub}^{(0)} + \frac{1}{\eta}I_{\rm bub}^{(1)} + \cdots\right]$$

- Analytical continuation defines $I(0)$

$$I(D;\{\nu_\alpha\};0) \equiv \lim_{\eta\to 0^+} I(D;\{\nu_\alpha\};\eta)$$

- **Physical Feynman integral can be defined as analytical continuation of a calculable series**

- The series contains only vacuum integrals, easy to obtain

New strategy to evaluate Feynman integrals

1) Construct series representation (Easy)

2) Perform analytical continuation (How?)

Analytical continuation: usually very hard

Yet another unsolved problem?

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Perturbative matching

$$\sum_{i=1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta)=0$$

- **Parametrization, finite unknown numbers**

$$Q_i(D,\vec{s},\eta)=\sum_{(\lambda_0,\vec{\lambda})\in\Omega^{r+1}_{d_i}} Q_i^{\lambda_0\ldots\lambda_r}(D)\,\eta^{\lambda_0}\,s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

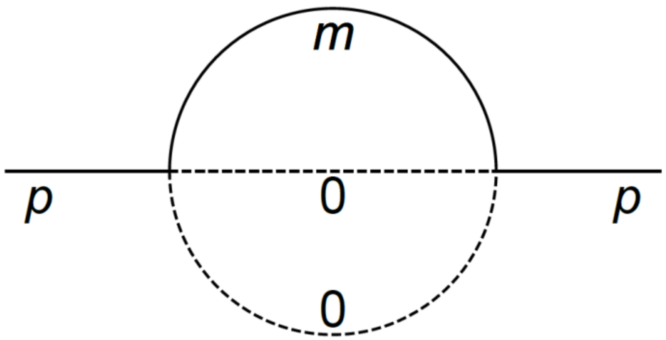
- **Determine unknown parameters by matching both sides of the relation at large η, using series representation**

- **Note: although obtained at large η, the relation is valid for any value of η**

Reduction: searching for effective relations

- Reduction of $I_{\nu 11}$ v.s. IBP

$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



ν	Our reduction		FIRE5	
	Time/s	# of relations	Time/s	# of relations
5	0.14	14	12	203
10	0.33	34	42	1313
15	0.48	54	346	4073
20	0.75	74	2169	9233
100	5.43	394	-	-

New method can generate more efficient relations comparing with IBP (FIRE 5), thus much faster

37 years after IBP, an independent (and more efficient) reduction method is finally available

Differential equations

- Perturbative matching reduces all loop integrals to MIs
  - Only need analytical continuation of MIs
- Perturbative matching can also set up differential equations for MIs

$$\frac{\partial}{\partial\eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$

- Boundary conditions at $\eta = \infty$: leading term of the series representation, known

Analytical continuation

➢ Solve DEs

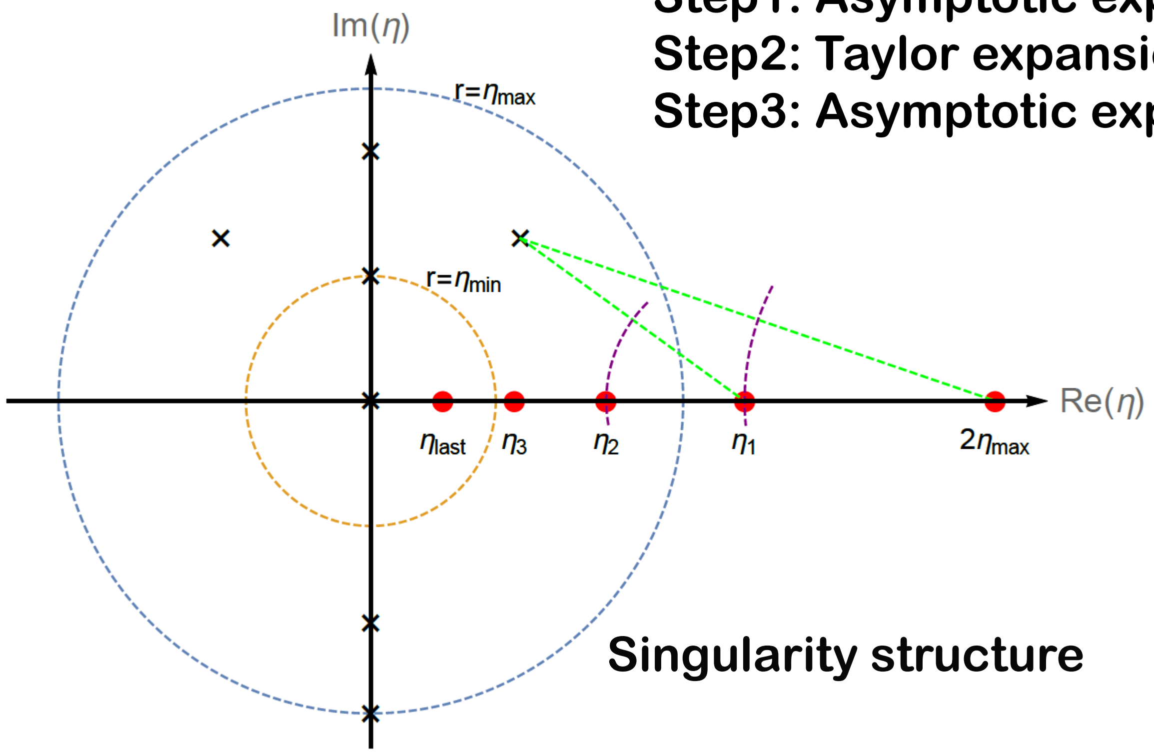
$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta) \quad \text{with known } \vec{I}(D;\infty)$$

Well-studied mathematic problem:

Step1: Asymptotic expansion at $\eta = \infty$

Step2: Taylor expansion at analytical points

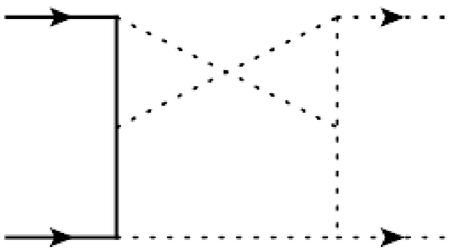
Step3: Asymptotic expansion at $\eta = 0$



Singularity structure

A 2-loop example

- **Nonplanar 2-loop box integral**



with $m^2 = 1, s = 4, t = -1$

- **168 master integrals, 26 steps to go from** $\eta = \infty$ **to** $\eta = 0^+$**, 30 orders in expansion**

A 2-loop example

- Result agree with sector decomposition

$$I_{\rm np}(4-2\epsilon) = 0.0520833\epsilon^{-4} - (0.131616 - 0.147262\mathrm{i})\epsilon^{-3} - (0.741857 + 0.185602\mathrm{i})\epsilon^{-2} + (3.73984 - 4.15756\mathrm{i})\epsilon^{-1} - (4.75677 - 12.0749\mathrm{i}) + (23.9674 - 55.4214\mathrm{i})\epsilon + \cdots ,$$

- A few minutes (7-9 significant digits)
- Sector decomposition code FIESTA4: $O(10^4)$ CPU core-hour (3-4 significant digits)

Faster by 10^5 times!!

Summary

- Computation of multi-loop integrals is very important and challenge
- A new strategy (with an auxiliary parameter)
  1) Construct series representation
  2) Perturbative matching to setup reduction relations and DEs
  3) Analytical continuation by solving DEs

Heavy particle effective theory

YQM, in progress

- Introduce auxiliary masses to Lagrangian

$$\widetilde{\mathcal{L}}_{\mathrm{QED}}(\lambda) = \mathcal{L}_{\mathrm{QED}} + \lambda\bar{\psi}\psi + \lambda^2 A^2$$

  - Construct HPET at $\lambda \to \infty$ (similar to but not the same as HQET)
  - Evaluate scattering amplitudes as power expansion in $\frac{1}{\lambda^2}$
  - Each term of the expansion can be very compact

- Recover physical results at $\lambda \to 0$ by analytical continuation

Thank you!

Integration-By-Parts (IBP)

➢ History

- **1981 K. G. Chetyrkin, F. V. Tkachov**
  "Integration by parts: the algorithm to calculate beta function in 4 loops"
- **2000 S. Laporta**
  "High precision calculation of multiloop Feynman integrals by difference equations"
- **2004 C. Anastasiou, A. Lazopoulos ->** AIR
  "Automatic Integral Reduction for higher order perturbative calculations"
- **2008 A. Smirnov ->** FIRE
  "Algorithm FIRE – Feynman Integral REduction"
- **2009 C. Studerus ->** Reduze
  "Reduze- Feynman integral reduction in C++"
- **2012 R. Lee ->** LiteRed
  "Presenting LiteRed: a tool for the Loop InTEgrals REDuction"
- **2015 A. Manteuffel, R. Schabinger**
  "A novel approach to integration by parts reduction"
- **2016 K. Larsen, Y. Zhang**
  "Integration-by-parts reductions from unitarity cuts and algebraic geometry"
- **2017 A. Georgoudis, K. Larsen, Y. Zhang ->** AZURITE
  "AZURITE: an algebraic geometry based package for finding bases of loop integrals"

➢ Integrand-level reduction

$$\text{Integrand} = \sum c_i \times I_i$$

Physical singularities $\Rightarrow$ **Coefficients**

$$\mathcal{M}^{(1)}(2 \to 2) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \left(\frac{\Delta_4(\ell)}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4} + \sum_{i_1 i_2 i_3} \frac{\Delta_{3,i_1 i_2 i_3}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{i_1 i_2} \frac{\Delta_{2,i_1 i_2}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \text{tadpoles} \right)$$

$$D_1 = D_2 = D_3 = D_4 = 0 \quad \Rightarrow \quad \Delta_4$$

$$D_{i_1} = D_{i_2} = D_{i_3} = 0 \quad \Rightarrow \quad \Delta_{3,i_1 i_2 i_3}$$

$$\vdots$$

Unitarity Cuts

➢ History

- **1994 Z. Bern, L. Dixon, D. Dunbar, D. Kosower**
  "One-loop n-point gauge theory amplitudes, unitarity and collinear limits"
- **2005 R. Britto, F. Cachazo, B. Feng**
  "Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills"
- **2007 G. Ossola, C. Papadopoulos, R. Pittan ->** OPP method
  "Reducing full one-loop amplitudes to scalar integrals at the integrand level"
- **2008 G. Ossola, C. Papadopoulos, R. Pittan ->** CutTools
  "CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes"
- **2011 P. Mastrolia, G. Ossola**
  "On the integrand-reduction method for two-loop scattering amplitudes"
- **2012 Y. Zhang**
  "Integrand-level reduction of loop amplitudes by computational algebraic geometry methods"
- **2017 J. Bosma, M. Sogaard, Y. Zhang**
  "Maximal cuts in arbitrary dimension"

➢ Feynman parametric representation

$$I(D;\{\nu_\alpha\}) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2)^{\nu_\alpha}}$$

where $q_\alpha = c_\alpha^i \ell_i + d_\alpha^i p_i$

$$I(D;\{\nu_\alpha\}) = (-1)^\nu \frac{\Gamma(\nu - LD/2)}{\prod_k \Gamma(\nu_k)} \int \prod_\alpha (x_\alpha^{\nu_\alpha - 1} \mathrm{d}x_\alpha) \times \delta(x-1) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\mathcal{F}^{\nu - LD/2}}$$

$$\mathcal{U}(\vec{x}) = \sum_{T\in T_1} \prod_{i \notin T_1} x_i$$

$$\mathcal{F}_0(\vec{x}) = -\sum_{T\in T_2} s_T \prod_{i\notin T_2} x_i$$

$$\mathcal{F}(\vec{x}) = \mathcal{F}_0(\vec{x}) + \mathcal{U}(\vec{x}) \sum_{\alpha=1}^{N} x_\alpha m_\alpha^2$$

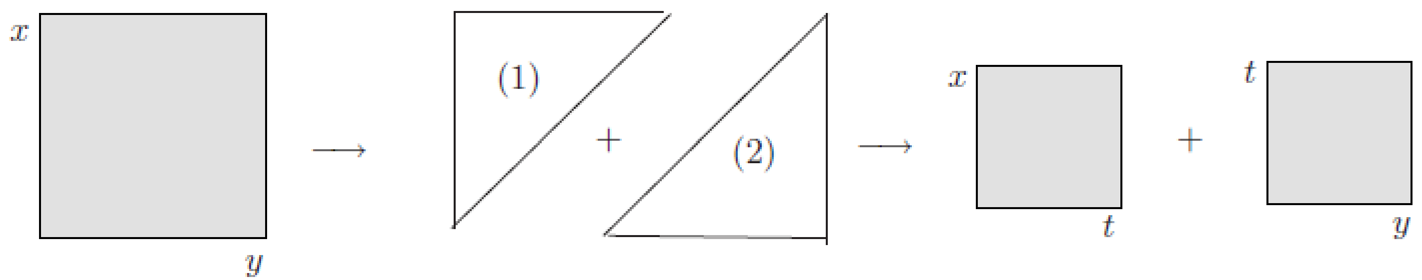
$\mathcal{U} \sim x^L$ **Spanning 1-tree, sub UV divergences**

$\mathcal{F} \sim x^{L+1}$ **Spanning 2-tree, IR divergences**

See e.g. [Heinrich2008]

➢ Sector decomposition: basic example

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \left(x + (1-x)\,y\right)^{-1}$$



$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \left(1 + (1-x)\,t\right)^{-1}$$

$$+ \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \left(1 + (1-y)\,t\right)^{-1}$$

Sector Decomposition

➢ Apply to Calculation of Feynman Integrals

- Generate primary sectors
- Generate subsectors iteratively
- Take epsilon expansion

[Binoth, Heinrich 2000]
…

$$I = (-1)^{\nu}\Gamma(\nu - LD/2)\sum_{i=1}^{N}\sum_{j=1}^{\Lambda(i)} I_{ij}, \quad I_{ij} = \sum_{k=-2L}^{r} C_{ij,k}\epsilon^{k} + \mathcal{O}(\epsilon^{r+1})$$

- Evaluate the finite integrals numerically

$$C_{ij,k} \quad \xrightarrow{\text{M-C}} \quad \text{number}$$

Sector Decomposition

➢ History

- **1966 K. Hepp (BPHZ)**

  "Proof of the Bogoliubov-Parasiuk Theorem on Renormalization"

- **2000 T. Binoth, G. Heinrich**

  "An automatized algorithm to compute infrared divergent multi-loop integrals"

- **2008 A. Smirnov, M.N. Tentyukov, et.al ->** FIESTA

  "Feynman Integral Evaluation by a Sector decomposiTion Approach (FIESTA)"

- **2010 J. Carter, G. Heinrich, et.al ->** SecDec

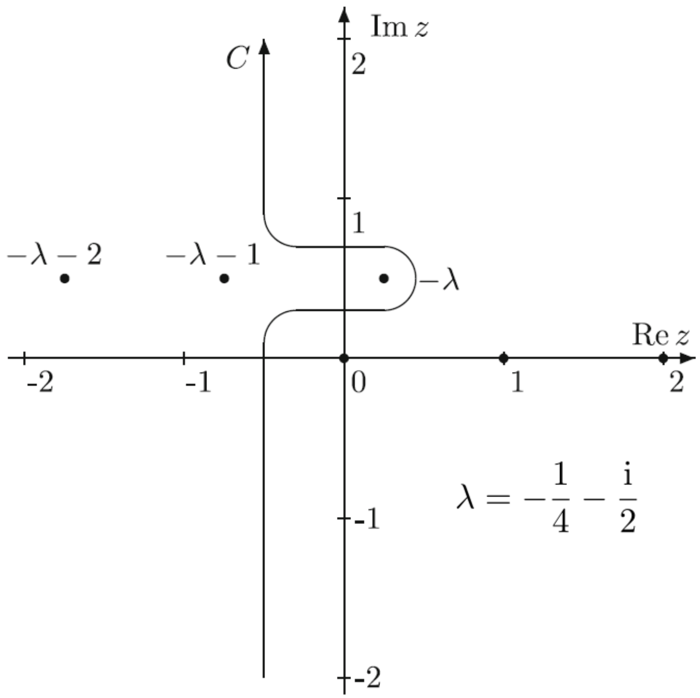
  "SecDec: A general program for sector decomposition"

- **2017 S. Borowka, G. Heinrich, et.al ->** pySecDec

  "pySecDec: a toolbox for the numerical evaluation of multi-scale integrals"

➢ Basic Relation

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}$$

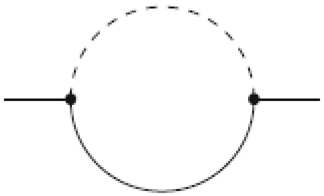


Rules:
Poles of $\Gamma(\cdots + z)$ are to the left of the contour.
Poles of $\Gamma(\cdots - z)$ are to the right of the contour.

Mellin-Barnes Representation

➢ **Apply to massive propagator**

$$\frac{1}{(\ell^2 - m^2)^\lambda} = \frac{1}{(\ell^2)^\lambda} \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \Gamma(\lambda + z) \Gamma(-z) \left(-\frac{m^2}{\ell^2}\right)^z$$



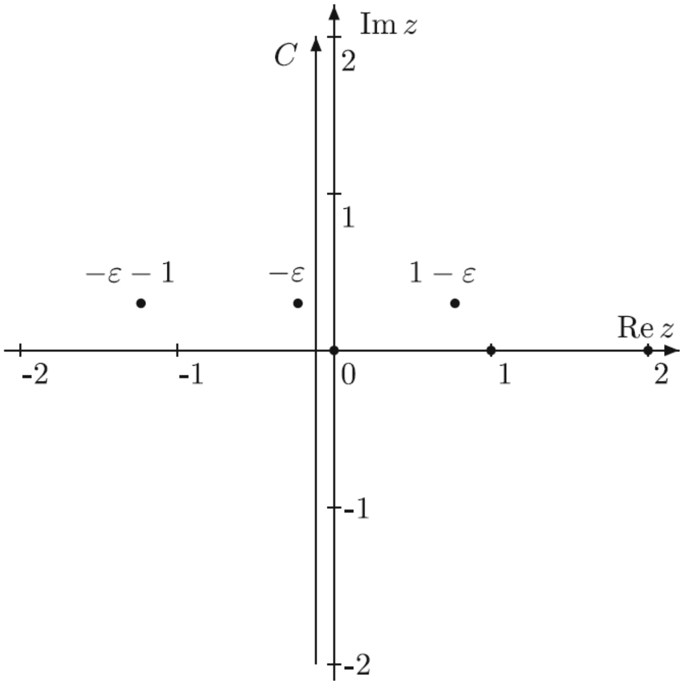
$$= \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)(\ell + p)^2}$$

$$\longrightarrow \int \mathrm{d}z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)} \left(-\frac{m^2}{p^2}\right)^z$$

The contour is pinched.

There is a UV divergence. We need to resolve the singularity.

Mellin-Barnes Representation



Strategy A: MBresolve.m [A. Smirnov, V. Smirnov 2009]

Deform the integration contours.

Strategy B: MB.m [M. Czakon 2005]

Fix the integration contours and tends ϵ to 0.

➢ Practical procedure

- Obtain MB representation
- Resolve epsilon singularities
- Perform epsilon expansion
- Evaluate the finite integrals numerically

[M. Czakon 2005]
[J.Gluza, et.al 2007]
[A. Smirnov, V. Smirnov 2009]
…

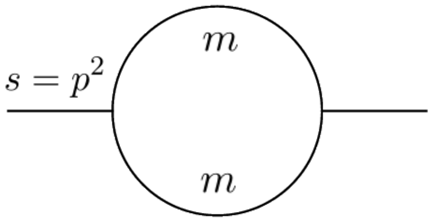
Mellin-Barnes Representation

➢ History

- **1975 N. Usyukina**
  "On a representation for the three-point function"
- **1999 V. Smirnov**
  "Analytical result for dimensionally regularized massless on-shell double box"
- **2005 M. Czakon->** MB.m
  "Automatized analytic continuation of Mellin-Barnes integrals"
- **2007 J. Gluza, K. Kajda, T. Riemann ->** AMBRE.m
  "AMBRE – a Mathematica package for the construction of Mellin-Barnes representations for Feynman integals"
- **2009 A. Smirnov, V. Smirnov, et.al ->** MBresolve.m
  "On the resolution of singularities of multiple Mellin-Barnes integrals"
- **2014 J. Blumlein, I. Dubovyk, et.al**
  "Non-planar Feynman integals, Mellin-Barnes representations, multiple sums"
- **2015 M. Ochman, T. Riemann ->** MBsums.m
  "Mbsums – a Mathematica package for the representation of Mellin-Barnes integrals by multiple sums"

Differential Equation Method

➢ **Differential Equation + Boundary Condition**



$$I(D;\{1,1\}) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)[(\ell+p)^2 - m^2]}$$

$$\frac{\partial}{\partial m^2} I(D;\{1,1\}) = I(D;\{2,1\}) + I(D;\{1,2\})$$

$$\stackrel{\text{IBP}}{=} \frac{2(D-3)}{4m^2 - s} I(D;\{1,1\}) - \frac{D-2}{m^2(4m^2-s)} I(D;\{1,0\})$$

$$\frac{\partial}{\partial m^2} I(D;\{1,0\}) = I(D;\{2,0\})$$

$$\stackrel{\text{IBP}}{=} \frac{D-2}{2m^2} I(D;\{1,0\})$$

$$I(D;\{1,1\})|_{m^2=0} = \Gamma(2-D/2)(-s)^{D/2-2} \frac{\Gamma(D/2-1)^2}{\Gamma(D-2)}, \quad I(D;\{1,0\})|_{m^2=0} = \cdots$$

Differential Equation Method

➢ Step1: Set up the differential equation

- Differentiate w.r.t. invariants, such as m^2, p^2
- IBP relations $$\frac{\partial}{\partial x}\vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

➢ Step2: Calculate boundary condition

- Calculate integrals at special value of m^2, p^2

➢ Step3: Solve the differential equation

- Canonical form $\partial_x \vec{I}(x;\epsilon) = \epsilon A(x)\vec{I}(x;\epsilon)$ [J. Henn 2013]
- …

Differential Equation Method

➢ History

- **1991 A. Kotikov**
  "Differential equations method: the calculation of N point Feynman diagrams"
- **1991 A. Kotikov**
  "Differential equations method: new technique for massive Feynman diagrams calculation"
- **1997 E. Remiddi**
  "Differential equations for Feynman graph ampltides"
- **2000 T. Gehrmann, E. Remiddi**
  "Differential equations for two-loop four-point functions"
- **2013 J. Henn ->** Canonical form
  "Multiloop integrals in dimensional regularization made simple"
- **2014 R. Lee**
  "Reducing differential equations for multiloop master integrals"
- **2017 L. Adams, E. Chaubey, S. Weinzierl**
  "Simplifying differential equations for multiscale Feynman integrals beyond multiple polylogarithms"

Analytic structure at infinity

- **Feynman parametric rep.**

$$I(\eta) = (-1)^{\nu} \frac{\Gamma(\nu - LD/2)}{\prod_i \Gamma(\nu_i)} \int \prod_{\alpha} (x_{\alpha}^{\nu_\alpha - 1} \mathrm{d}x_{\alpha})\, \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U} - \mathrm{i}\eta)^{\nu - LD/2}}$$

  - $\mathcal{U}$: graph polynomial of 1-tree
  - $\mathcal{F}$: graph polynomial of 2-tree

- **Observation:** $|\mathcal{F}/\mathcal{U}|$ is bounded in the Feynman parameter space!

$$|\mathcal{F}_i| < |t_i||\mathcal{U}_i| < |t_i||\mathcal{U}| \text{ and } |\mathcal{F}| < \textstyle\sum_i |t_i||\mathcal{U}|$$

- **Thus:** $J(D;\eta) \equiv \eta^{\nu - LD/2} I(D;\eta)$ is analytic at $\eta = \infty$

Review of QCD factorization

- Factorize observables to nonperturbative functions; RGEs for nonperturbative functions

$$\sigma = \hat{\sigma} \otimes f, \qquad df = C \otimes f$$

- Calculate quantities in perturbative region

$$\sigma = \sum_n \sigma^{(n)} \alpha^n, \quad df = \sum_n (df)^{(n)} \alpha^n, \quad f = \sum_n f^{(n)} \alpha^n$$

- Perturbative matching to determine coefficients of nonperturbative relations
  - For $n = 0$: $\sigma^{(0)} = \hat{\sigma}^{(0)} \otimes f^{(0)} \rightarrow \hat{\sigma}^{(0)} = \sigma^{(0)}/f^{(0)}$
  - For $n = 1$: $\sigma^{(1)} = \hat{\sigma}^{(1)} \otimes f^{(0)} + \hat{\sigma}^{(0)} \otimes f^{(1)} \rightarrow \hat{\sigma}^{(1)} = (\sigma^{(1)} - \hat{\sigma}^{(0)} \otimes f^{(1)})/f^{(0)}$
  - And so on. Similar for $C^{(n)}$
- Everything is determined by BC of f

Step1: Expansion at the infinity

- **Transformation**

$$J(D;\eta) = \eta^{\nu - LD/2} I(D;\eta)$$

- $\eta \to x^{-1}$

$$x\frac{\partial}{\partial x}\vec{J}(x) = B_1(x)\vec{J}(x)$$

- **"Outside of the large circle"**

$$\vec{J}(x) = \sum_{n=0}^{\infty} \vec{J}_n x^n, \quad B_1(x) = \sum_{n=0}^{\infty} B_{1n} x^n$$

Step1: Expansion at the infinity

- Recurrence relations

$$(n - B_{10})\vec{J}_n = \sum_{k=0}^{n-1} B_{1n-k}\vec{J}_k$$

- Can be used to determine any order of $\vec{J}_n$

- Estimation of $\vec{J}(x)$

$$\vec{J}(x) \sim \sum_{n=0}^{n_0} \vec{J}_n x^n$$

e.g. at $x = \frac{1}{2}r$, $\vec{J}\left(\frac{\mathrm{r}}{2}\right) \Rightarrow \vec{I}\left(\frac{2}{r}\right)$

Step2: Expansion at analytical points

- **At $\eta = \eta_k$:**
  - Expand the differential equation and obtain the recurrence relations
  - Solve for high-order expansion coefficients
  - Estimate the value of $\vec{I}(\eta)$ at $\eta = \eta_{k+1}$

- **End if we have entered the small circle**

Step3: Expansion at $\eta = 0$

- $\vec{I}(\eta_0)$ is known. How to determine $\vec{I}(0)$?
- DE

$$\eta \frac{\partial}{\partial \eta} \vec{I} = \tilde{A} \vec{I}$$

- Asymptotic behavior

$$\vec{I}(\eta) \sim \eta^{\tilde{A}(0)} \vec{v}_0$$

with $\vec{v}_0$ being constant

- In general

$$\vec{I}(\eta) \equiv P(\eta) \eta^{\tilde{A}(0)} \vec{v}_0$$

Step3: Expansion at $\eta = 0$

- Expand and obtain recurrence relations

$$nP_n + [P_n, \tilde{A}_0] = \sum_{k=0}^{n-1} \tilde{A}_{n-k} P_k$$

- Can be used to determine any order of P_n
- $\vec{v}_0$ contains all information of boundary
- Determine $\vec{v}_0$ via matching

$$\vec{I}(\eta_0) = P(\eta_0)\eta_0^{\tilde{A}(0)}\vec{v}_0$$

then

$$\vec{I}(0) = \lim_{\eta\to 0} \eta^{A(0)}\vec{v}_0$$