# Disentangling Covariant Wigner Functions for Chiral fermion

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JHG, Z.T. Liang, Q. Wang, X.N. Wang, arXiv:1802.06216

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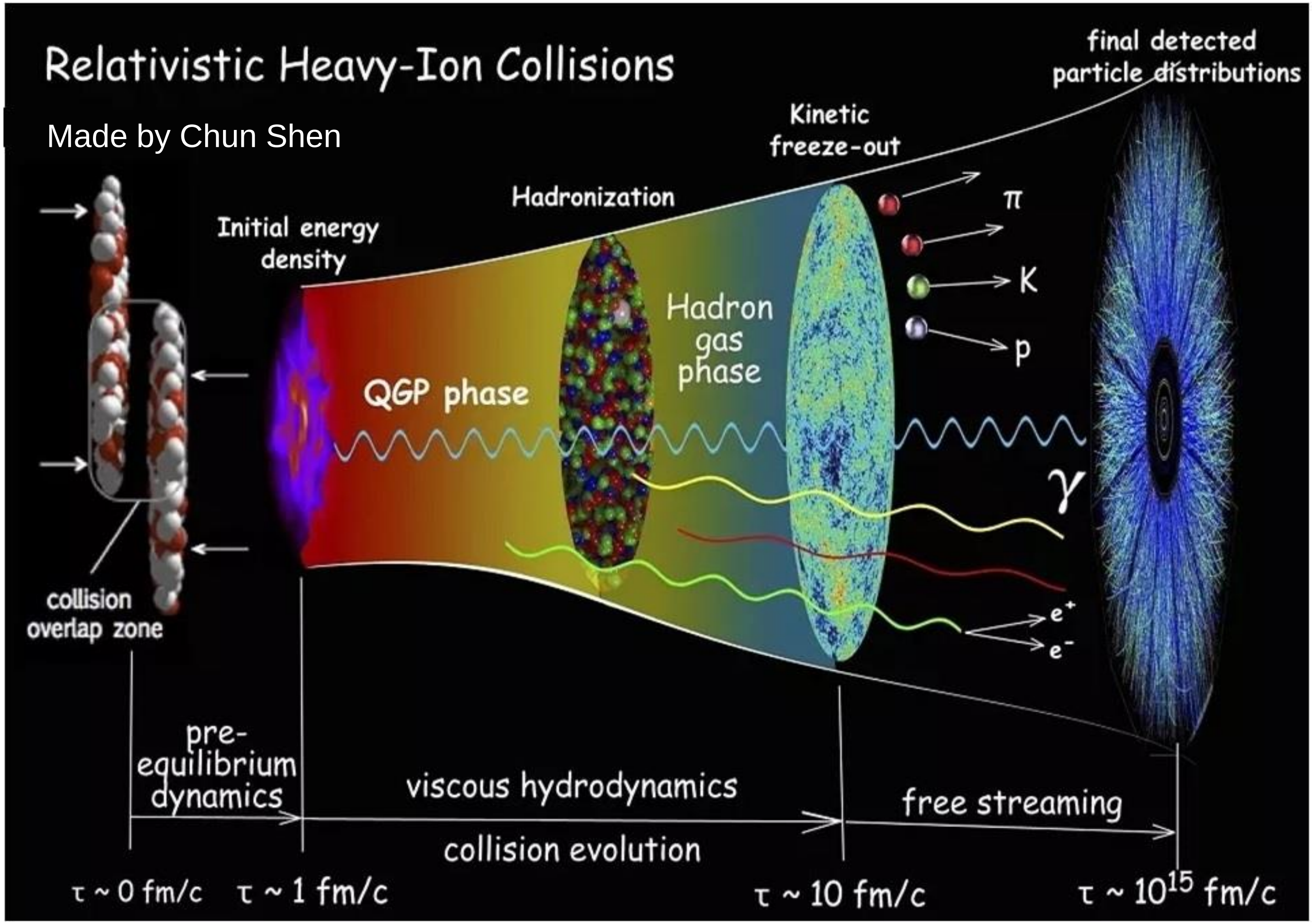
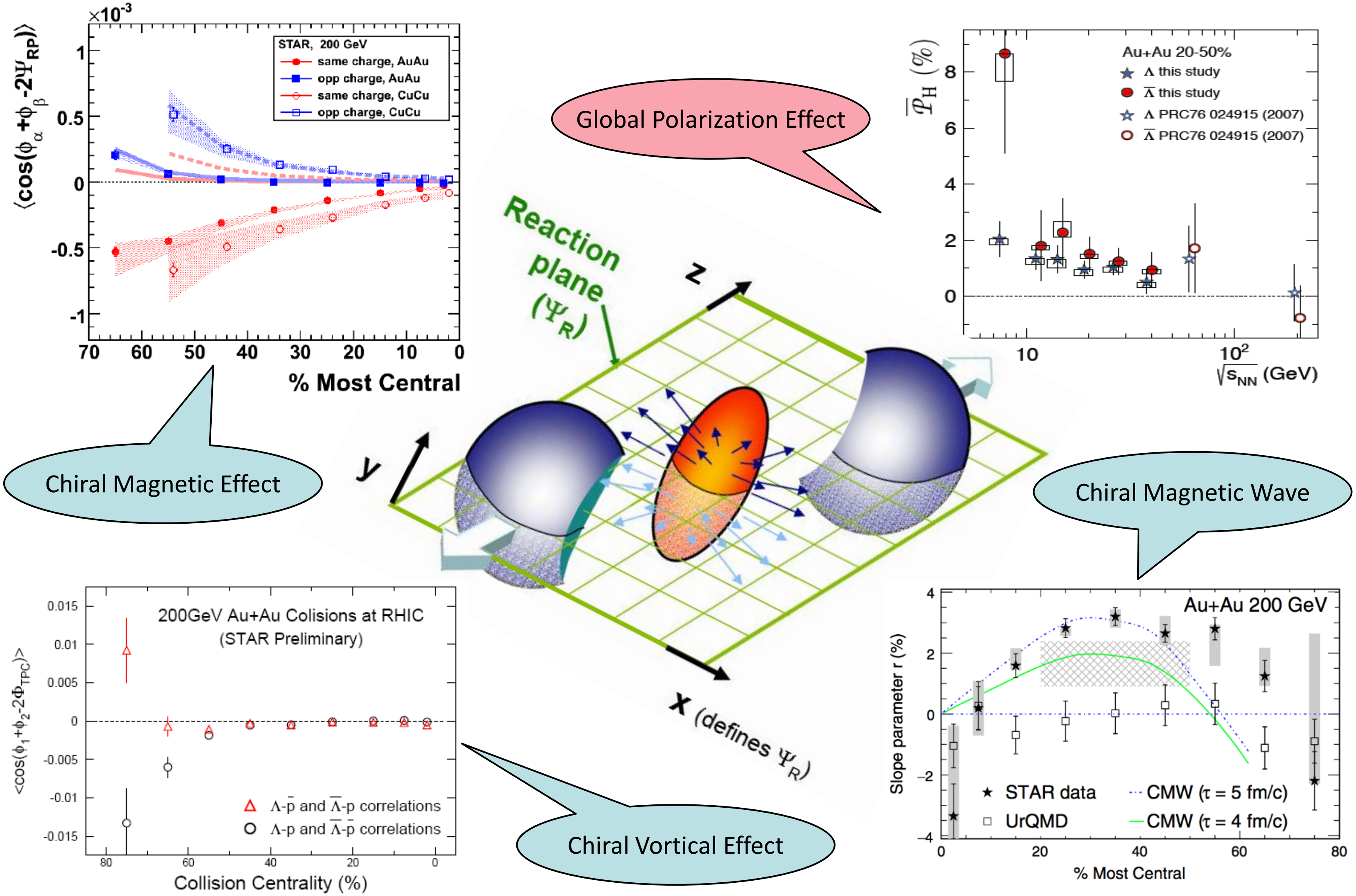
# Outline

- Introduction
- Entangle covariant Wigner functions
- Chiral kinetic equation and Chiral anomaly
- Rewrite our formalism in covariant form
- Summary

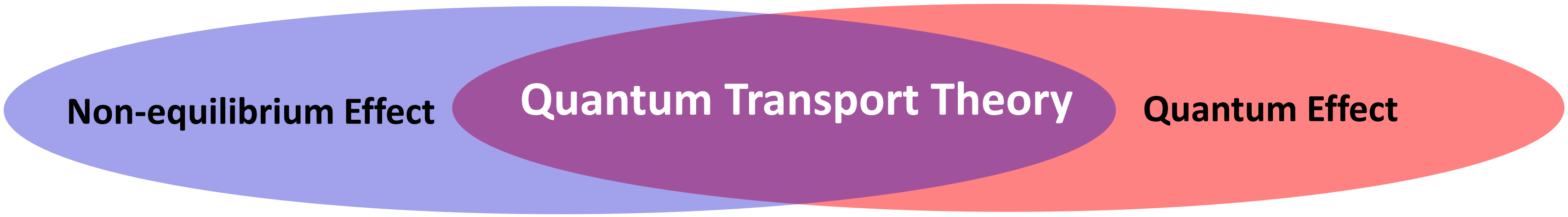
# Non-Equilibrium Effect in HIC



# Quantum Effects in HIC



# Quantum Transport Theory

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# Classical Transport Theory

Distribution function in phase-space: $f(t,\vec{x},\vec{p})$

The classical Boltzmann equation with the background EM fields

$$p^\mu\left(\partial_\mu - F_{\mu\nu}\partial_p^\nu\right) f(t,\vec{x},\vec{p}) \;=\; \mathcal{C}[f]$$

→ collision term

Conserved current:

$$j^\mu(x) \;\equiv\; \int \frac{d^3p}{2E_p} p^\mu f(t,\vec{x},\vec{p})$$

Energy-momentum tensor:

$$T^{\mu\nu}(x) \equiv \int \frac{d^3p}{2E_p} p^\mu p^\nu f(t,\vec{x},\vec{p})$$

Classical Transport Theory:

**One distribution function, one equation!** **Simple and Intuitive!**

# Quantum Transport Theory: Wigner Functions

Quantum transport theory for spin-1/2 fermion with Abelian gauge field

Wigner operator: $$\hat{W}_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \bar{\psi}_\beta(x_2)\, U(x_2,x_1)\, \psi_\alpha(x_1)$$

$$x \equiv \frac{x_2+x_1}{2}$$

$$y \equiv x_2 - x_1$$

Gauge link: $$U(x_2,x_1) \equiv \mathcal{P}\mathrm{Exp}\left(-ie\int_{x_1}^{x_2} dx^\mu A_\mu(x)\right)$$

Straight line:


Particle density at $x$ with $p$: $$\hat{W}(x,p) = \bar{\psi}(x)\delta^4\left(p - \hat{\pi}(x)\right)\psi(x)$$

Kinetic Momentum: $$\hat{\pi}_\mu = \hat{p}_\mu - eA_\mu(x)$$

Wigner functions: $$W_{\alpha\beta}(x,p) = <: \hat{W}(x,p)_{\alpha\beta} :>$$

$$\begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}$$

16 independent components due to: $$\hat{W}^\dagger(x,p) \;=\; \gamma^0 \hat{W}(x,p)\gamma^0$$

D.Vasak, M.Gyulassy, H. Elze Annals Phys. 173 (1987)

# Covariant Wigner Equations

Wigner Equations:

D.Vasak, M.Gyulassy, H. Elze Annals Phys. 173 (1987)

$$\left[m-\gamma\cdot\left(p_\mu+\frac{1}{2}i\partial_\mu\right)\right]\hat{W}(x,p)$$
$$= ie\frac{\partial}{\partial p_\mu}\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\bar{\psi}(x_1)\otimes\mathcal{P}\left[\int_0^1 ds\,(1-s)\,F_{\mu\nu}\left(x-\frac{1}{2}y+sy\right)U(A;x_1,x_2)\right]\gamma^\nu\psi(x_2)$$

Background EM Fields

$$(\gamma\cdot K-m)\,W(x,p)=0$$

$$K^\mu\equiv\Pi^\mu+\frac{1}{2}iG^\mu \qquad \Pi^\mu \equiv p^\mu-\frac{1}{2}j_1\left(\frac{1}{2}\triangle\right)F^{\mu\nu}\partial^p_\nu \qquad G^\mu \equiv \partial^\mu_x-j_0\left(\frac{1}{2}\triangle\right)F^{\mu\nu}\partial^p_\nu$$

$j_0, j_1$ :Spherical Bessel Functions $\triangle\equiv\partial^p\cdot\partial_x$ $\partial_x$ only act on EM fields

**16 independent functions and 32 coupled equations:**

**More complicated, abstract and highly constrained!**

# Decoupled Equations under Chiral Limit

Wigner functions spinor decomposition:

$$W=\frac{1}{4}\left[\mathscr{F}+i\gamma^5\mathscr{P}+\gamma^\mu\mathscr{V}_\mu+\gamma^5\gamma^\mu\mathscr{A}_\mu+\frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

$$\begin{aligned}
\Pi^\mu\mathscr{V}_\mu=m\mathscr{F},\quad G^\mu\mathscr{A}_\mu &= -2m\mathscr{P},\\
\Pi_\mu\mathscr{F}+\frac{1}{2}G^\nu\mathscr{S}_{\mu\nu} &= m\mathscr{V}_\mu,\\
G_\mu\mathscr{P}-\epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathscr{S}^{\rho\sigma} &= 2m\mathscr{A}_\mu,\\
\frac{1}{2}(G_\mu\mathscr{V}_\nu-G_\nu\mathscr{V}_\mu)-\epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathscr{A}^\sigma &= m\mathscr{S}_{\mu\nu}.
\end{aligned}$$

$$\begin{aligned}
G^\mu\mathscr{V}_\mu=0,\quad \Pi^\mu\mathscr{A}_\mu &= 0,\\
\frac{1}{2}G_\mu\mathscr{F}-\Pi^\nu\mathscr{S}_{\mu\nu} &= 0,\\
\Pi_\mu\mathscr{P}+\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}G^\nu\mathscr{S}^{\rho\sigma} &= 0,\\
(\Pi_\mu\mathscr{V}_\nu-\Pi_\nu\mathscr{V}_\mu)+\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^\rho\mathscr{A}^\sigma &= 0.
\end{aligned}$$

Chiral limit $m$=0

$$\begin{aligned}
\Pi^\mu\mathscr{V}_\mu &= 0,\quad \Pi^\mu\mathscr{A}_\mu=0,\\
G^\mu\mathscr{V}_\mu &= 0,\quad G^\mu\mathscr{A}_\mu=0,\\
\epsilon_{\mu\nu\rho\sigma}G^\rho\mathscr{A}^\sigma &= -2\left(\Pi_\mu\mathscr{V}_\nu-\Pi_\nu\mathscr{V}_\mu\right),\\
\epsilon_{\mu\nu\rho\sigma}G^\rho\mathscr{V}^\sigma &= -2\left(\Pi_\mu\mathscr{A}_\nu-\Pi_\nu\mathscr{A}_\mu\right).
\end{aligned}$$

8 functions & 16 equations

$$\begin{aligned}
\Pi_\mu\mathscr{F}+\frac{1}{2}G^\nu\mathscr{S}_{\mu\nu} &= 0,\\
\frac{1}{2}G_\mu\mathscr{F}-\Pi^\nu\mathscr{S}_{\mu\nu} &= 0,\\
-G_\mu\mathscr{P}+\epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathscr{S}^{\rho\sigma} &= 0,\\
\Pi_\mu\mathscr{P}+\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}G^\nu\mathscr{S}^{\rho\sigma} &= 0.
\end{aligned}$$

8 functions & 16 equations

# Decoupled Equations under Chiral Limit

Our interest:

$$j^\mu = \int d^4p \mathscr{V}^\mu, \quad j_5^\mu = \int d^4p \mathscr{A}^\mu, \quad T^{\mu\nu} = \int d^4p p^\mu \mathscr{V}^\nu$$

We focus on:

$$\begin{aligned} \Pi^\mu \mathscr{V}_\mu &= 0, \quad \Pi^\mu \mathscr{A}_\mu = 0, \\ G^\mu \mathscr{V}_\mu &= 0, \quad G^\mu \mathscr{A}_\mu = 0, \\ \epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{A}^\sigma &= -2\left(\Pi_\mu \mathscr{V}_\nu - \Pi_\nu \mathscr{V}_\mu\right), \\ \epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{V}^\sigma &= -2\left(\Pi_\mu \mathscr{A}_\nu - \Pi_\nu \mathscr{A}_\mu\right). \end{aligned}$$

Right/Left ($s = \pm 1$) $\quad \mathscr{J}_s^\mu \equiv \frac{1}{2}(\mathscr{V}_\mu + s\mathscr{A}_\mu)$

$$\Pi_\mu \mathscr{J}_s^\mu = 0, \quad G_\mu \mathscr{J}_s^\mu = 0, \quad \epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{J}_s^\sigma = -2s\left(\Pi_\mu \mathscr{J}_{s\nu} - \Pi_\nu \mathscr{J}_{s\mu}\right)$$

**4 independent Wigner functions and 8 coupled equations**

S.Ochs, U. Heinz Annals Phys. 266 (1998), JHG, Q. Wang Phys.Lett. B749 (2015)

# $\hbar$ Expansion in Wigner Equations

4D Covariant Wigner equations in background EM field:

$$\begin{aligned} \Pi_\mu \mathscr{J}^\mu &= 0, \quad G_\mu \mathscr{J}^\mu = 0, \\ \hbar \epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{J}^\sigma &= -2s\left(\Pi_\mu \mathscr{J}_\nu - \Pi_\nu \mathscr{J}_\mu\right) \end{aligned}$$

Wigner functions, $G^\mu, \Pi^\mu$ operators expand in $\hbar$ :

$$\mathscr{J}^\mu = \sum_{k=0}^{\infty} \hbar^k \mathscr{J}^\mu_{(k)} \quad G^\mu = \sum_{k=0}^{\infty} \hbar^k G^\mu_{(k)} \quad \Pi^\mu = \sum_{k=0}^{\infty} \hbar^k \Pi^\mu_{(k)}$$

*k*=0:

$$\begin{aligned} G^\mu_{(0)} &\equiv \partial_x^\mu - F^{\mu\nu}\partial^p_\nu \\ \Pi^\mu_{(0)} &\equiv p^\mu \end{aligned}$$

*k*>0:

$$\begin{aligned} G^\mu_{(k)} &\equiv \frac{\left[1+(-1)^k\right](-1)^{\frac{k+2}{2}}}{2^{k+1}(k+1)!}\left(\partial_x \cdot \partial_p\right)^k F^{\mu\nu}\partial^p_\nu \\ \Pi^\mu_{(k)} &\equiv \frac{\left[1+(-1)^k\right]k(-1)^{\frac{k}{2}}}{2^{k+1}(k+1)!}\left(\partial_x \cdot \partial_p\right)^{k-1} F^{\mu\nu}\partial^p_\nu, \end{aligned}$$

For odd number *k*, $G^\mu_{(k)}, \Pi^\mu_{(k)}$ vanish.

JHG,Z.T. Liang, Q. Wang, X.N. Wang, arXiv:1802.06216

# 3D vector forms for Wigner equations

4D covariant Wigner equation at $O(\hbar^n)$: JHG,Z.T. Liang, Q. Wang, X.N. Wang, arXiv:1802.06216

$$
\begin{aligned}
\sum_{k=0}^{n} \Pi_{\mu}^{(k)} \mathscr{J}_{(n-k)}^{\mu} &= 0, \quad \sum_{k=0}^{n} G_{\mu}^{(k)} \mathscr{J}_{(n-k)}^{\mu} = 0, \\
\epsilon_{\mu\nu\rho\sigma} \sum_{k=0}^{n} G_{(k)}^{\rho} \mathscr{J}_{(n-k)}^{\sigma} &= -2s \sum_{k=0}^{n} \left( \Pi_{\mu}^{(k)} \mathscr{J}_{\nu}^{(n+1-k)} - \Pi_{\nu}^{(k)} \mathscr{J}_{\mu}^{(n+1-k)} \right)
\end{aligned}
$$

3D vector Wigner equation at $O(\hbar^n)$: $\mathscr{J}^{\mu} = \left(\mathscr{J}^0, \vec{\mathscr{J}}\right) \quad G^{\mu} = \left(G^0, -\vec{G}\right), \quad \Pi^{\mu} = \left(\Pi^0, \vec{\Pi}\right)$

Evolution equations:

$$
\begin{aligned}
&\sum_{k=0}^{n} \left[ G_0^{(k)} \mathscr{J}_0^{(n-k)} + \vec{G}^{(k)} \cdot \vec{\mathscr{J}}^{(n-k)} \right] = 0, \qquad G_0^{(0)} = \partial_t + \vec{E} \cdot \vec{\nabla}_p \\
&\sum_{k=0}^{n} \left[ G_0^{(k)} \vec{\mathscr{J}}^{(n-k)} + \vec{G}^{(k)} \mathscr{J}_0^{(n-k)} \right] = 2s \sum_{k=0}^{n+1} \vec{\Pi}_{(k)} \times \vec{\mathscr{J}}^{(n-k+1)}
\end{aligned}
$$

Constraint equations:

$$
\begin{aligned}
\sum_{k=0}^{n} \left[ \Pi_0^{(k)} \mathscr{J}_0^{(n-k)} - \vec{\Pi}^{(k)} \cdot \vec{\mathscr{J}}^{(n-k)} \right] &= 0, \\
\sum_{k=0}^{n} \vec{G}^{(k)} \times \vec{\mathscr{J}}^{(n-k)} &= -2s \sum_{k=0}^{n+1} \left[ \vec{\Pi}_{(k)} \mathscr{J}_0^{(n-k+1)} - \Pi_{(k)}^0 \vec{\mathscr{J}}^{(n-k+1)} \right]
\end{aligned}
$$

# Wigner Equations at $O(\hbar^0)$

3D vector forms at $O(\hbar^0)$ $\quad G_0^{(0)} = \partial_t + \vec{E}\cdot\vec{\nabla}_p \qquad \vec{G}^{(0)} = \vec{\nabla}_x + \vec{E}\partial_{p_0} + \vec{B}\times\vec{\nabla}_p$

E0 Evolution equation $\mathscr{J}_0^{(0)}$ :

$$G_0^{(0)}\mathscr{J}_0^{(0)} + \vec{G}^{(0)}\cdot\vec{\mathscr{J}}^{(0)} = 0$$

Ei Evolution equation $\vec{\mathscr{J}}^{(0)}$ :

$$G_0^{(0)}\vec{\mathscr{J}}^{(0)} + \vec{G}^{(0)}\mathscr{J}_0^{(0)} = -2s\vec{p}\times\vec{\mathscr{J}}^{(1)}$$

C1 Constraint equation:

$$p_0\mathscr{J}_0^{(0)} - \vec{p}\cdot\vec{\mathscr{J}}^{(0)} = 0,$$

C2 Constraint equation:

$$0 = -2s\left[\vec{p}\mathscr{J}_0^{(0)} - p^0\vec{\mathscr{J}}^{(0)}\right] \Rightarrow \vec{\mathscr{J}}^{(0)} = \frac{\vec{p}}{p_0}\mathscr{J}_0^{(0)}$$

$$\vec{G}^{(0)}\times\vec{\mathscr{J}}^{(0)} = -2s\left[\vec{p}\mathscr{J}_0^{(1)} - p^0\vec{\mathscr{J}}^{(1)}\right] \Rightarrow \vec{\mathscr{J}}^{(1)} = \frac{\vec{p}}{p_0}\mathscr{J}_0^{(1)} + \frac{s}{2}\vec{G}^{(0)}\times\vec{\mathscr{J}}^{(0)}$$

Remarkable property: E0 + C1 + C2 $\Longrightarrow$ Ei

**Only $\mathscr{J}_0^{(0)}$ is independent!** **Only E0 & C1 are necessary!**

# Wigner Equations at $O(\hbar^n)$

With the help of mathematical induction, we can prove that only the time component $\mathscr{J}_0$ of Wigner functions $\mathscr{J}^\mu$ is independent up to any $O(\hbar^n)$, Space components $\vec{\mathscr{J}}$ can be given by the iterative relation:

$$\vec{\mathscr{J}}^{(n+1)} = \frac{\vec{p}}{p_0}\mathscr{J}_0^{(n+1)} + \frac{s}{2p_0}\sum_{k=0}^{n}\vec{G}^{(k)}\times\vec{\mathscr{J}}^{(n-k)} + \frac{1}{p_0}\sum_{k=1}^{n+1}\left[\vec{\Pi}_{(k)}\mathscr{J}_0^{(n-k+1)} - \Pi^0_{(k)}\vec{\mathscr{J}}^{(n-k+1)}\right]$$

$$\vec{\mathscr{J}}^{(0)} = \frac{\vec{p}}{p_0}\mathscr{J}_0^{(0)} \quad\Longrightarrow\quad \vec{\mathscr{J}}^{(n+1)} = \frac{\vec{p}}{p_0}\mathscr{J}_0^{(n+1)} + \vec{C}\left[\mathscr{J}_0^{(n)}, \mathscr{J}_0^{(n-1)}, \cdots, \mathscr{J}_0^{(0)}\right]$$

Evolution equation for $\mathscr{J}_0^{(n)}$ :

$$\sum_{k=0}^{n}\left[G_0^{(k)}\mathscr{J}_0^{(n-k)} + \vec{G}^{(k)}\cdot\vec{\mathscr{J}}^{(n-k)}\right] = 0$$

Constraint equation for $\mathscr{J}_0^{(n)}$ :

$$\sum_{k=0}^{n}\left[\Pi_0^{(k)}\mathscr{J}_0^{(n-k)} - \vec{\Pi}^{(k)}\cdot\vec{\mathscr{J}}^{(n-k)}\right] = 0$$

**At any $O(\hbar^n)$, one distribution function and two equations.**

# Peel the Last Constraint Equation

From the constraint equation

$$\sum_{k=0}^{n}\left[\Pi_0^{(k)}\mathscr{J}_0^{(n-k)}-\vec{\Pi}^{(k)}\cdot\vec{\mathscr{J}}^{(n-k)}\right] = 0$$

$O(\hbar^0)$ :

$$\mathscr{J}_0^{(0)} = p_0 f_p^{(0)}\delta\left(p^2\right),$$

$O(\hbar^1)$ :

$$\mathscr{J}_0^{(1)} = p_0 f_p^{(1)}\delta\left(p^2\right)+s\left(\vec{B}\cdot\vec{p}\right)f_p^{(0)}\delta'\left(p^2\right)$$

$O(\hbar^2)$ :

$$\begin{aligned}\mathscr{J}_0^{(2)} = {} & p_0 f_p^{(2)}\delta\left(p^2\right)+s\left(\vec{B}\cdot\vec{p}\right)f_p^{(1)}\delta'\left(p^2\right)+\frac{1}{2p_0}\left(\vec{B}\cdot\vec{p}\right)^2 f_p^{(0)}\delta''\left(p^2\right)\\ & +\frac{1}{p^2}\vec{p}\cdot\left(\vec{\Pi}^{(2)}\mathscr{J}_0^{(0)}-\Pi_0^{(2)}\vec{\mathscr{J}}^{(0)}\right)-\frac{1}{p^2}p_0\left(\Pi_0^{(2)}\mathscr{J}_0^{(0)}-\vec{\Pi}_0^{(2)}\cdot\vec{\mathscr{J}}^{(0)}\right)\\ & +\frac{1}{4p^2}\vec{p}\cdot\left[\vec{G}^{(0)}\times\left(\frac{1}{p_0}\vec{G}^{(0)}\times\frac{\vec{p}\mathscr{J}_0^{(0)}}{p_0}\right)\right]\end{aligned}$$

Now $f_p^{(0)}, f_p^{(1)}, f_p^{(2)}, \cdots, f_p^{(n)}, \cdots$ are free functions.

$$f_p = f_p^{(0)}+\hbar f_p^{(1)}+\hbar^2 f_p^{(2)}+\cdots$$

**The analog of distribution functions in classical transport theory**

# Chiral kinetic equations in 4D up to $O(\hbar^2)$

Boltzmann-like equation in 4D ($p_0, \vec{p}$)

$O(\hbar^0)$ :

$$p^\mu G_\mu^{(0)}\left[f_p(x,p)\delta\left(p^2\right)\right] = 0 \qquad G_\mu^{(0)} \equiv \partial_\mu^x - F_{\mu\nu}\partial_p^\nu$$

$O(\hbar^0) + O(\hbar^1)$ :

$$\left[p^\mu G_\mu^{(0)} + \frac{s\hbar}{2}\vec{G}^{(0)}\cdot\left(\frac{1}{p_0}\vec{G}^{(0)}\times\vec{p}\right)\right]\left[f_p(x,p)\delta\left(p^2+\frac{s\hbar\vec{B}\cdot\vec{p}}{p_0}\right)\right] = 0$$

$O(\hbar^0) + O(\hbar^1) + O(\hbar^2)$ :

$$p_0 = |\vec{p}|\left(1-\hbar\vec{B}\cdot\vec{\Omega}_p\right), \qquad \vec{\Omega}_p \equiv \frac{s\hat{\vec{p}}}{2\vec{p}^2}$$

$$\left[p^\mu G_\mu^{(0)} + \frac{s\hbar}{2}\vec{G}^{(0)}\cdot\left(\frac{1}{p_0}\vec{G}^{(0)}\times\vec{p}\right)\right]\left[f_p(x,p)\delta\left(p^2+\frac{s\hbar\vec{B}\cdot\vec{p}}{p_0}\right)\right] + \hbar^2 C(f) = 0$$

$$C(f) = \frac{1}{4}\left(p^\mu G_\mu^{(0)}\frac{\mathbf{p}}{p^2} + \mathbf{G}^{(0)}\right)\cdot\left[\frac{1}{p_0}\mathbf{G}^{(0)}\times\left(\frac{1}{p_0}\mathbf{G}^{(0)}\times\mathbf{p}f\delta\left(p^2\right)\right) + \frac{4}{p_0}\left(\mathbf{\Pi}^{(2)}p_0 - \Pi_0^{(2)}\mathbf{p}\right)f\delta\left(p^2\right)\right]$$
$$-p^\mu G_\mu^{(0)}\left[\frac{1}{p^2}\Pi_\nu^{(2)}p^\nu f\delta\left(p^2\right)\right] + G_\mu^{(2)}p^\mu f\delta\left(p^2\right)$$

# From 4D to 3D by integrating over $p_0$

**Boltzmann-like equation in 3D can be obtained by integrating over $p_0$ !**

Particle: integrate over $p_0$ from $0$ to $+\infty$

Antiparticle: integrate over $p_0$ from $-\infty$ to $0$

$$\int_0^{+\infty} dp_0 \left\{ \left[p_0\left(\partial_t + \vec{E}\cdot\vec{\nabla}_p\right) + \vec{p}\cdot\left(\vec{\nabla}_x + \vec{E}\partial_{p_0} + \vec{B}\times\vec{\nabla}_p\right)\right]\left[f_p\delta\left(p^2 + \frac{s\hbar\vec{B}\cdot\vec{p}}{p_0}\right)\right] \right.$$
$$-\frac{s\hbar}{2p_0^2}\left[\vec{E}\times\left(\vec{\nabla}_x + \vec{B}\times\vec{\nabla}_p\right)\right]\cdot\left[\vec{p}f_p^{(0)}\delta\left(p^2 + \frac{s\hbar\vec{B}\cdot\vec{p}}{p_0}\right)\right]$$
$$\left. +\frac{s\hbar}{2p_0}\left\{\left(\vec{\nabla}_x\times\vec{E}\right)\partial_{p_0} + \left[\left(\vec{\nabla}_p\cdot\vec{\nabla}_x\right)\vec{B}\right]\right\}\cdot\left[\vec{p}f_p^{(0)}\delta\left(p^2 + \frac{s\hbar\vec{B}\cdot\vec{p}}{p_0}\right)\right] \right\} = 0$$

**It should be noted that we can not drop all the derivative terms $\partial_{p_0}$ when integrating over $p_0$ because some singular terms arise as $p_0 \to 0$.**

**We drop the terms which will vanish as $p_0 \to 0$ together with momentum measure $d^3\vec{p}$ and keep the ones that will not vanish.**

# Wigner Equations in 3D vector forms

Chiral kinetic equation in 3D forms up to $O(\hbar^1)$ $\vec{v} = \left(1+2\hbar\vec{B}\cdot\vec{\Omega}_p\right)\hat{\vec{p}} - \hbar\left(\hat{\vec{p}}\cdot\vec{\Omega}_p\right)\vec{B}$

Large $|\vec{p}|$:

$$\left(1+\hbar\vec{B}\cdot\vec{\Omega}_p\right)\partial_t f_p + \left[\vec{v}+\hbar\vec{E}\times\vec{\Omega}_p+\hbar\left(\hat{\vec{p}}\cdot\vec{\Omega}_p\right)\vec{B}\right]\cdot\vec{\nabla}_x f_p + \left[\vec{E}+\vec{v}\times\vec{B}+\hbar|\vec{p}|\vec{\nabla}_x\left(\vec{B}\cdot\vec{\Omega}_p\right)+\hbar\left(\vec{E}\cdot\vec{B}\right)\vec{\Omega}_p\right]\cdot\vec{\nabla}_p f_p = 0$$

Small $|\vec{p}|$: New terms appear from infrared region of momentum

$$\hbar\left(\vec{E}\cdot\vec{B}\right)\left(\vec{\nabla}_p\cdot\vec{\Omega}_p\right)f_p - \frac{2s\hbar}{\Lambda}\left(\vec{E}\cdot\vec{p}\right)\left(\vec{B}\cdot\vec{p}\right)\delta'\left(\Lambda^2-|\vec{p}|^2\right)f_p \rightarrow 2\pi s\hbar\left(\vec{E}\cdot\vec{B}\right)\delta^3(\vec{p})f_p - \frac{3s\hbar}{4}\left(\vec{E}\cdot\hat{\vec{p}}\right)\left(\vec{B}\cdot\hat{\vec{p}}\right)\delta''(|\vec{p}|)f_p$$

Where does the chiral anomaly come from ?

$$\partial_\mu j_s^\mu = \frac{\hbar}{2}\int d^3\vec{p}\left[\left(\vec{E}\cdot\vec{B}\right)\vec{\Omega}_p\cdot\vec{\nabla}_p f_p + \left(\vec{E}\cdot\vec{B}\right)\left(\vec{\nabla}_p\cdot\vec{\Omega}_p\right)f_p - \frac{2s}{\Lambda}\left(\vec{E}\cdot\vec{p}\right)\left(\vec{B}\cdot\vec{p}\right)\delta'\left(\Lambda^2-|\vec{p}|^2\right)f_p\right]$$

previous term: (1) New term: (2) New term: (3)

(2)+(3)=0 → (1) contributes.

(1)+(2)~$\int d^3\vec{p}\,\vec{\nabla}\cdot\left(\vec{\Omega}_p f_p\right)$=0 → (3) contributes.

# Rewrite all formula in a general Lorentz frame

Introduce a time-like 4-vector $u^\mu$ with normalization $u^2 = 1$

Decompose any vector as:

$$X^\mu = (X \cdot u)u^\mu + \bar{X}^\mu$$

Decompose EM tensor as:

$$F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} u_\rho B_\sigma$$

Wigner equations at the zeroth order

$$\begin{aligned} u \cdot \nabla \left(u \cdot \mathscr{J}^{(0)}\right) + \bar{\nabla} \cdot \bar{\mathscr{J}}^{(0)} &= 0, \quad (u \cdot p)(u \cdot \mathscr{J}^{(0)}) + \bar{p} \cdot \bar{\mathscr{J}}^{(0)} = 0, \\ \bar{p}_\mu (u \cdot \mathscr{J}^{(0)}) - (u \cdot p) \bar{\mathscr{J}}_\mu^{(0)} &= 0, \quad \bar{p}_\mu \bar{\mathscr{J}}_\nu^{(0)} - \bar{p}_\nu \bar{\mathscr{J}}_\mu^{(0)} = 0, \end{aligned}$$

Wigner equations at the first order

$$\begin{aligned} u \cdot \nabla \left(u \cdot \mathscr{J}^{(1)}\right) + \bar{\nabla} \cdot \bar{\mathscr{J}}^{(1)} &= 0, \quad (u \cdot p)(u \cdot \mathscr{J}^{(1)}) + \bar{p} \cdot \bar{\mathscr{J}}^{(1)} = 0, \\ 2s \left[\bar{p}_\mu (u \cdot \mathscr{J}^{(1)}) - (u \cdot p) \bar{\mathscr{J}}^{(1)\mu}\right] &= -\epsilon_{\mu\nu\rho\sigma} u^\nu \bar{\nabla}^\rho \bar{\mathscr{J}}^{(0)\sigma}, \\ 2s \left(\bar{p}_\mu \bar{\mathscr{J}}_\nu^{(1)} - \bar{p}_\nu \bar{\mathscr{J}}_\mu^{(1)}\right) &= -\epsilon_{\mu\nu\rho\sigma} u^\rho \left[(u \cdot \nabla) \bar{\mathscr{J}}^{(0)\sigma} - \bar{\nabla}^\sigma (u \cdot \mathscr{J}^{(0)})\right] \end{aligned}$$

# Rewrite all formula in a general Lorentz frame

Express $\bar{\mathscr{J}}_\mu$ as $u\cdot\mathscr{J}^{(0)}$

$$\bar{\mathscr{J}}_\mu^{(0)} = \bar{p}_\mu \frac{u\cdot\mathscr{J}^{(0)}}{u\cdot p}, \qquad \bar{\mathscr{J}}_\mu^{(1)} = \bar{p}_\mu \frac{u\cdot\mathscr{J}^{(1)}}{u\cdot p} - \frac{s}{2(u\cdot p)}\epsilon^{\mu\nu\rho\sigma}u_\nu\bar{\nabla}_\sigma\left(\bar{p}_\rho\frac{u\cdot\mathscr{J}^{(0)}}{u\cdot p}\right)$$

On-shell condition yields

$$\frac{u\cdot\mathscr{J}^{(0)}}{u\cdot p} = f^{(0)}\delta\left(p^2\right), \qquad \frac{u\cdot\mathscr{J}^{(1)}}{u\cdot p} = f^{(1)}\delta\left(p^2\right) - sQ\frac{B\cdot p}{u\cdot p}f^{(0)}\delta'\left(p^2\right)$$

Wigner functions in covariant form

$$\mathscr{J}_\mu = \delta\left(p^2 - \hbar sQ\frac{B\cdot p}{u\cdot p}\right)\left[p_\mu f - \hbar sQ\frac{B_\mu}{u\cdot p}f - \hbar s\frac{1}{2(u\cdot p)}\epsilon_{\mu\nu\rho\sigma}u^\nu p^\rho\nabla^\sigma f\right]$$

Chiral kinetic equations in covariant form

$$\nabla_\mu\left[\delta\left(p^2 - \hbar sQ\frac{B\cdot p}{u\cdot p}\right)\left(p^\mu - \hbar sQ\frac{B^\mu}{u\cdot p} - \hbar s\frac{1}{2(u\cdot p)}\epsilon^{\mu\nu\rho\sigma}u_\nu p_\rho\nabla_\sigma\right)f\right] = 0$$

# Summary

- **Chiral systems in a background EM field can be described sufficiently by one distribution function and one Boltzmann-like equation up to any order of $\hbar$ in 4D.**

- **This property will significantly simplify the description and simulation of chiral effects in HIC and Dirac/Weyl semimetals**

- **We present unintegrated chiral kinetic equations in 4D forms up to $O(\hbar^2)$ and integrated form in 3D up to $O(\hbar)$**

- **We find other possible contributions to the chiral anomaly in very infrared momentum region, in contrast to the well-known Berry curvature term.**

- **Our formalism can be written in covariant form very naturally.**

**Thanks for your attention！**