Chiral kinetic theory in general spacetime

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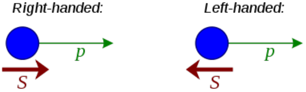
Table of contents

Chiral magnetic effect and chiral vortical effect

- Chirality is the same as helicity for massless particles: the sign of the projection of the spin vector onto the momentum vector (picture from wikipedia)



- Chiral magnetic effect(CME)(Kharzeev, Mclerran, Warringa, Fukushima 2008; Son, Zhitnitsky 2004)

$$\begin{aligned} \mathbf{J} &= \sigma \mathbf{B}, \quad \sigma = \frac{1}{2\pi^2}\mu_5, \\ \mathbf{J}_5 &= \sigma_5 \mathbf{B}, \quad \sigma_5 = \frac{1}{2\pi^2}\mu, \end{aligned} \tag{1}$$

with **B** the magnetic field.

- Chiral vortical effect(CVE)(Erdmenger etal 2008; Barnerjee etal 2008, Son, Surowka 2009; Landsteiner etal 2011)

$$\begin{aligned} \mathbf{J} &= \xi \overrightarrow{\omega}, \quad \xi = \frac{1}{\pi^2}\mu\mu_5, \\ \mathbf{J}_5 &= \xi_5 \overrightarrow{\omega}, \quad \xi_5 = \frac{(\mu^2+\mu_5^2)}{2\pi^2} + \frac{T^2}{6}, \end{aligned} \tag{2}$$

with $\overrightarrow{\omega}$ the fluid vorticity.

Rotating fluid vs. rotating frame

- Rotating fluid vs. rotating frame, CVE and Coriolis force

In magnetic field, Lorentz force: $\boldsymbol{F} = e(\dot{\boldsymbol{x}} \times \boldsymbol{B})$

In rotating frame, Coriolis force: $\boldsymbol{F} = 2\boldsymbol{\varepsilon}(\dot{\boldsymbol{x}} \times \boldsymbol{\omega}) + \boldsymbol{O}(\boldsymbol{\omega}^2)$

- It suggests the following replacement $\mathbf{B} \Longleftrightarrow 2\epsilon\vec{\omega}$(Stephanov, Yin 2012)

$$\mathbf{J}_{CME} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\mathbf{B}(\mathbf{b}\cdot\hat{\mathbf{p}})f \quad \Longrightarrow \quad \mathbf{J}_{CVE} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}2\epsilon\vec{\omega}(\mathbf{b}\cdot\hat{\mathbf{p}})f \qquad (3)$$

where $\mathbf{b} = \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}$ is the Berry curvature. This (+ antiparticle) indeed gives expected result.

- A kinetic theory in general spacetime would be helpful to study rotating frame and CVE

Wigner operator and Horizontal lift

- Wigner operator

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int \frac{\sqrt{-g}d^4y}{(2\pi)^4} e^{-ip\cdot y/\hbar} \left[\bar{\psi}(x) e^{1/2 y\cdot \overleftarrow{\nabla}}\right]_\beta \left[e^{-1/2 y\cdot\nabla}\psi(x)\right]_\alpha . \quad (4)$$

Where the derivative $\overleftarrow{\nabla}_\mu(\nabla_\mu)$ acting to the left(right).

- We emphasize that x in equation (4) is the coordinate of point(P) in curved spacetime, and y is vector in the tangent space of point P, and p is vector in cotangent space of P.
- Horizontal lifted covariant derivatives (Winter J. 1985; Calzetta E, Habib S, Hu B L. 1988; Fonarev O A. 1994)

$$\nabla_\mu \equiv D_\mu - \Gamma^\lambda_{\mu\nu} y^\nu \frac{\partial}{\partial y^\lambda} + \Gamma^\lambda_{\mu\nu} p_\lambda \frac{\partial}{\partial p_\nu} \underbrace{+\Gamma_\mu + \frac{i}{\hbar}A_\mu}_{\text{connection for spinor}} , \quad (5)$$

$$\overleftarrow{\nabla}_\mu \equiv \overleftarrow{D}_\mu - \frac{\overleftarrow{\partial}}{\partial y^\lambda}\Gamma^\lambda_{\mu\nu} y^\nu + \frac{\overleftarrow{\partial}}{\partial p_\nu}\Gamma^\lambda_{\mu\nu} p_\lambda \overbrace{-\Gamma_\mu - \frac{i}{\hbar}A_\mu}, \quad (6)$$

where D_μ is the usual covariant derivative operator, A_μ is gauge field, $\Gamma_\mu \equiv -\frac{i}{4}\omega^{ab}_\mu \sigma_{ab}$ is spin connection with $\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$ and ω^{ab}_μ the vierbein connection.

- Vierbein: $e^a = e^a_\mu \partial^\mu$.

Dynamics of Wigner function

- Wigner function: the ensemble average of Wigner operator

$$W(x,p) \equiv \langle : \hat{W}(x,p) : \rangle. \tag{7}$$

- Dirac equations

$$(i\gamma^{\mu}\nabla_{\mu} - m)\psi = 0 = \overline{\psi}(i\gamma^{\mu}\overleftarrow{\nabla}_{\mu} + m), \tag{8}$$

- Dynamic equation of Wigner function

$$\begin{aligned} \gamma^{\mu}(\frac{i}{2}\nabla_{\mu} + \frac{p_{\mu}}{\hbar})W(x,p) &= mW(x,p) - i\gamma^{\mu}\hbar\hat{H}_{\mu}\left(x, -\frac{1}{2}i\hbar\partial_p\right) \otimes W(x,p) \\ &\quad + \frac{i\hbar}{2}\gamma^{\mu}\left[\hat{G}_{\mu}\left(x, -\frac{1}{2}i\hbar\partial_p\right), W(x,p)\right]_{\otimes} \end{aligned} \tag{9}$$

  - This equation is exact.
  - Operators

$$\begin{aligned} \hat{H}_{\mu}(x,y) &= \sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{[y\cdot\nabla,\cdots,[y\cdot\nabla}\,,\nabla_{\mu}]\cdots], \\ \hat{G}_{\mu}(x,y) &= \sum_{n=1}^{\infty}\frac{1}{(n+1)!}\underbrace{[y\cdot\nabla,\cdots,[y\cdot\nabla}\,,\nabla_{\mu}]\cdots]. \end{aligned} \tag{10}$$

Clifford algebra decomposition and $\hbar$ expansion

- Decomposition of Winger function

$$W(x,y) = \frac{1}{4}[\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}], \tag{11}$$

- Up to $O(\hbar)$ order we can obtain

$$(\nabla_\mu - F_{\mu\nu}\partial_p^\nu)\mathcal{L}^\mu = 0, \qquad p^\mu\mathcal{L}_\mu = 0, \tag{12}$$

$$(\nabla_\mu - F_{\mu\nu}\partial_p^\nu)\mathcal{R}^\mu = 0, \qquad p^\mu\mathcal{R}_\mu = 0. \tag{13}$$

$$\hbar\epsilon_{\mu\nu\rho\sigma}(\nabla^\rho - F^{\rho\lambda}\partial_\lambda^p)\mathcal{L}^\sigma = 2(p_\mu\mathcal{L}_\nu - p_\nu\mathcal{L}_\mu), \tag{14}$$

$$\hbar\epsilon_{\mu\nu\rho\sigma}(\nabla^\rho - F^{\rho\lambda}\partial_\lambda^p)\mathcal{R}^\sigma = -2(p_\mu\mathcal{R}_\nu - p_\nu\mathcal{R}_\mu), \tag{15}$$

where $\mathcal{L}_\mu, \mathcal{R}_\mu = \frac{1}{2}(\mathcal{V}_\mu \pm \mathcal{A}_\mu)$.

Kinetic theory for chiral fermions

- Exact form of $\mathcal{L}_\mu$ up to $O(\hbar)$ order

$$\mathcal{L}_\mu = p_\mu f_l \delta(p^2) - \hbar \widetilde{F}_{\mu\sigma} p^\sigma f_l \delta'(p^2) - \hbar s_{\mu\nu} \triangle^\nu f_l \delta(p^2) \quad (16)$$

  - $\triangle_\mu \equiv \nabla_\mu - qF_{\mu\nu}\partial_p^\nu$.
  - $s_{\mu\nu} \equiv \frac{1}{2p_5}\epsilon_{\mu\nu\rho\sigma} p^\rho n^\sigma$ is the spin operator as (Chen J Y, Son D T, Stephanov M A. 2015) discussed and n^μ is a frame choosing vector.
- kinetic equation for f_l

$$0 = \delta(p^2 - \frac{\hbar}{p \cdot n} \widetilde{F}_{\nu\sigma} n^\nu p^\sigma)\Big(p_\mu \triangle^\mu f_l - \triangle^\mu(s_{\mu\nu}\triangle^\nu f_l)\Big). \quad (17)$$

- Current

$$J_l^\mu = \int d^4p \delta(p^2 + \frac{\hbar p^\sigma B_\sigma}{p \cdot n})\Big\{p^\mu f_l + \hbar s^{\mu\nu}(\frac{E_\nu}{p \cdot n} - \nabla_\nu) f_l\Big\} \quad (18)$$

where

$$E_\mu = n^\nu F_{\mu\nu},$$
$$B_\mu = n^\nu \widetilde{F}_{\mu\nu}. \quad (19)$$

Abelian gauge field in Minkowski space

- kinetic equation with $n^\mu = (1,0,0,0)$

$$
\begin{aligned}
0 &= \Bigg\{\left(1+\frac{\hbar(\mathbf{B}\cdot\mathbf{p})}{2|\mathbf{p}|^3}\right)\partial_t + \left(\mathbf{v}+\frac{\hbar\mathbf{B}}{2|\mathbf{p}|^2}+\frac{\hbar}{2|\mathbf{p}|^3}[(\mathbf{E}-\nabla\epsilon_p)\times\mathbf{p}]\right)\cdot\nabla \\
&\quad +(\mathbf{E}-\nabla\epsilon_p)\cdot\nabla_{\mathbf{p}} + \mathbf{v}\times\mathbf{B}\cdot\nabla_{\mathbf{p}} + \frac{\hbar}{2|\mathbf{p}|^3}[(\mathbf{E}-\nabla\epsilon_p)\cdot\mathbf{B}]\mathbf{p}\cdot\nabla_{\mathbf{p}}\Bigg\} f_l \qquad (20)
\end{aligned}
$$

where $\epsilon_p \equiv p_0 = |\mathbf{p}| - \frac{\hbar\mathbf{B}\cdot\mathbf{p}}{2|\mathbf{p}|^2}$ and $\mathbf{v} \equiv \frac{\partial\epsilon_p}{\partial\mathbf{p}}$, it is accord with the results in (Son D T, Yamamoto N. 2013; Hidaka Y, Pu S, Yang D L. 2017).

- current

$$
\begin{aligned}
J^0 &= \int d^3\mathbf{p}\left(1+\frac{\hbar(\mathbf{B}\cdot\mathbf{p})}{2|\mathbf{p}|^3}\right)f_l, \\
\mathbf{J} &= \int d^3\mathbf{p}\left(\mathbf{v}+\frac{\hbar\mathbf{B}}{2|\mathbf{p}|^2}+\frac{\hbar}{2|\mathbf{p}|^3}\mathbf{E}\times\mathbf{p}-\frac{\hbar}{2|\mathbf{p}|^3}\epsilon_{\mathbf{p}}\mathbf{p}\times\nabla\right)f_l \qquad (21)
\end{aligned}
$$

Rotating fluid in Minkowski space

- $\epsilon_p \equiv p_0 = |\mathbf{p}|$ and $\mathbf{v} \equiv \frac{\partial \epsilon_p}{\partial \mathbf{p}}$.
- kinetic equation with $n^\mu = u^\mu = (1, \mathbf{\Omega} \times \mathbf{x})$ the velocity vector of fluid

$$0 = \sqrt{G}\left[\partial_t + \left(\mathbf{v} + \frac{\hbar \mathbf{\Omega}}{2|\mathbf{p}|} - \frac{\hbar \mathbf{p} \cdot \mathbf{\Omega}}{2|\mathbf{p}|^3}\mathbf{p}\right) \cdot \nabla\right] f_l, \tag{22}$$

with $\mathbf{\Omega}$ the angular velocity of fluid and $\sqrt{G} \equiv \left(1 + \hbar \frac{\mathbf{p} \cdot \mathbf{\Omega}}{|\mathbf{p}|^2}\right)$.

- Single particle EOM

$$\sqrt{G}\dot{\mathbf{x}} = \frac{\partial \widetilde{\epsilon_\mathbf{p}}}{\partial \mathbf{p}} + \frac{\hbar \mathbf{\Omega}}{|\mathbf{p}|}, \qquad \sqrt{G}\dot{\mathbf{p}} = 0, \tag{23}$$

where $\widetilde{\epsilon_\mathbf{p}} \equiv \epsilon_p - \frac{\hbar}{2}\hat{\mathbf{p}} \cdot \mathbf{\Omega} = |\mathbf{p}| - \frac{\hbar}{2}\hat{\mathbf{p}} \cdot \mathbf{\Omega}$.

- Equilibrium distribution function

$$f_l^{eq} = f_l(\mu + |\mathbf{p}| - \mathbf{p} \cdot (\mathbf{\Omega} \times \mathbf{x}) - \frac{\hbar}{2|\mathbf{p}|}\mathbf{p} \cdot \mathbf{\Omega}) \tag{24}$$

- The CVE current (Chen J Y, Son D T, Stephanov M A, et al. 2014)

$$\mathbf{J} = \int d^3\mathbf{p}\left\{\mathbf{v} - \hbar \frac{\mathbf{p}}{2|\mathbf{p}|^2} \times \nabla\right\} f_l. \tag{25}$$

Rotating frame with angular velocity $\boldsymbol{\Omega}$

- Metric

$$g_{00} = 1 - (\boldsymbol{\Omega} \times \mathbf{x})^2, \quad g_{0i} = g_{i0} = -(\boldsymbol{\Omega} \times \mathbf{x})^i, \quad g_{ij} = -\delta_{ij}. \tag{26}$$

$$g^{00} = 1, \quad g^{0i} = g^{i0} = -(\boldsymbol{\Omega} \times \mathbf{x})^i, \quad g^{ij} = -\delta^{ij} + (\boldsymbol{\Omega} \times \mathbf{x})^i (\boldsymbol{\Omega} \times \mathbf{x})^j. \tag{27}$$

- Nonzero connections

$$\Gamma^i{}_{00} = [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})]^i, \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \epsilon^{ilj} \Omega^l. \tag{28}$$

- Riemann curvature is identically zero in rotating frame.

Setup

- Momentum

$$\mathbf{p} \equiv -(p_1, p_2, p_3) \tag{29}$$

- Energy

$$\epsilon_{\mathbf{p}} \equiv p_0 = |\mathbf{p}| - (\boldsymbol{\Omega} \times \mathbf{x}) \cdot \mathbf{p}, \qquad p^0 = g^{0\nu} p_\nu = g^{00} p_0 + g^{0i} p_i = |\mathbf{p}|. \tag{30}$$

- Effective velocity

$$\mathbf{v} \equiv \frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|} - (\boldsymbol{\Omega} \times \mathbf{x}) \tag{31}$$

Co-rotating observer and Inertial fluid

- Kinetic equation with $n^\mu = u^\mu = (1, -(\boldsymbol{\Omega} \times \mathbf{x}))^\mu$

$$0 = \left\{\partial_t + \mathbf{v} \cdot \nabla + \mathbf{p} \times \boldsymbol{\Omega} \frac{\partial}{\partial \mathbf{p}}\right\} f_l. \tag{32}$$

- EOM

$$\begin{aligned} \dot{\mathbf{x}} &= \hat{\mathbf{p}} - (\boldsymbol{\Omega} \times \mathbf{x}) = \frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= \mathbf{p} \times \boldsymbol{\Omega} = -\frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{x}} \end{aligned} \tag{33}$$

- Current

$$J^0 = \int d^3\mathbf{p} f_l \tag{34}$$

$$\mathbf{J} = \int d^3\mathbf{p} \left(\mathbf{v} f_l - \frac{\hbar \mathbf{p}}{2|\mathbf{p}|^2} \times \nabla f_l\right) \tag{35}$$

- Equilibrium distribution function

$$f_l = f_l(\mu + |\mathbf{p}|) \tag{36}$$

we find there is no CVE.

Co-rotating observer and Co-rotating fluid

- kinetic equation with $n^\mu = u^\mu = (1, 0, 0, 0)$

$$0 = \left\{\sqrt{G}\partial_t + \sqrt{G}\left[\mathbf{v} + \hbar\left(\frac{\mathbf{\Omega}}{2|\mathbf{p}|} - \frac{(\mathbf{p}\cdot\mathbf{\Omega})\mathbf{p}}{2|\mathbf{p}|^3}\right)\right]\cdot\nabla + \mathbf{p}\times\mathbf{\Omega}\cdot\frac{\partial}{\partial\mathbf{p}}\right\} f_l . \quad (37)$$

with $\sqrt{G} \equiv (1 + \hbar\frac{\mathbf{p}\cdot\mathbf{\Omega}}{|\mathbf{p}|^2})$.

- Single particle EOM

$$\begin{aligned} \sqrt{G}\dot{\mathbf{x}} &= \frac{\partial\widetilde{\epsilon_{\mathbf{p}}}}{\partial\mathbf{p}} + \frac{\hbar\mathbf{\Omega}}{|\mathbf{p}|}, \\ \sqrt{G}\dot{\mathbf{p}} &= \mathbf{p}\times\mathbf{\Omega}, \end{aligned} \quad (38)$$

with $\widetilde{\epsilon_{\mathbf{p}}} \equiv \epsilon_{\mathbf{p}} - \frac{\hbar}{2}\hat{\mathbf{p}}\cdot\mathbf{\Omega} = |\mathbf{p}| - \mathbf{p}\cdot(\mathbf{\Omega}\times\mathbf{x}) - \frac{\hbar}{2}\hat{\mathbf{p}}\cdot\mathbf{\Omega}$

- Current

$$\mathbf{J} = \int d^3\mathbf{p}\left(\mathbf{v}f_l - \frac{\hbar\mathbf{p}}{2|\mathbf{p}|^2}\times\nabla f_l\right) \quad (39)$$

- Equilibrium state

$$f_l^{eq} = f_l(\mu + |\mathbf{p}| - \mathbf{p}\cdot(\mathbf{\Omega}\times\mathbf{x}) - \frac{\hbar}{2}\hat{\mathbf{p}}\cdot\mathbf{\Omega}), \quad (40)$$

it induces CVE.

Co-rotating observer and Co-rotating fluid

One can find the Coriolis force

$$|\mathbf{p}|\ddot{\mathbf{x}} = 2\mathbf{p}\times\mathbf{\Omega}. \tag{41}$$

Redefining the three components momentum $\mathbf{k} = \mathbf{p} - |\mathbf{p}|\mathbf{\Omega}\times\mathbf{x}$

- Kinetic equation with $\sqrt{G} = \left(1 + \hbar\frac{\mathbf{k}\cdot\mathbf{\Omega}}{|\mathbf{k}|^2}\right)$.

$$0 = \left\{\sqrt{G}\partial_t + \sqrt{G}\left[\hat{\mathbf{k}} + \hbar\left(\frac{\mathbf{\Omega}}{2|\mathbf{k}|} - \frac{(\mathbf{k}\cdot\mathbf{\Omega})\mathbf{k}}{2|\mathbf{k}|^3}\right)\right]\cdot\nabla + 2\mathbf{k}\times\mathbf{\Omega}\cdot\frac{\partial}{\partial\mathbf{k}}\right\}f_l \tag{42}$$

- Single particle EOM

$$\begin{aligned}\sqrt{G}\dot{\mathbf{x}} &= \frac{\partial\widetilde{\epsilon}_k}{\partial\mathbf{k}} + \frac{\hbar\mathbf{\Omega}}{|\mathbf{k}|},\\ \sqrt{G}\dot{\mathbf{k}} &= 2\mathbf{k}\times\mathbf{\Omega},\end{aligned} \tag{43}$$

with $\widetilde{\epsilon}_{\mathbf{k}} = |\mathbf{k}| - \frac{\hbar\mathbf{\Omega}\cdot\mathbf{k}}{2|\mathbf{k}|}$

Compare with magnetic field case

$$\begin{aligned}\sqrt{G_{\mathbf{B}}}\dot{\mathbf{x}} &= \frac{\partial\epsilon_k}{\partial\mathbf{k}} + \frac{\hbar\mathbf{B}}{2|\mathbf{k}|^2}\\ \sqrt{G_{\mathbf{B}}}\dot{\mathbf{k}} &= \hat{\mathbf{k}}\times\mathbf{B}\end{aligned} \tag{44}$$

with $\sqrt{G_{\mathbf{B}}} = \left(1 + \frac{\hbar(\mathbf{B}\cdot\mathbf{k})}{2|\mathbf{k}|^3}\right)$ and $\epsilon_k \equiv |\mathbf{k}| - \frac{\hbar\mathbf{B}\cdot\mathbf{k}}{2|\mathbf{k}|^2}$

Correspondence: $\mathbf{B} \Longleftrightarrow 2|\mathbf{k}|\mathbf{\Omega}$

Summary and outlook

Summary

- We have derived covariant chiral kinetic theory up to $O(\hbar)$ order in general spacetime.
- We found that CVE is due to rotating of the fluid in inertial frame, and a rotating frame itself does not generate CVE.
- We have derived the correspondence between magnetic field and vorticity.

Outlook

- Quantum correction to the collision term.
- Massive fermions.
- Expanding fluid.
- Spin dynamics.

Thank you!