An Introduction to Resummation: LECTURE III

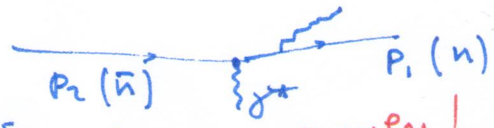
In this lecture we will introduce an alternative approach that can be applied in (most of) the cases in which we can find a <u>factorized</u> expression for the observable.

The main observation is that in the soft/collinear limits the QCD Lagrangian can contain contributions which are subdominant (power suppressed), and therefore can be integrated out. This will result in an effective theory of soft and collinear modes (SCET) that can be used to control the logarithmic terms at all orders in perturbation theory.

In the following I outline the main ideas which are needed for the application we are considering (i.e. the thrust distribution) and refer the reader to the given references for more details.

In the case of thrust, the emission's momenta essentially probe three types of regimes:

⊕ let us consider a single emission:



1) non-soft/non-collinear region: $\tau \sim \frac{k_t^2}{Q^2 z^{(\ell_1)}(1-z^{(\ell_1)})} \sim 1 \to \begin{cases} k_t \sim Q \\ z^{(\ell_1)} \sim 1 \end{cases}$ not singular!

2) soft region ($z^{(\ell_1)} \sim \boxed{k_t/Q}$): $\tau \sim \frac{k_t^2}{Q^2 z^{(\ell_1)}} \ll 1 \to \begin{cases} z^{(\ell_1)} \sim k_t/Q \\ k_t \sim \tau Q \ll Q \end{cases}$

lower kinematic bound

3) hard-collinear region ($z^{(\ell_1)}$ finite and ~ 1): $\tau \sim \frac{k_t^2}{Q^2 z^{(\ell_1)}(1-z^{(\ell_1)})} \ll 1 \to \begin{cases} z^{(\ell_1)} \sim 1 \\ k_t \sim \sqrt{\tau} Q \ll Q \end{cases}$

regions 2 and 3 probe the singularities of the QCD (perturbative) amplitudes and therefore are ~~an~~ dominant when $\tau \to 0$.

To perform the expansion at the Lagrangian level, we need to study how the fields scale in the various regions.

NB We then parametrize all momenta with respect to the thrust axis, as

$$k^\mu = \frac{k^+}{2}\bar{n}^\mu + \frac{k^-}{2} n^\mu + k_\perp^\mu = (k^+, k^-, k_\perp)$$

where $k^+ = \underline{k \cdot n}$, $k^- = k \cdot \bar{n}$, $k_\perp^2 = -k_T^2$, $n \cdot \bar{n} = 2$

→ We saw that in the collinear region: $k_T \sim \sqrt{\tau}\, Q \sim \underline{\lambda}\, Q$

↑ small parameter

- if the gluon is collinear to n, then $k^- \sim Q$
- imposing $k^2 = 0$ we find that

$$k^\mu \sim Q(\lambda^2, 1, \lambda) \qquad \text{along } n^\mu$$

- analogously, if k were collinear to $\bar{n}$ we'd have

$$k^\mu \sim Q(1, \lambda^2, \lambda)$$

→ In the soft region we have $k_T \sim k^+ \sim k^- \sim \tau Q \sim \lambda^2 Q$ therefore

$$k^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)$$

We can now deduce the scaling of the fields by looking at the propagators.

We thus define the corresponding soft and collinear fields by decomposing the gauge ($A^\mu(x)$) and fermion ($\psi(x)$) fields into a soft and a collinear component

$$A^\mu(x) \to A^\mu_c(x) + A^\mu_s(x)$$

$$\psi(x) \to \psi_c(x) + \psi_s(x)$$

each collinear field will have a small component (along $\bar{n}$) and a large component (along n)

$$\psi_c(x) \equiv \chi(x) + \eta(x)$$

defined as

$$\chi(x) = P_+ \psi_c(x) \quad , \quad P_+ = \frac{\not{n}\not{\bar{n}}}{4} \quad \text{(large component)}$$

$$\eta(x) = P_- \psi_c(x) \quad , \quad P_- = \frac{\not{\bar{n}}\not{n}}{4} \quad \text{(small component)}$$

where $P_\pm$ are projectors $\left(P_+ + P_- = 2\frac{\bar{n}\cdot n}{4} = 1\right)$ such that

$$P_- \chi(x) = P_+ \eta(x) = 0$$

We consider then the two-point correlators of the various components

⊕ $$\langle 0|T\{\chi(x)\bar{\chi}(0)\}|0\rangle = \frac{\not{n}\not{\bar{n}}}{4}\int \frac{d^4k}{(2\pi)^4}\frac{i\not{k}}{k^2+i0}e^{-ik\cdot x}\frac{\not{\bar{n}}\not{n}}{4} \sim \lambda^2$$

$$\left(\frac{\not{n}\not{\bar{n}}}{4}\not{k}\frac{\not{\bar{n}}\not{n}}{4} = \frac{\not{n}\not{\bar{n}}}{4}\left(\bar{n}\cdot k\frac{\not{n}}{2} + n\cdot k\frac{\not{\bar{n}}}{2} + \not{k}_\perp\right) = \bar{n}\cdot k\frac{\not{n}}{2} \sim \lambda^0\right).$$

we deduce that

$$\boxed{\chi(x) \sim \lambda}$$

⊕ $\langle 0|T\{\eta(x)\bar{\eta}(0)\}|0\rangle = \frac{\bar{\not n}\not n}{4}\int\frac{d^4k}{(2\pi)^4}\frac{i\not k}{k^2+i0}e^{-ik\cdot x}\frac{\not n\bar{\not n}}{4} \sim \lambda^4$

$\left(\frac{\bar{\not n}\not n}{4}\not k\frac{\not n\bar{\not n}}{4} = n\cdot k\frac{\bar{\not n}}{2} \sim \lambda^2\right)$

from which

$$\boxed{\eta(x) \sim \lambda^2}$$

⊕ In the soft limit we have

$$\langle 0|T\{\psi_s(x)\bar{\psi}(0)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i\not k}{k^2+i0}e^{-ik\cdot x} \sim \lambda^6$$

$$\boxed{\psi_s(x) \sim \lambda^3}$$

⊕ gluon fields

$$\langle 0|T\{A^\mu(x)A^\nu(0)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i0}\left[-g^{\mu\nu}+\xi\frac{k^\mu k^\nu}{k^2}\right]e^{-ik\cdot x}$$

which gives ($A_s^\mu \sim k_s^\mu$; $A_c^\mu \sim k_c^\mu$)

$$\boxed{\begin{aligned} &\bar{n}\cdot A_c \sim \lambda^0 \\ &n\cdot A_c \sim \lambda^2 \\ &A_{c\perp} \sim \lambda \\ &A_s^\mu \sim \lambda^2 \end{aligned}}$$

We can now use the above power counting to expand the Lagrangian while keeping only leading power terms (remember we are still considering a single colinear direction)

$$\mathcal{L}_{SCET} = \bar{\psi}_s \, i \not{D}_s \psi_s - \frac{1}{4} \left(F_{\mu\nu}^{s,a}\right)^2 - \frac{1}{4} \left(F_{\mu\nu}^{c,a}\right)^2 +$$

$$+ \bar{\chi} \frac{\not{\bar{n}}}{2} \left[i n \cdot D + i \not{D}_c^{\perp} \frac{1}{i \bar{n} \cdot D_c} i \not{D}_c^{\perp} \right] \chi$$

where we defined

$$i D_s^{\mu} = i \partial^{\mu} + g A_s^{\mu}$$

$$i D_c^{\mu} = i \partial^{\mu} + g A_c^{\mu}$$

$$i D^{\mu} = \left[i n \cdot \partial + g n \cdot A_c + g n \cdot A_s \right] \frac{\bar{n}^{\mu}}{2}$$

$$+ i \bar{n} \cdot D_c \frac{n^{\mu}}{2} + i D_c^{\perp}$$

and

$$i g F_{\mu\nu}^{s} = \left[i D_{s,\mu} , i D_{s,\nu} \right]$$

$$i g F_{\mu\nu}^{c} = \left[i D_{\mu} , i D_{\nu} \right]$$

Notice that the only interaction between soft gluons and colinear fields occurs through the $n \cdot A$ components, which scale in the same fashion $\mathcal{O}(\lambda^2)$.

REMARK: this, effectively, allows one to evaluate the soft field in the interaction terms with the colinear fields at $x \simeq x \cdot \bar{n} = x_-$.

We also observe that the small component of the collinear field $\eta(x)$ can be completely integrated out using the equations of motion.

Some important remarks, useful in what follows:

1) Gauge symmetry is preserved in each sector separately.
We can define two families of gauge transformation

$$V_s(x) = e^{i\delta_s^a(x)\, t^a} \quad , \quad V_c(x) = e^{i\delta_c^a(x)\, t^a}$$

where $\delta_s^a(x)$ and $\delta_c^a(x)$ have soft and collinear scaling, respectively.

The soft fields transform as:

$$\psi_s(x) \xrightarrow{V_s} V_s(x)\,\psi_s(x)$$

$$A_s^\mu(x) \xrightarrow{V_s} V_s(x)\, A_s^\mu(x)\, V_s^\dagger(x) + \frac{i}{g} V_s(x)\left[\partial^\mu, V_s^\dagger(x)\right]$$

$$\psi_s(x) \xrightarrow{V_c} \psi_s(x)$$

$$A_s^\mu(x) \xrightarrow{V_c} A_s^\mu(x)$$

The collinear fields transform as:

$$\left.\begin{aligned} \chi(x) &\xrightarrow{V_s} V_s(x_-)\,\chi(x) \\ A_c^\mu(x) &\xrightarrow{V_s} V_s(x_-)\, A_c^\mu(x)\, V_s^\dagger(x_-) \end{aligned}\right] \text{ neglecting subleading powers.}$$

$$\chi(x) \xrightarrow{V_c} V_c(x)\,\chi(x)$$

$$A_c^\mu(x) \xrightarrow{V_c} V_c(x)\, A^\mu_c(x)\, V_c^\dagger(x) + \frac{1}{g} V_c(x)\left[i\partial^\mu + \underline{g\frac{\bar{n}^\mu}{2} n\cdot A_s(x_-)}, V_c^\dagger(x)\right].$$

2) The derivation of the SCET Lagrangian relies on the possibility to identify a fixed reference direction (Thrust axis in our case) to parametrize the observable under consideration.

The SCET Lagrangian is then invariant under small modifications (at least $O(\lambda)$) of the reference axis

$\rightarrow$ reparametrization invariance

3) In order to achieve full factorization, soft and collinear terms in the Lagrangian must be completely separated. However, they still interact in the terms containing the covariant derivative component $\underline{D \cdot n}$:

$$\bar{\chi} \frac{\bar{n}\!\!\!/}{2} g\, n \cdot A_s \chi \quad , \quad -\frac{1}{4} \left(F_{\mu\nu}^{c,a}\right)^2$$

The term $F_{\mu\nu}^2$ is separately gauge invariant. Therefore in a physical gauge where $n \cdot A_s = 0$ the interaction disappears.

On the other hand, the term $\bar{\chi} \frac{\bar{n}\!\!\!/}{2} g\, n \cdot A_s \chi$ is not separately gauge invariant, and the interaction between the soft gluon and the collinear quark remains.

To get rid of it, we redefine the fields as follows

$$\chi(x) \longrightarrow Y_n(x_-)\, \chi(x)$$

$$A_c^{\mu}(x) \longrightarrow Y_n(x_-)\, A_c^{\mu}(x)\, Y_n^{\dagger}(x_-)$$

BPS field redefinition

where $Y_n(x)$ is a soft Wilson line

$$Y_n(x) = \mathbb{P} \exp\left\{ ig \int_{-\infty}^{0} dy\, n \cdot A_s(x + yn) \right\}$$

with the above redefinition the new soft and colinear fields $\chi(x)$ and $A(x)$ do not interact. The interaction is now encoded in the soft Wilson lines.

Together with $Y_n(x)$ we introduce a colinear Wilson line defined as

$$W_n(x) = \mathbb{P}\exp\left\{ ig \int_{-\infty}^{0} dy\, \underbrace{\bar{n}\cdot A_c(x+y\bar{n})}_{\text{large field component}} \right\}$$

which transforms as follows under a colinear gauge transformation

$$W_n(x) \longrightarrow V_c(x)\, W_n(x)\, V_c^{\dagger}(-\infty\bar{n}) = V_c(x)\, W_n(x)$$

this is an important property since it allows us to build the following gauge-invariant quark fields (under colinear gauge)

$$\xi_n(x) = W_n^{\dagger}(x)\chi(x), \quad \bar{\xi}_n(x) = \bar{\chi}(x)\, W_n(x).$$

4) The extension of the above Lagrangian to multiple colinear directions (2 in our case: $n, \bar{n}$) is simply achieved by creating copies of the colinear bits of the SCET Lagrangian, with n^μ replaced by the proper reference vector.

An important consequence is that there is no interaction between fields along different colinear directions at leading power.

The only cross talk between them occurs through the exchange of soft gluons.

Factorization and resummation in SCET:

We consider a $q\bar{q}$ system of momenta p_1 and p_2 respectively, produced in a e^+e^- annihilation.

When extra final-state emissions occur, in the soft and/or colinear limit, the thrust axis remains aligned with the direction of the $q\bar{q}$ system, i.e.

$$n^\mu \parallel p_1^\mu \,; \quad \bar{n}^\mu \parallel p_2^\mu$$

We can write the normalized differential distribution as

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \sum_{|h\rangle} \frac{|M(e^+e^- \to h)|^2}{\sigma_0} (2\pi)^4 \delta^{(4)}(q - p_h)\, \delta(\tau - v(h))$$

sum over states

where

$$|M(e^+e^- \to h)|^2 = \sum_{i=V,A} L^{(i)}_{\mu\nu} \langle 0 | j_i^{\mu\,\dagger} | h\rangle \langle h | j_i^\nu | 0\rangle$$

and $j_V^\mu = \bar{\psi}_f^a \gamma^\mu \psi_f^a$, $j_A^\mu = \bar{\psi}_f^a \gamma^\mu \gamma^5 \psi_f^a$.

Using the integral representation of the $\delta^{(4)}$ we can recast the distribution as

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \sum_{|h\rangle} \frac{1}{\sigma_0} \int d^4x\, e^{iq\cdot x} \sum_i L^{(i)}_{\mu\nu} \langle 0 | j_i^{\mu\,\dagger}(x) | h\rangle \langle h | j_i^\nu(0) | 0\rangle \times \delta(\tau - v(h))$$

The precise form of the leptonic tensor is irrelevant since we are dividing by σ_0.

In the following we discuss the procedure for the vector current as the axial case is exactly identical.

We now want to formulate the above quantity in SCET.
We proceed in two steps:

1) First, we need to "match" the QCD current onto the EFT. We write

$$J_i^\mu = C_{n\bar n}(p_1, p_2)\, O^\mu_{n\bar n, i}(x, p_1, p_2)$$

where $C_{n\bar n}$ is a Wilson coefficient that encodes all the hard (non-IRC) degrees of freedom that are not described in the leading-power effective Lagrangian.

In the $\tau \to 0$ limit, these d.o.f. can only be of virtual nature, therefore $C_{n\bar n}$ simply coincides with the quark form factor in QCD.

$O_{n\bar n, i}$ is an SCET operator that approximates the QCD current when q is a fast-moving quark along the $\bar n^\mu$ direction.
Since the current

$$\bar\chi_n(x)\, \Gamma_i^\mu\, \chi_{\bar n}(x)$$

is not gauge invariant, we use collinear Wilson lines to redefine the fermionic fields as above, i.e.

$$\xi_n(x) = W_n^\dagger(x)\chi_n(x), \quad \xi_{\bar n}(x) = W_{\bar n}^\dagger(x)\chi_{\bar n}(x)$$

Moreover, we perform the BPS field redefinition and obtain the following SCET current

$$O^\mu_{n\bar n, i}(x, p_1, p_2) = e^{i(p_1 - p_2)\cdot x}\, \bar\xi_n(x)\, Y_n^\dagger(x)\, \Gamma_i^\mu\, Y_{\bar n}(x)\, \xi_{\bar n}(x)$$

where $\Gamma_V^\mu = \gamma_\perp^\mu$, $\Gamma_A^\mu = \gamma_\perp^\mu \gamma^5$.

2) We need to decompose each state $|h\rangle$ into a soft and two colinear components,

$$|h\rangle = |h_s\rangle|h_n\rangle|h_{\bar n}\rangle$$

Moreover, we notice once again that the observable (thrust) is additive, implying

$$v(h) = v(h_s) + v(h_n) + v(h_{\bar n})$$

where $v(h)$ is the value of thrust for a state $|h\rangle$.
Therefore we can use the following expression for the $\delta(\tau - v(h))$

$$\int d\tau_n \int d\tau_{\bar n} \int d\tau_s \,\delta(\tau - \tau_s - \tau_n - \tau_{\bar n})\,\delta(\tau_s - v(h_s))\,\delta(\tau_n - v(h_n))\,\delta(\tau_{\bar n} - v(h_{\bar n}))$$

Plugging these definitions into our initial expression for the cross section (we use the fact that p_1, p_2 are back-to-back) we obtain

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \frac{|C_{n\bar n}(p_1,p_2)|^2}{\sigma_0}\int d^4x \int d\tau_s\, d\tau_n\, d\tau_{\bar n}\, \delta(\tau - \tau_s - \tau_n - \tau_{\bar n})$$

$$\times \boxed{\frac{1}{N_c}\mathrm{Tr}} \Big\{ \sum_{|h_n\rangle} \langle 0|\xi_n(x)_\beta |h_n\rangle\langle h_n|\bar\xi_n(0)_\gamma|0\rangle\,\delta(\tau_n - v(h_n))\Big\}$$

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$$\times \boxed{\frac{1}{N_c}\mathrm{Tr}} \Big\{ \sum_{|h_{\bar n}\rangle} \langle 0|\bar\xi_{\bar n}(x)_\alpha |h_{\bar n}\rangle\langle h_{\bar n}|\xi_{\bar n}(0)_\delta|0\rangle\,\delta(\tau_{\bar n} - v(h_{\bar n}))\Big\}$$

$$\times \boxed{\frac{1}{N_c}\mathrm{Tr}} \Big\{ \sum_{|h_s\rangle} \langle 0|Y_{\bar n}^\dagger(x)\, Y_n(x)|h_s\rangle\langle h_s|Y_n^\dagger(0)\, Y_{\bar n}(0)|0\rangle \times \delta(\tau_s - v(h_s))\Big\}$$

$$\times \Big\{ \gamma_{\perp\,\alpha\beta}^{\mu\dagger}\, \gamma^\nu_{\perp\,\gamma\delta}\, L^{(v)}_{\mu\nu} + (\gamma_\perp^\mu\gamma^5)^\dagger_{\alpha\beta}\,(\gamma_\perp^\nu\gamma^5)_{\gamma\delta}\, L^{(A)}_{\mu\nu}\Big\}$$

from which we see that the cross section is completely factorized.

We thus define:

$\oplus$ $$\frac{1}{N_c}\mathrm{Tr}\Big\{\sum_{|k_n\rangle}\langle 0|\xi_n(x)_\beta|k_n\rangle\langle k_n|\bar{\xi}_n(0)_\gamma|0\rangle\delta(\tau_n - V(k_n))\Big\}$$

$$= \int \frac{dk^+ dk^- d^2k_\perp}{2(2\pi)^4} e^{-ik\cdot x}\mathcal{J}_n(\tau_n, k^+)\left(\frac{\not{\bar{n}}}{2}\right)_{\beta\gamma}$$

$\oplus$ $$\frac{1}{N_c}\mathrm{Tr}\Big\{\sum_{|k_{\bar{n}}\rangle}\langle 0|\bar{\xi}_{\bar{n}}(x)_\alpha|k_{\bar{n}}\rangle\langle k_{\bar{n}}|\xi_{\bar{n}}(0)_\delta|0\rangle\delta(\tau_{\bar{n}} - V(k_{\bar{n}}))\Big\}$$

$$= \int \frac{dl^+ dl^- d^2l_\perp}{2(2\pi)^4} e^{-il\cdot x}\mathcal{J}_{\bar{n}}(\tau_{\bar{n}}, l^-)\left(\frac{\not{n}}{2}\right)_{\delta\alpha}$$

$\oplus$ $$\frac{1}{N_c}\mathrm{tr}\Big\{\sum_{|k_s\rangle}\langle 0|Y_{\bar{n}}^\dagger(x)Y_n(x)|k_s\rangle\langle k_s|Y_n^\dagger(0)Y_{\bar{n}}(0)|0\rangle$$

$$\times\delta(\tau_s - V(k_s))\Big\} = \int\frac{d^4r}{(2\pi)^4}e^{-ir\cdot x}S(\tau_s, r)$$

We can now integrate inclusively over $k_\perp, l_\perp, l^+, k^-$. These integrals produce δ functions that can be used to get rid of the d^4x integration. The result becomes

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = |C_{n\bar{n}}(p_1,p_2)|^2\int d\tau_s\, d\tau_n\, d\tau_{\bar{n}}\,\delta(\tau - \tau_s - \tau_n - \tau_{\bar{n}})$$

$$\times J_n(\tau_n)J_{\bar{n}}(\tau_{\bar{n}})S(\tau_s)$$

where we defined

$$\left.\begin{aligned} J_n(\tau_n) &= \int\frac{dk^+}{2\pi}\mathcal{J}_n(\tau_n, k^+) \\ J_{\bar{n}}(\tau_{\bar{n}}) &= \int\frac{dl^-}{2\pi}\mathcal{J}_{\bar{n}}(\tau_{\bar{n}}, l^-) \\ S(\tau_s) &= \int\frac{d^4r}{(2\pi)^4}S(\tau_s, r)\end{aligned}\right\}\begin{aligned}&n \text{ jet function}\\ &\bar{n} \text{ jet function}\\ &\text{soft function}\end{aligned}$$

the final thing left to obtain a factorised cross section is to deal with the $\delta(\tau-\tau_s-\tau_n-\tau_{\bar{n}})$ that mixes soft and collinear modes.

This is a simple task and can be solved by simply taking a Laplace Transform, i.e.

$$\delta(\tau-\tau_s-\tau_n-\tau_{\bar{n}}) = \frac{1}{2\pi i}\int_{C} e^{u\tau}\, e^{-u\tau_s}\, e^{-u\tau_n}\, e^{-u\tau_{\bar{n}}}\, du$$

to the right of all singularities of the integrand

that allows us to write the cross section differential in τ as

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \underbrace{\left|C_{n\bar{n}}(p_1,p_2)\right|^2}_{\mathcal{H}(Q^2)\ \text{(hard function)}} \frac{1}{2\pi i}\int du\, e^{u\tau}\cdot \tilde{S}(u)\,\tilde{J}_n(u)\,\tilde{J}_{\bar{n}}(u)$$

Laplace transform of the soft and jet functions

SOME REMARKS:

1) Each of the ingredients of the above factorization theorem contains both real and virtual radiative corrections. This implies that $\tilde{S}$ and $\tilde{J}$ are separately IRC finite. Conversely, in the EFT picture the c.o.m. scale (hard scale Q) of the process is seen as infinitely large by all components of the momenta but those that scale like Q (e.g. large component of a collinear momentum).

This induces new UV divergences in the soft and jet function that can be renormalised away in the factorized space.

$\Rightarrow$ FULL FACTORIZATION AT THE PRICE OF EXTRA SINGULARITIES.

This operation will induce renormalization-group-equations (RGEs) for all ingredients of the factorized formula, i.e.

$$\frac{d\ln \mathcal{H}(Q^2,\mu^2)}{d\ln\mu^2} = \Gamma_H(\alpha_s(\mu^2)) \; ; \; \frac{d\ln \tilde{S}(u,\mu^2)}{d\ln\mu^2} = \Gamma_S(\alpha_s(\mu^2))$$

$$\frac{d\ln \tilde{J}_n(u,\mu^2)}{d\ln\mu^2} = \Gamma_J(\alpha_s(\mu^2)) \; ; \; \frac{d\ln \tilde{J}_{\bar{n}}(u,\mu^2)}{d\ln\mu^2} = \Gamma_J(\alpha_s(\mu^2))$$

identical for the two jet functions.

From the RG-invariance of the cross section we also get the following consistency relation

$$\boxed{\Gamma_H + 2\Gamma_J + \Gamma_S = 0}$$

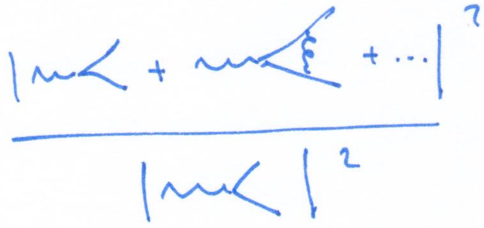
These RGEs can be ultimately used to resum the logarithms. This relies on the fact that each ingredient depends on a single characteristic scale through logarithms of its ratio to the renormalization scale μ.

2) The presence of the UV divergences leads to spurious contributions (i.e. of non-IRC nature) to the anomalous dimensions Γ_F ($F = H, S, J$) defined above.
Although these eventually cancel in the factorization theorem they lead to a logarithmic structure for the individual soft and jet function that differs from the QCD counterpart.
e.g. the logarithms contained in the soft function $\boxed{\text{are not}}$ the soft logarithms in QCD. Only the combination $\mathcal{H}\tilde{S}\tilde{J}_n\tilde{J}_{\bar{n}}$ makes physical sense (unless extra regulators are included).

An example: thrust at LL.

We now work out our usual case study.

⇒ we start from the hard function. As explained above, this coincides with the quark form factor in QCD (squared), whose one-loop expression reads

$$\mathcal{H}^{(\text{bare})}(Q^2, \mu^2) =$$ 

$$\begin{pmatrix}\text{coupling}\\ \text{renormalized}\\ \text{in } \overline{MS} \text{ scheme}\end{pmatrix} = 1 + C_F \frac{\alpha_s(\mu)}{2\pi}\left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 2\frac{\ln \mu^2/Q^2}{\varepsilon} - \ln^2\frac{\mu^2}{Q^2}\right.$$

$$\left. - 3\ln\frac{\mu^2}{Q^2} - 8 + \frac{7}{6}\pi^2\right) + \mathcal{O}(\alpha_s^2(\mu))$$

we can renormalize it by an additional multiplicative renormalization constant such as

$$\mathcal{H} = Z_H^{-1}\, \mathcal{H}^{(\text{bare})} \quad \longleftarrow \text{finite for } \varepsilon \to 0$$

with $$Z_H = 1 + C_F\frac{\alpha_s(\mu)}{2\pi}\left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 2\frac{\ln \mu^2/Q^2}{\varepsilon}\right)$$

to derive the corresponding RGE, we exploit the fact that $\mathcal{H}^{(\text{bare})}$ is μ invariant

$$\frac{d\mathcal{H}^{(\text{bare})}}{d\ln\mu^2} = \frac{\partial \ln \mathcal{H}^{(\text{bare})}}{\partial \ln \mu^2} + \underbrace{\beta(\alpha_s)}_{\beta \text{ function in } 4-2\varepsilon \text{ dimensions}} \frac{\partial \ln \mathcal{H}^{(\text{bare})}}{\partial\, \alpha_s}$$

which leads to

$$\frac{d\ln \mathcal{H}}{d\ln\mu^2} = -\frac{d\ln Z_H}{d\ln\mu^2} = -\frac{\alpha_s(\mu^2)}{\pi} C_F\left(2\ln\frac{\mu^2}{Q^2} + 3\right)$$

$$\boxed{\beta(\alpha_s) = -2\varepsilon\,\alpha_s + \beta^{(4D)}(\alpha_s)}$$

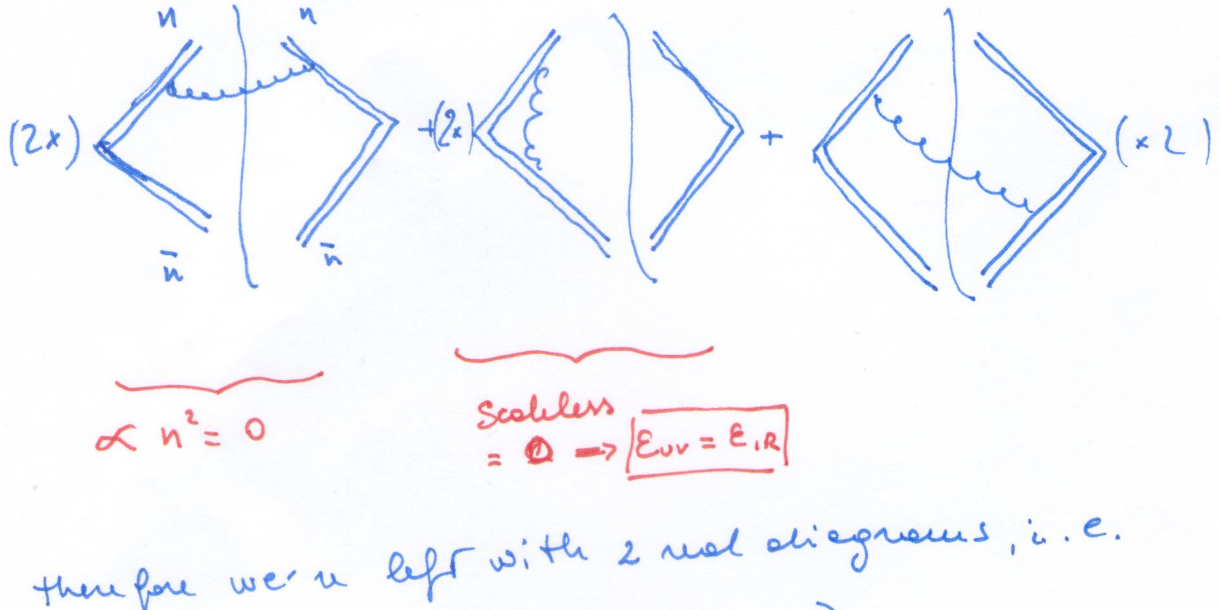
from the above calculation we observe that the hard function has no logarithms at $\mu = Q$. Therefore we can use this scale to set the boundary condition for the above differential equation that leads to the following resummed hard function

$$\mathcal{H}(Q^2, \mu^2) = \mathcal{H}(Q^2, Q^2)\, \exp\left\{ - \int_{Q^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \frac{\alpha_s(\mu'^2)}{\pi} C_F \left[2 \ln \frac{\mu'^2}{Q^2} + 3 \right] \right\}$$

where $$\mathcal{H}(Q^2, Q^2) = 1 + \underbrace{C_F \frac{\alpha_s(Q^2)}{2\pi} \left(-8 + \frac{7}{6}\pi^2 \right)}_{\text{this contribution is NNLL and we ignore it in the following.}} + \mathcal{O}(\alpha_s^2(Q))$$

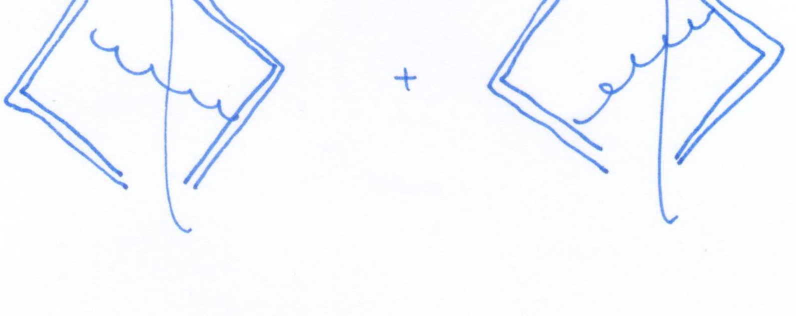
$\Rightarrow$ soft function. This is defined as an expectation value of Wilson lines (soft), as seen in the formulation of the factorization theorem.

At $\mathcal{O}(\alpha_s)$ the contributing diagrams are



$\underbrace{\qquad}_{\propto n^2 = 0}$ $\underbrace{\qquad}_{\text{Scaleless} = 0 \Rightarrow \boxed{\varepsilon_{UV} = \varepsilon_{IR}}}$

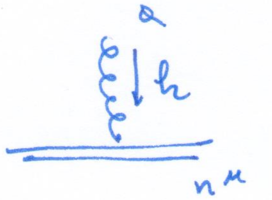
therefore we're left with 2 real diagrams, i.e.



which read:

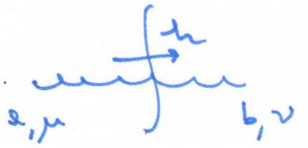
Remember that:

1) The Feynman rule for a soft gluon emission off a Wilson line is



$$= -g\, T^a \frac{n^\mu}{k\cdot n}$$

and a cut propagator (gluon) corresponds to (Feynman gauge)



$$\frac{-i\, g^{\mu\nu}\delta_{ab}}{k^2+i0} \times (k^2+i0)(-2\pi i)\,\delta^{(+)}(k^2)$$

$$= -2\pi\, g^{\mu\nu}\delta_{ab}\,\delta^{(+)}(k^2)$$

2) the expression of thrust in the soft limit ($z^{(e)} \to 0$ in the previous lectures) is

$$\delta(\tau - k^+/Q)\,\Theta(k^- - k^+) + \delta(\tau - k^-/Q)\,\Theta(k^+ - k^-)$$

3) $d^d k = \frac{dk^+ dk^-}{2} d^{d-2}k_\perp$; $dk^+dk^- d^{d-2}k_\perp \,\delta(k^2) = dk^+dk^-(k^+k^-)^{-\varepsilon}\Theta(k^+)\Theta(k^-) \times \frac{\pi^{1-\varepsilon}}{\Gamma(1-\varepsilon)}$

the one-loop soft function becomes ($\overline{MS}$ coupling)

$$S^{\text{bare}}(\tau,\mu^2) = \cancel{1+\frac{\alpha_s(\mu)}{4\pi}C_F\left(\ldots\frac{1}{\epsilon}\ldots\right)}\ \delta(\tau) + 2 g_s^2 \mu^{2\varepsilon} C_F\, n\cdot\bar n \int \frac{d^d k}{(2\pi)^d}(2\pi)\delta(k^2)$$

$\overline{MS}$ scheme $\left(\alpha_s \mu^{2\varepsilon}(4\pi)^{\varepsilon} \to \alpha_s(\mu)\mu^{2\varepsilon} e^{\gamma_E \varepsilon}\right)$

$$\times \frac{1}{k^+k^-}\,\delta\!\left(\frac{\min(k^+,k^-)}{Q} - \tau\right) \overset{\downarrow}{=} \delta(\tau) + 2\, C_F \frac{\alpha_s(\mu)}{\pi}\left(\frac{\mu}{Q}\right)^{2\varepsilon} \tau^{-1-2\varepsilon} \frac{e^{\gamma_E \varepsilon}}{\Gamma(1-\varepsilon)}\, \frac{1}{\varepsilon}$$

Remember that this quantity can be renormalized in a multiplicative way only in Laplace space, where the process is entirely factorized.

The Laplace transform therefore reads

$$\tilde{S}^{\text{bare}}(u,\mu^2) = 1 + C_F \frac{\alpha_s(\mu^2)}{\pi}\left[-\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{Q u_0}{\mu u} - 2\ln^2\frac{Q u_0}{\mu u} - \frac{\pi^2}{4}\right]$$

with $u_0 = e^{-\gamma_E}$. The renormalization constant thus reads

$$Z_S = 1 + C_F\frac{\alpha_s(\mu^2)}{\pi}\left[-\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{Q u_0}{\mu u}\right]$$

which leads to

$$\frac{d\ln\tilde{S}(u,\mu^2)}{d\ln\mu^2} = 4\frac{\alpha_s(\mu^2)}{\pi}C_F\ln\frac{Q u_0}{u\mu}$$

we see that its boundary condition can be set at $\mu = \frac{Q u_0}{u}$.

This leads to

$$\tilde{S}(u,\mu^2) = \tilde{S}\left(u, \frac{Q^2 u_0^2}{u^2}\right)\exp\left\{\int_{\frac{Q^2 u_0^2}{u^2}}^{\mu^2}\frac{d\mu'^2}{\mu'^2}4\frac{\alpha_s(\mu'^2)}{\pi}\ln\frac{Q u_0}{u\mu'}\right\}$$

where $\tilde{S}\left(u, Q^2\frac{u_0^2}{u^2}\right) = 1 + C_F\underbrace{\frac{\alpha_s(\mu^2)}{\pi}\left[-\frac{\pi^2}{4}\right]}_{\text{this is NNLL and we can ignore it.}}$

→ We're left with the jet function. Instead of computing it, we remember that we can use the consistency relation to extract its anomalous dimension and RGE. Therefore

$$\frac{d\ln\tilde{J}_n(u,\mu^2)}{d\ln\mu^2} = -\frac{1}{2}\left(\Gamma_H + \Gamma_S\right) = -\frac{\alpha_s(\mu^2)}{2\pi}C_F\left[2\ln\frac{Q^4}{\mu^4}\frac{u_0^2}{u^2} + 3\right]$$

$$= -\frac{\alpha_s(\mu^2)}{\pi}C_F\left[2\ln\frac{Q^2 u_0}{\mu^2 u} + \frac{3}{2}\right]$$

from which we see that the initial condition can be set at $\mu = \frac{Q\sqrt{u_0}}{\sqrt{u}}$

The solution reads

$$\tilde{J}_n(u,\mu^2) = \underbrace{\tilde{J}_n\left(u, Q^2\frac{u_0}{u}\right)} \exp\left\{-\int_{Q^2\frac{u_0}{u}}^{\mu^2} \frac{d\mu'^2}{\mu'^2}\frac{\alpha_s(\mu'^2)}{\pi} C_F\left[2\ln\frac{Q^2 u_0}{\mu^2 u} + \frac{3}{2}\right]\right\}$$

the boundary condition (non-logarithmic term) cannot be extracted from RG arguments, and must be computed directly. Since it differs from 1 starting at NNLL, we don't consider it here.

We can choose any value for the common renormalization scale μ. We therefore set it to $\mu = Q$, so that the hard function is set to one at this order.

However, since we are only interested in LL terms, we neglect the factor $3/2\, C_F \frac{\alpha_s(\mu^2)}{\pi}$ in the jet function's RGE, since this enters at NLL.

The factorization theorem hence becomes

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \frac{1}{2\pi i}\int du\, e^{u\tau} \exp\left\{\int_{Q^2\frac{u_0^2}{u^2}}^{Q^2} \frac{d\mu^2}{\mu^2}\, 4 C_F \frac{\alpha_s(\mu^2)}{\pi}\ln\frac{Q u_0}{u\mu} - \int_{Q^2\frac{u_0}{u}}^{\mu^2}\frac{d\mu^2}{\mu^2}\frac{\alpha_s(\mu^2)}{\pi} C_F\, 2\ln\frac{Q^2 u_0}{\mu^2 u}\right.$$

$$= \frac{1}{2\pi i}\int du\, e^{u\tau} e^{-R(u_0/u)}$$

where the functional form of R is the same that we obtained with the branching algorithm.

REMARK:

the inverse Laplace transform has to deal with the Landau pole present in $R(u_0/u)$. Several prescriptions can be adopted in this case.

To invert the Laplace Transform, we can safely expand (this is allowed by the kinematics of the observable) u_0/u about τ.

This operation is completely analogous to the Taylor expansion that we performed in the branching formalism i.e.

$$R'(\tau_i) \simeq R'(\tau) + \dots$$

with this prescription we get

$$e^{-R(u_0/u)} = e^{-R(\tau)} e^{-R'(\tau)\ln\frac{u\tau}{u_0} + \overbrace{\mathcal{O}(R'')}^{NNLL}}$$

leading to

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \frac{e^{-R(\tau)}}{2\pi i}\int du\, e^{u\tau} e^{-R'\ln\frac{u\tau}{u_0}}$$

we observe that now the only Landau pole is hit when τ is small, and the remaining integration is trivial!
Since we previously worked at the cumulative level, we obtain

$$\Sigma(\tau) = \int_0^\tau d\tau' \frac{1}{\sigma_0}\frac{d\sigma}{d\tau'} = \frac{e^{-R(\tau)}}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{du}{u} e^{u\tau} e^{-R'(\tau)\ln\frac{u\tau}{u_0}}$$

$$= \boxed{e^{-R(\tau)}\frac{e^{-\gamma_E R'(\tau)}}{\Gamma(1+R'(\tau))}}$$

which reproduces the result obtained in the previous lecture.

CONCLUDING REMARKS:

- The above Taylor expansion can be directly avoided if we decide to set the initial evolution scales directly in thrust space. However, in the more general case this is a dangerous operation as it can lead to spurious singularities in the momentum-space answer (e.g. transverse momentum of a Z boson at the LHC).

- Nowadays, the state of the art of resummations is much more advanced than what we saw in this class and some observables in 2-scale, 2-leg problems (like thrust itself) are now known to N^3LL.

  This often serves as a tool to perform precise predictions in regions of phase space affected by large logarithms, as well as to study and improve on more realistic (exclusive) simulations based on Parton-Shower event generators.

- Lately, several directions are being considered in this research field:

  - study of multileg/multiscale problems
  - structure of higher-order anomalous dimensions
  - encoding higher-order corrections in event generators
  - study of non-global observables
  - ...