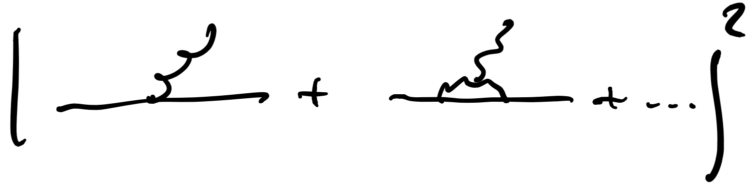
$$\Sigma(\tau) = \frac{1}{\cancel{\sigma_0}} \int \cancel{[d\Phi_B]\, |M_B(p_1,p_2)|^2}$$

$$\times\, \mathcal{H}(Q^2) \rightarrow$$

$$\times \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i]\, |M(k_1, \ldots, k_n)|^2$$

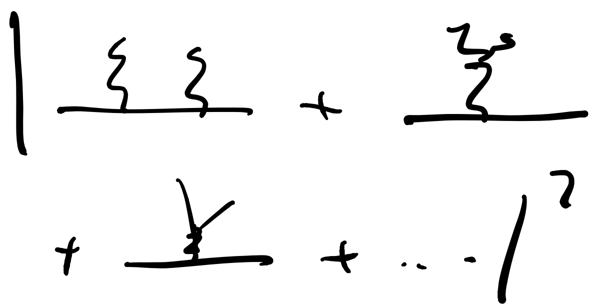
$$\times\, \Theta(\tau - V(k_1, \ldots, k_n))$$

$$|M(k)|^2 =$$



$$= \left| M^{(0)}(k) + M^{(1)}(k) + \ldots \right|^2$$

$$|M(k_1, k_2)|^2 =$$



$$= \boxed{|M(k_1)|^2 \, |M(k_2)|^2} \to |1|^2 \, |3|^2$$

$$+ |\tilde{M}(k_1, k_2)|^2$$

$$|M(k_1, k_2, k_3)|^2 = \prod_{i=1}^{3} |M(k_i)|^2$$

$$+ |M(k_1)|^2 \, |\tilde{M}(k_2, k_3)|^2 + \text{perm.}$$

$$+ |\hat{M}(k_1, k_2, k_3)|^2$$

$$|M(h_1, \dots h_n)|^2 = \prod_{i=1}^{n} |M(h_i)|^2$$

$$+ \sum_{a>b} |\tilde{M}(h_a, h_b)|^2 \sum_{\substack{i=1 \\ i\neq a,b}}^{n} |M(h_i)|^2$$

$$+ \sum_{\substack{a>b \\ c>d}} |\tilde{M}\underline{(h_a, h_b)}|^2 |\tilde{M}(h_c, h_d)|^2 \sum_{\substack{i=1 \\ i\neq a,b,c,d}}^{n} |M(h_i)|^2 + \dots$$

Q: for what observable?

e.g. 1 : <u>thrust</u> $(\Sigma(\tau))$

$$|M(k)|^2 \to \underline{\alpha_s} L^2$$

$$|M(k_1,k_2)|^2 = \overbrace{|M(k_1)|^2 \left(M(k_2)\right)^2}^{\alpha_s^2 L^4}$$

$$+ \underbrace{|\tilde{M}(k_1,k_2)|^2}_{\alpha_s^2 L^3} \quad (\to 1/L)$$

$$|\tilde{M}(k_1,\ldots,k_n)|^2 \sim \alpha_s^n L^{n+1}$$

GREAT!

GLOBAL <u>OBSERVABLES</u>

$$|M(k)|^2 \to \alpha_s L$$

$$|\tilde{M}(k_1, k_2)|^2 \to \alpha_s^2 L^2$$

$$|\hat{M}(k_1, k_2, k_3)|^2 \to \alpha_s^3 L^3$$

NON-GLOBAL OBSERVABLE

$$|M(k)|^2 = |M^{(0)}(k) + M^{(1)}(k) + \dots|^2$$

$$= 1PC = 1PC^{(0)} \frac{\alpha_s}{2\pi} + 1PC^{(1)} \left(\frac{\alpha_s}{2\pi}\right)^2$$

$+ \ldots$

$$|\tilde{M}(h_1, \ldots, h_n)|^2 = nPC$$

$$= \sum_{j=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^{n+j} nPC^{(j)}$$

GLOBAL OBS.

$nPC^{(j)}$	soft limit	hard. coll. limit	why?
LL	$n+j \leq 1$	—	
NLL	$n+j \leq 2$	$n+j \leq 1$	
NNLL	$n+j \leq 3$	$n+j \leq 2$	
$\vdots$	$\vdots$	$\vdots$	

$$nPC^{(j)} \Rightarrow nPC_s^{(j)} + nPC_{h.c.}^{(j')}$$

$$[dk_i]|M(k_i)|^2 = \sum_{\ell=1}^{L} 2\,\frac{\alpha(\mu)}{2\pi}\frac{dk_t}{k_t}\frac{dz^{(\ell)}}{z^{(\ell)}}$$

μ: k_t

$$\times\Big[\underbrace{2C_F}_{\text{soft}} + \underbrace{\left(z^{(\ell)}P(z^{(\ell)}) - \lim_{z^{(\ell)}\to 0} z^{(\ell)}P(z^{(\ell)})\right)}_{\text{h.c.}}\Big]$$

* PLEASE ASK DURING DISCUSSION:

LL_Σ: $\alpha_s^n L^{2n}$ $\to$ $\mu = k_t$

LL: $\ln \Sigma(v) \sim \alpha_s^n L^{n+1}$

recursively IRC safe

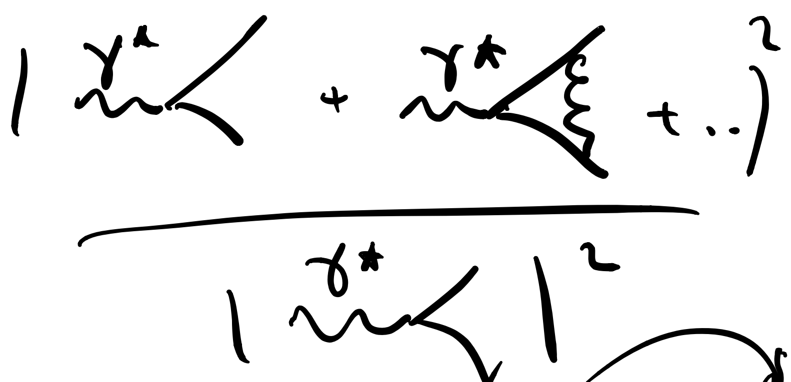
(e.g. thrust)

example: $\underline{LL}$ resummation

$$\sum_{n=1}^{\infty}\prod_{i=1}^{n}\int [dk_i]\, |M(k_1 \dots k_n)|^2$$

$$= \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{1}{n!}\int [dk_i]\,\underline{|M(k_i)_{soft}|^2}$$

(REAL CORRECTIONS)

$\mathcal{H}(a^2) =$ 

$$\overset{LL}{\simeq} \sum_{n=0}^{\infty}\prod_{i=1}^{n}\frac{1}{n!}\left(-\int[dk_i]\,|M(k_i)_{soft}|^2\right)$$

$+$ ~~contours~~ $\rightarrow \underbrace{\alpha_s L^0}_{NNLL} + \underbrace{\alpha_s^2 L^0}_{N^3LL}$

$$= \exp\left\{ -\int \underbrace{[\mathrm{d}\, k]\, |M_{\mathrm{soft}}(k)|^2}_{\text{divergent}} \right\}$$

$\uparrow$ constants

ALL SINGULARITIES OF THE FORM FACTOR EXPONENTIATE

DIXON, MAGNEA, STERMAN 0805.3515

$$[\mathrm{d}\, k_i]\, |M_{\mathrm{soft}}(k_i)|^2 = \sum_{\ell=1}^{2} \int \frac{\mathrm{d} k_{T i}}{k_{T i}}\, 2 C_F \frac{\alpha_s(k_{T i})}{\pi} \times \frac{\mathrm{d} z_i^{(\ell)}}{z_i^{(\ell)}} \underbrace{\frac{\mathrm{d}\phi_i}{2\pi}}_{\ell}$$

$$\ln \Sigma(\tau) - \ln \Sigma_{exact} \sim \alpha_s^n L^n$$

$\rightarrow$ control over $\alpha_s^n L^{n+1}$

$$\Sigma(\tau) = e^{-\sum_{\ell=1}^{2}\int \frac{dk_t}{k_t}\frac{dz^{(\ell)}}{z^{(\ell)}}\, 2C_F \frac{\alpha_s(k_t)}{\pi}}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \left(\sum_{\ell=1}^{2} C_F \int \frac{dk_{ti}}{k_{ti}} \frac{dz_i^{(\ell)}}{z_i^{(\ell)}}\, 2\frac{\alpha_s(k_{ti})}{\pi} \right)$$

$$\times\, \Theta\left(\tau - V(k_1, \dots, k_n)\right)$$

$$\Theta(\tau_i - \omega\tau) + \Theta(\omega\tau - \tau_i) = 1 \quad \text{thm of}$$

$$V(k_1, \dots, k_n) = \sum_{\substack{i=1 \\ i\in \mathcal{H}_p}}^{n} \frac{k_{ti}^2}{Q^2 z_i^{(1)}} + \sum_{\substack{j=1 \\ j\in \mathcal{H}_{\bar p}}}^{n} \frac{k_{tj}^2}{Q^2 z_j^{(2)}}$$

$$= \sum_{k=1}^{n} \frac{\hbar \omega_k^2}{U^2 \chi_k}(\ell_k) = \sum_{k=1} \tau_k$$

④ slicing method

$$\tau_k \sim \tau$$

$$\tau_k \ll \tau$$

$$\tau_k < \omega \tau$$

$$\omega \ll 1$$

$$V(h_1, \dots, h_k, \dots, h_n)$$

$$\simeq V(h_1, \dots, h_n) + \omega \tau$$

$$\omega \to 0$$

$$\bar{\Sigma}(\tau) = \exp\left\{ -\sum_{\ell=1}^{2} \int \frac{dk_t}{k_t} \frac{dz^{(\ell)}}{z^{(\ell)}} \, 2C_F \frac{\alpha_s(k_t)}{\pi} \times \left(1 - \Theta\left(\omega\tau - \tau(k)\right) \right) \right\}$$

$$\times \left(\text{REAL RESOLVED} \right) \Theta\left[\tau - V(k_1, \ldots, k_n)\right]$$

$$\Theta\left(\tau(k) - \omega\tau\right) \qquad (\text{soft}) \qquad \frac{k_t^2}{Q^2 z^{(\ell)}}$$

$$R(\omega\tau) = \sum_{\ell=1}^{2} \int_0^{Q} \frac{dk_t}{k_t} \int_{k_t/Q}^{1} \frac{dz^{(\ell)}}{z^{(\ell)}} \, 2C_F \frac{\alpha_s(k_t)}{\pi} \times \Theta\left(\frac{k_t^2}{z^{(\ell)} Q^2} - \omega\tau \right)$$

RADIATOR

$$\alpha_s(k_t) = \frac{\alpha_s(Q)}{1 - \alpha_s(Q)\beta_0 \ln\frac{Q^2}{k_t^2}}$$

$$R'(\tau_i) = \tau_i \sum_{\ell=1}^{2} \int \frac{dk_t}{k_t} \frac{dz^{(\ell)}}{z^{(\ell)}}$$

$$\times\, 2C_F \frac{\alpha_s(k_t)}{\pi} \delta\left(\tau_i - \tau(k_t)\right)$$

$$[dk]\, |M(k)|^2 = \int \frac{d\tau}{\tau} [dk]\, |M(k)|^2$$

$$\times\, \delta\left(\ln\tau - \ln\tau(k)\right)$$

$$\Sigma(\tau) = e^{-R(\omega\tau)}$$

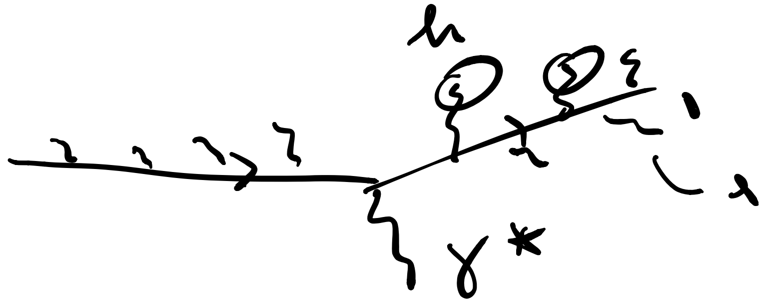
radiator

$$\sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\omega\tau}^{\infty} \frac{d\tau_i}{\tau_i} R'(\tau_i)$$

$$\times\, \Theta\left(\tau - \sum_{i=1}^{n} \tau_i\right) + O(\omega\tau)$$

$\omega \to 0$

$$|M(k_1 \dots k_n)|^2 = \prod_{i=1}^{n} |M(k_i)|^2$$



k has a $\tau(k) = \dfrac{k_T^2}{Q^2 z(q)} \gtrless \omega\tau$

$$\tau(q) < \omega\tau$$

2 approximations:

$$R(\omega\tau) = R(\tau) + R'(\tau) \ln \frac{1}{\omega} + \frac{R''(\tau)}{2!} \ln^2 1/\omega + \dots$$

$$R'(\tau) = \frac{d R(\tau)}{d \ln 1/\tau}$$

$$R(\tau) \sim \alpha_s L^2 + \alpha_s^2 L^3 + \cdots$$

$$\cdots \sim \alpha_s^n L^{n+1} \sim LL$$

$$R' = \frac{dR}{dL} \sim \alpha^n L^n \sim NLL$$

$$R'' \sim \alpha^n L^{n-1} \sim NNLL$$

$\tau_i \sim \tau$

NNLL

$$R'(\tau_i) \simeq R'(\tau) + R''(\tau) \ln \frac{\tau}{\tau_i} + \cdots$$

$$\Sigma(\tau) = e^{-R(\tau) - \boxed{R'(\tau) \ln \frac{1}{\omega}} + \cdots O(NNLL)}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{m} \int_{\omega\tau}^{1} \frac{d\tau_i}{\tau_i} R'(\tau) \boxed{\Theta\left(\tau - \sum_{i=1}^{m} \tau_i\right)}$$

$\ln 1/\omega$

$$+ O(NNLL)$$

SOLUTION:

1) ANALYTIC SOLUTION

$$\Theta\left(\tau - \sum_{i=1}^{n} \tau_i\right) =$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{du}{u} e^{u\tau} \left(\prod_{i=1}^{n} e^{-u\tau_i} \right)$$

LAPLACE TRANSFORM

$u \in \mathbb{C}$

γ is parallel to imaginary axis, to the right of all singularities of the integrand

$$\Sigma(\tau) = \frac{e^{-R(\tau)}}{2\pi i} \int_\gamma \frac{du}{u}\, e^{u\tau}$$

$$e^{-R'(\tau)\ln 1/\omega} \sum_{n=0}^{\infty} \left(\frac{1}{n!} \prod_{i=1}^{n} \int_{\omega\tau}^{1} \frac{d\tau_i}{\tau_i} R'(\tau) \times e^{-u\tau_i} \right)$$

$$= \frac{e^{-R(\tau)}}{2\pi i} \int_\gamma \frac{du}{u}\, e^{u\tau}\, e^{\int_{\omega\tau}^{1} \frac{d\tau'}{\tau'} R'(\tau) \left(e^{-u\tau'} - 1\right)}$$

$$\omega \to 0$$

$$e^{-u\tau'} - 1 \simeq -\Theta\left(\tau' - \frac{u_0}{u}\right)$$

$$u_0 = e^{-\gamma_E}$$

$$= \frac{e^{-R(\tau)}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{du}{u}\, e^{u\tau}\, e^{-R'(\tau)\ln \frac{u}{u_0}}$$

$$= e^{-R(\tau)} \underbrace{\frac{e^{-\gamma_E R'(\tau)}}{\Gamma(1+R'(\tau))}}_{}$$

RESUMMED CROSS SECTION FOR THRUST

$$R'(\tau) \sim \alpha_s L + \alpha_s^2 L^2 + \dots$$

2) numerical solution

$$\Sigma(\tau) = e^{-R(\tau)}\, e^{-R'(\tau)\ln\frac{1}{\omega}}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\omega\tau} \frac{d\tau_i}{\tau_i} R'(\tau)\, \Theta\!\left(\tau - \sum_{i=1}^{n} \tau_i\right)$$

$$e^{-R'(\tau)\ln\frac{1}{\omega}} = e^{-R'(\tau)\ln\frac{\tau}{\tau_1}} \times e^{-R'(\tau)\ln\frac{\tau_1}{\tau_2}} \cdots e^{-R'(\tau)\ln\frac{\tau_{n-1}}{\tau_n}}$$

$$\tau_1 > \tau_2 > \ldots > \tau_n$$

REAL EMISSIONS (RESOLVED)

$$\sum_{n=0}^{\infty}\left(\int_{\omega\tau}^{\tau}\frac{d\tau_1}{\tau_1}R'(\tau) + \int_{\omega\tau}^{\tau}\frac{d\tau_1}{\tau_1}R'(\tau)\int_{\omega\tau}^{\tau_1}\frac{d\tau_2}{\tau_2}R'(\tau) + \ldots\right)\omega\left[\tau - \sum_{i=1}^{\infty}\tau_i\right)$$

$$\frac{d}{d\tau_i}\left(e^{-R'(\tau)\ln\frac{\tau_{i-1}}{\tau_i}}\right) \leftarrow$$

$$= e^{-R'(\tau) \ln \frac{\tau_{i-1}}{\tau_i}} R'(\tau) \frac{d\tau_i}{\tau_i}$$

ALGORITHM:

1) start with no emissions
$\tau_0 = \tau$

2) generate 1^{st} emission:

solve $e^{-\frac{R'}{\partial\tau} \ln \frac{\tau_0}{\tau_1}} = \underline{\text{random}} \in [0,1]$

for $\underline{\tau_1}$.

3) is $\underline{\tau_1 < \omega\tau}$
yes : stop
NO : $\underline{\text{repeat}}$

generate τ_2

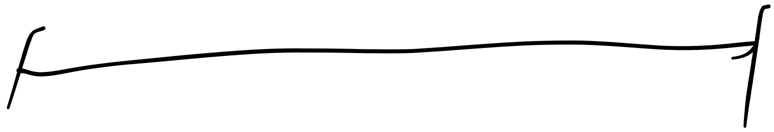
$$e^{-R'(\tau)\ln\frac{\tau_1}{\tau_2}} = \text{rand!}$$

for τ_2

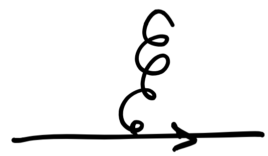
compute $\sum_{i=1}^{n} \tau_i$

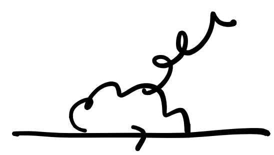
$< \tau$ accept

$> \tau$ reject

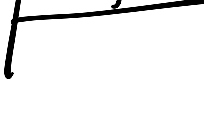


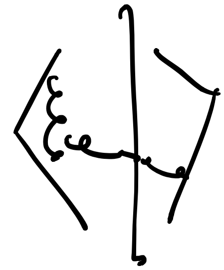
$$|M(h)|^2 = |M^{(0)}(h) + M^{(1)}(h) + \cdots|^2$$

$M^{(0)}(h) \to$ 

$M^{(1)}(h) \to$ 

$$= \frac{\alpha_s}{2\pi} \, 1PC^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2 1PC^{(1)} + \cdots$$

$1PC^{(0)} = \Big|$  $\Big|^2$

$1PC^{(1)} = 2\,\mathrm{Re}\Big[$  $\Big]$

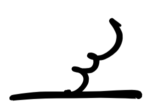
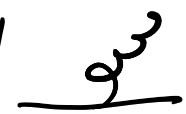
$\frac{1}{\varepsilon^4}$

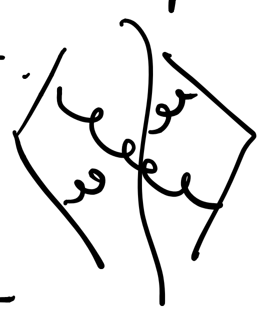
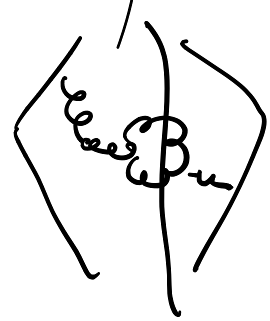
divergences

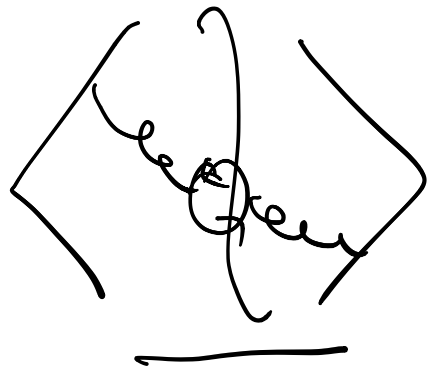
$(1PC^{(0)})^2$

$$\Big|M(k_1, k_2)\Big|^2 = |M(k_1)|^2\,|M(k_2)|^2$$

$$+\ |\tilde{M}(k_1, k_2)|^2$$

$O(\alpha_s^2) =$ |  $|^2$ |  $|^2$

$+\Big[$  $+$  $\Big]$

$+$  $]$

$$= 2PC^{(0)}\left(\frac{\alpha_0}{2\pi}\right)^2 + 2PC^{(1)}\left(\frac{\alpha_0}{2\pi}\right)^3 + \dots$$

$1PC^{(0)}$ finite

$1PC^{(1)}$ and $2PC^{(0)}$ is finite

$1+1$ $\qquad$ $2+0$

$n + j = \underline{\text{const}}$

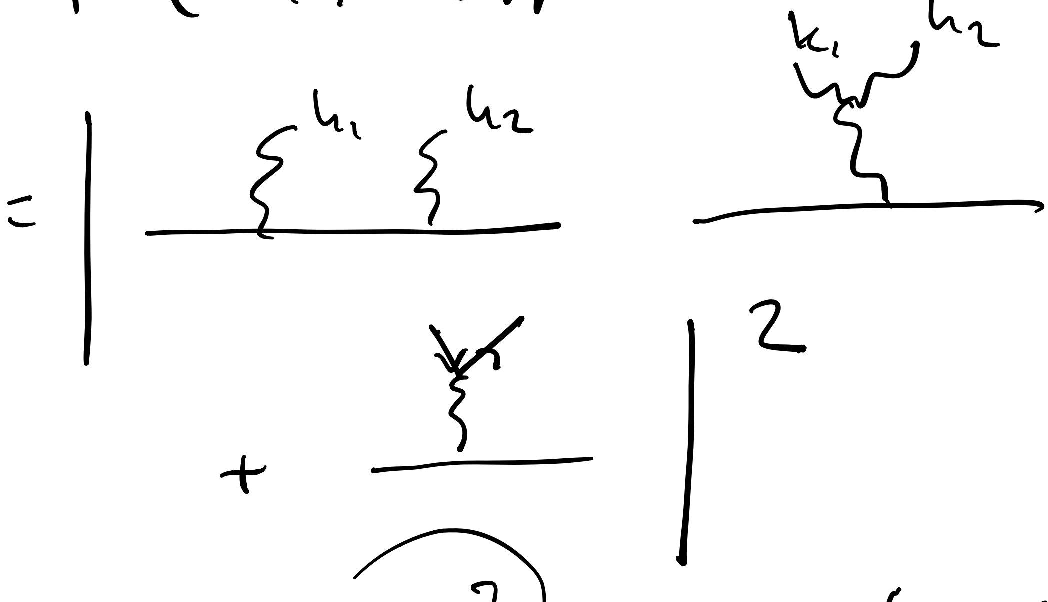
$nPC^{(j)}$

→ n is the number of emissions

→ $j+n$ is the order in α_s

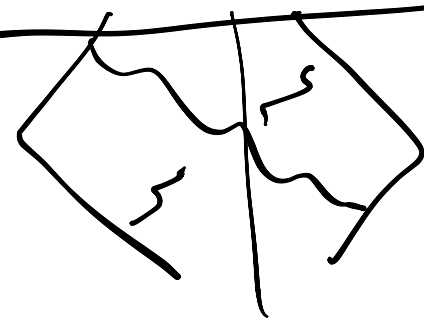
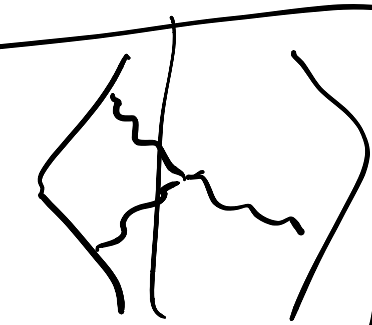
$$|M(k)|^2 = \frac{\alpha_s}{2\pi}\, IPC^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2 IPC^{(1)} + \dots$$

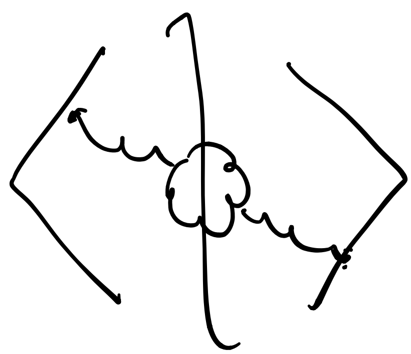
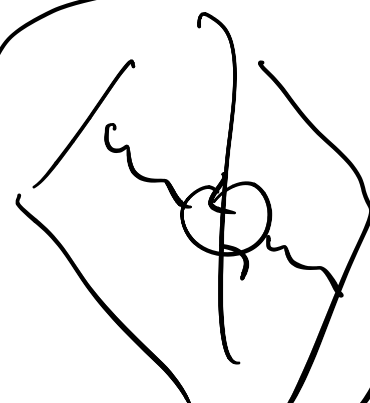
$$|M(k_1, k_2)|^2 =$$



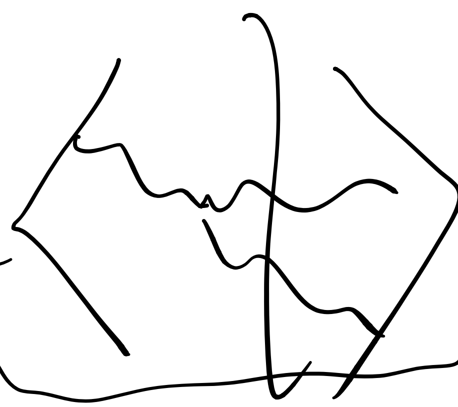
$$= \left(IPC^{(0)}\right)^2$$

= ] C_F^2 $\alpha_s^2 L^4$

+  + 

+  + 

$C_F C_A$ $C_F n_F$

+  + mirror

$\alpha_s^2 L^3$

$+ O(\alpha_s^3)$

$$= |M(k_1)|^2 |M(k_2)|^2$$

$$+ |\tilde{M}(k_1, k_2)|^2$$

$$|\underline{\tilde{M}}(k_1, k_2)|^2 = |M(k_1, k_2)|^2 - |M(k_1)|^2 |M(k_2)|^2$$

$$= \left(\frac{\alpha_s}{2\pi}\right)^2 2PC^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)^3 2PC^{(1)} + \cdots$$

$$1PC^{(0)} \to \alpha_s^n L^{2n} \to LL$$

$$2PC^{(0)}, 1PC^{(1)} \to \alpha_s^n L^{2n-1}$$

$$2\,PC^{(1)};\; 1PC^{(2)},\; 3PC^{(0)} \to \alpha_s^n L^{2n-2}$$

$$|M(h_1, \ldots, h_n)| =$$

$$= \prod_{i=1}^{m} \left(1PC^{(0)} \frac{\alpha_s}{2\pi} + 1PC^{(1)} \left(\frac{\alpha_s}{2\pi}\right)^2 + \right.$$

$$\left. \cdots \right)$$

$$+ \sum_{\substack{\text{perm.} \\ a,b}} \left(2PC^{(0)} \left(\frac{\alpha_s}{2\pi}\right)^2 + 2PC^{(1)} \left(\frac{\alpha_s}{2\pi}\right)^3 \right.$$

$$\left. + \cdots \right) \prod_{\substack{i=1 \\ i \neq a,b}}^{m} \left(1PC^{(0)} \frac{\alpha_s}{2\pi} \right.$$

$$\left. + 1PC^{(1)} \left(\frac{\alpha_s}{2\pi}\right)^2 + \cdots \right)$$

$n=1$