An Introduction to Resummation: LECTURE II

From now on we focus on global observables in $e^+e^- \to 2$ jets. (i.e. a 2-leg process). In particular, let us keep working with the Thrust event shape.

In the previous lecture we established that the squared amplitude for n emissions can be decomposed as

$$|M(p_1,p_2,k_1,\dots,k_n)|^2 \simeq |M(p_1,p_2)|^2 \Big\{ \overbrace{\prod_{i=1}^{n} |M(k_i)|^2}^{LL} +$$

$$+ \overbrace{\sum_{a>b} |\tilde{M}(k_a,k_b)|^2 \prod_{\substack{i=1\\ i\neq a,b}}^{n} |M(k_i)|^2}^{NLL} + \dots$$

$$\vdots$$

$$+ \overbrace{\sum_{a>b>c} |\tilde{M}(k_a,k_b,k_c)|^2 \prod_{\substack{i=1\\ i\neq a,b,c}}^{n} |M(k_i)|^2}^{NNLL} + \dots \Big\}$$

Each amplitude in this decomposition receives virtual corrections

$$M(k) = \overset{g_s}{M^{(0)}(k)} + \overset{g_s^3}{M^{(1)}(k)} + \dots$$

$$\tilde{M}(k_1,k_2) = \overset{g_s^2}{\tilde{M}^{(0)}(k_1,k_2)} + \overset{g_s^4}{\tilde{M}^{(1)}(k_1,k_2)} + \dots$$

$$\vdots$$

it is convenient to define a little notation to keep track of the various squared amplitudes above.

We introduce the concept of n-particle-correlated block $nPC^{(i)}$ of order i defined as follows

$$1PC = |M(k)|^2 = 1PC^{(0)}\frac{\alpha_s}{2\pi} + 1PC^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^2 + \dots$$

$$2PC = |\tilde{M}(k_1, k_2)|^2 = 2PC^{(0)}\left(\frac{\alpha_s}{2\pi}\right)^2 + 2PC^{(1)}\left(\frac{\alpha_s}{2\pi}\right)^3 + \dots$$

$$\vdots$$

$$nPC = |\tilde{M}(k_1, \dots, k_n)|^2 = \sum_{j=0}^{\infty}\left(\frac{\alpha_s}{2\pi}\right)^{n+j} nPC^{(j)}$$

with this notation we keep track of the order of virtual corrections as well.

We can summarize what we've learnt so far in the following table

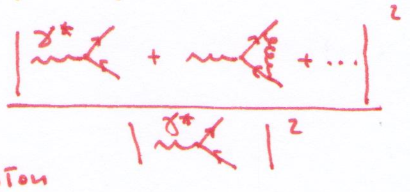
$nPC^{(j)}$	soft limit	hard-collinear limit
LL	$n+j \leq 1$	—
NLL	$n+j \leq 2$	$n+j \leq 1$
NNLL	$n+j \leq 3$	$n+j \leq 2$
N^kLL	$n+j \leq k+1$	$n+j \leq k$

Although we derived the squared amplitude decomposition in the soft limit, the same procedure can be applied ~~the~~ to the hard-collinear limit, where the squared amplitudes are simply obtained from the AP splitting kernels, after subtracting their soft contribution to avoid double counting. In this limit each correlated block looses one logarithmic power

In this lecture we apply the above decomposition to resum the thrust distribution. We work at the level of the cumulative cross section that we express as

$$\Sigma(\tau) = \frac{1}{\sigma_0}\underbrace{\int [d\Phi_B]\,|M_B(p_1,p_2)|^2}_{\text{Born PS and squared amplitude}}\,\underbrace{\mathcal{H}(Q^2)}_{\text{virtual form factor}} \times$$

virtual form factor:



Q^2: energy of the virtual photon

$$\times \sum_{n=0}^{\infty} \underbrace{\int \prod_{i=1}^{n} [dk_i]\,|M(k_1,\dots,k_n)|^2}_{\text{Real radiation phase space and matrix element}}\,\Theta\big(\tau - \underbrace{V(k_1,\dots,k_n)}_{\text{Observable as a function of the emissions' four-momenta}}\big)$$

Let's study all factors in more detail:

1) Born PS: the Born matrix element and phase space are completely factorised from the radiation phase space in a $e^+e^- \to 2$ jet event. Hence we can use

$$\int [d\Phi_B]\,|M_B(p_1,p_2)|^2 = \sigma_0 \to \text{Born cross section}$$

2) Real radiation:

For the sake of clarity, we can work in the independent emission approximation, that is within the first term of the squared-amplitude decomposition. In particular, we consider $IPC^{(0)}$ terms as defined above.

By looking at the table with the logarithmic counting, we see that this is not enough for a full NLL resummation, for which all blocks with $n+j \leq 2$ must be included.

We thus write (remember Sudakov parametrization)

$$\prod_{i=1}^{n} [dk_n] \, |M(k_1, \ldots, k_n)|^2 \simeq \frac{1}{n!} \prod_{i=1}^{n} \left(\sum_{\ell=1}^{2} 2 \frac{\alpha_s(\mu)}{2\pi} \frac{dk_t}{k_t} \frac{dz^{(\ell)}}{z^{(\ell)}} z^{(\ell)} P_{gq}(z^{(\ell)}) \right)$$

with $P_{gq}(z) = C_F \frac{1+(1-z)^2 - \varepsilon z^2}{z} \xrightarrow[4D]{} C_F \frac{1+(1-z)^2}{z}$

$\simeq C_F \frac{2}{z}$ in the soft limit

it reproduces the eikonal factor for $z \to 0$. Both soft and collinear limits encoded in the same splitting function.

2.a) scale (μ) of the running coupling and logarithmic order:

the scale of the coupling μ can be set by recalling the definition of LL we gave at the beginning of the first lecture.

We want to have control over all terms of the form

$$\boxed{\ln \Sigma \sim \alpha_s^n \ln^{n+1} \frac{1}{\tau}}$$

however, we have seen that the $1PC^{(0)}$ terms control contributions of the type

$$\Sigma \sim \alpha_s^n \ln^{2n} 1/\tau$$

and the combination of $1PC^{(1)}$ and $2PC^{(0)}$ ~~with~~ starts at order

pure soft limit $\longrightarrow$ $\Sigma \sim \alpha_s^n \ln^{n+1} 1/\tau$ with $n = 2$

Since we are neglecting these terms, we are not able to get a full LL with the above parametrization for the amplitude. However, one can fix the scale of the running coupling in such a way that the $\alpha^2 L^3$ terms are absorbed into the independent-emission picture, ~~[illegible]~~ In this way the remaining contribution from the $1PC^{(1)}$ and $2PC^{(0)}$ blocks will be $\alpha_s^n L^n$, i.e. NLL in the exponential counting.

To see if this is possible we compute in the soft limit

$$\int [dk] \Big\{ |M^{(0)}(k) + M^{(1)}(k)|^2 + \int [dk_a][dk_b] |\tilde{M}^{(0)}(k_a, k_b)|^2 \times$$

$$\times \underline{\delta^{(2)}(\vec{k}_t - \vec{k}_{ta} - \vec{k}_{tb})\, \delta(Y - Y_{ab})} \Big\}$$

$\uparrow$

I exploit the fact that the exclusive treatment of the $|\tilde{M}^{(0)}(k_a, k_b)|^2$ term enters at NNLL, therefore I can treat it inclusively at LL.

The physical picture is an ensemble of gluon emissions inclusively integrated over the subsequent branchings

$$= \int [dk] \frac{\alpha_s(\mu)}{2\pi} \, \mathbb{P}C^{(0)}(k) \left[1 + \alpha_s(\mu) \left(\beta_0 \ln \frac{k_t^2}{\mu^2} + \frac{K}{2\pi} \right) + \ldots \right]$$

where $\beta_0 = \frac{11 C_A - 2 n_F}{12\pi}$, and $K = \left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} n_F$

the contribution of K is NLL $(\alpha_s^2 L^2)$ as it can be ~~seen~~ by integrating over $z^{(e)}$ and k_t with the thrust measurement function. The $\alpha_s^2 L^3$ term can be absorbed into the $\mathbb{P}C^{(0)}(k)$ block by simply choosing

$$\boxed{\mu^2 = k_t^2}$$

therefore the real matrix element squared becomes (including phase space factors)

$$\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{\ell=1}^{2} 2C_F\frac{\alpha_s(k_t)}{\pi}\frac{dk_t}{k_t}\frac{dz^{(\ell)}}{z^{(\ell)}}\right)$$

which is LL accurate in the exponential counting.

It is instructive to perform the full calculation in the soft approximation, that is LL accurate. However, the systematic inclusion of the hard-collinear limit as well as of higher-multiplicity blocks allows for the extension to any logarithmic order.

3) virtual corrections: $\mathcal{V}(Q^2)$

Since we are performing a LL resummation, we are expected to include all terms with $n+j\leq 1$ in the above table. For purely virtual corrections ($n=0$) this clearly means that we need the one-loop form factor in the soft approximation (i.e. eikonal Feynman rules)

This can be easily computed, however we can use a simpler unitarity argument to bootstrap it.

From the KLN theorem we know that the IRC singularities cancel between real and virtual corrections. Given that the finite terms of the form factor contribute at NNLL (e.g. $\alpha_s(Q)\times C$), we're only interested in the singular structure at LL.

This is identical to the one of the real radiation so that I can express the all-order form factor as

$$\mathcal{V}(Q^2) \overset{LL}{\simeq} \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\left(-\int[dk_i]\,|M_s^{(0)}(k_i)|^2\right) = e^{-\int[dk]\,\underbrace{|M_s^{(0)}(k)|^2}}$$

this ensures cancellation of uncertainties.

Soft limit of the $IRC^{(0)}$ block in D dimensions.

In 4 dimensions it becomes

$$\mathcal{H}(Q^2) \overset{LL}{\simeq} e^{-\sum_{\ell=1}^{2}\int^{Q} \frac{dk_t}{k_t}\frac{dz^{(\ell)}}{z^{(\ell)}} 2 C_F \frac{\alpha_s(k_t)}{\pi}}$$

(arXiv: 0805.3515)
(see Dixon, Magnea, Sterman)

Althoug the above integral is IRC-divergent, the singularities will cancel against the ones in the real radiation, as we will see in a while.

Putting everything together we have

$$\Sigma(\tau) = e^{-\sum_{\ell=1}^{2}\int^{Q} \frac{dk_t}{k_t}\frac{dz^{(\ell)}}{z^{(\ell)}} 2 C_F \frac{\alpha_s(k_t)}{\pi}} \times \sum_{n=0}^{\infty}\left(\frac{1}{n!}\prod_{i=1}^{n}\sum_{\ell=1}^{2}\int^{Q} \frac{dk_{ti}}{k_{ti}}\frac{dz_i^{(\ell)}}{z_i^{(\ell)}} 2\frac{\alpha_s(k_{ti})}{\pi} C_F\right) \times$$

$$\times\, \Theta(\tau - V(k_1, \ldots, k_n)), \quad \text{with } 1 \geq z_i^{(\ell)} \geq \frac{k_{ti}}{Q}$$

The last step is to cancel the divergences of the real and virtual corrections as $k_t \to 0$ and $z \to 0$.

One efficient way to do this is to introduce a slicing parameter $\omega\tau$, such that in the limit $\omega \to 0$ all emissions with $V(k_i) < \omega\tau$ can be ignored in the calculation of the observable, i.e.

$$\left|\begin{array}{l} V(k_1, \ldots, k_n) \simeq V(k_1, \ldots, k_{n-1}) + \omega^p \tau, \quad p>0 \\ \text{if } V(k_n) < \omega\tau \end{array}\right.$$

I can therefore rewrite the above expression as

$$\Sigma(\tau) = e^{-\sum_{\ell=1}^{2}\int^{Q} \frac{dk_t}{k_t}\frac{dz^{(\ell)}}{z^{(\ell)}} 2 C_F \frac{\alpha(k_t)}{\pi}\left(1-\Theta(\omega\tau - V(k))\right)}$$

unresolved emissions and virtual corrections

only resolved emissions

$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{\ell=1}^{2}\int^{Q}\frac{dk_{ti}}{k_{ti}}\frac{dz_i^{(\ell)}}{z_i^{(\ell)}} 2\frac{\alpha_s(k_{ti})}{\pi} C_F\right)\Theta(V(k_i) - \omega\tau)\Big)$$

$$\times\, \Theta(\tau - V(k_1, \ldots, k_n))$$

The ω dependence cancels exactly in the $\omega \to 0$ limit.

the above equation looks quite familiar. I simplify the notation and introduce

$$R(\tau) = \sum_{\ell=1}^{2} \int^{Q} \frac{dk_t}{k_t} \int_{k_t/Q}^{1} \frac{dz^{(\ell)}}{z^{(\ell)}}\, 2C_F \frac{\alpha_s(k_T)}{\pi}\, \Theta(V(k) - \tau)$$

SUDAKOV RADIATOR

(give formula here; 4 pages ahead)

moreover I introduce a convenient parametrization for the real emissions

$$R'(\tau) = \tau \sum_{\ell=1}^{2} \int^{Q} \frac{dk_t}{k_t} \int_{k_t/Q}^{1} \frac{dz^{(\ell)}}{z^{(\ell)}}\, 2C_F \frac{\alpha_s(k_t)}{\pi}\, \delta(\tau - V(k)) = \frac{dR(\tau)}{d\ln\frac{1}{\tau}}$$

Finally, I use the fact that in the soft limit we're interested in the expression of the observable reads (explain why)

$$V(k_1, \dots, k_n) = \sum_{\substack{i=1\\ i\in \mathcal{H}_p}}^{n} \frac{k_{ti}^2}{Q^2 z_i^{(1)}} + \sum_{\substack{i=1\\ i\in \mathcal{H}_{\bar p}}}^{n} \frac{k_{ti}^2}{Q^2 z_i^{(2)}} + \underbrace{\dots}$$

the recoil of the Born legs is negligible in the soft limit.

$$= \sum_{i=1}^{n} V(k_i) = \sum_{i=1}^{n} \tau_i$$

where $V(k_i)$ is the Thrust computed on a single emission in its soft-collinear limit.

The above formula for $\Sigma(\tau)$ can be expressed as

$$\Sigma(\tau) = e^{-R(\omega\tau)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{\omega\tau}^{1} \frac{d\tau_i}{\tau_i} R'(\tau_i)\, \Theta\Big(\tau - \sum_{i=1}^{n} \tau_i\Big)$$

this is a (one-dimensional) parton-shower equation!

The evolution variable is τ_i. We introduce the Sudakov that describe the no-emission probability between τ_i and $\tau_{i+1} < \tau_i$

$$\Delta(\tau_i, \tau_{i+1}) = \frac{e^{-R(\tau_i)}}{e^{-R(\tau_{i+1})}}$$

so that the above equation becomes ($\tau_{max} = 1$)

$$\Sigma(\tau) = \Bigg(\Delta(1, w\tau) + \int_{w\tau}^{1} \frac{d\tau_1}{\tau_1} \Delta(1, \tau_1) R'(\tau_1) \Delta(\tau_1, w\tau) +$$

$$+ \int_{w\tau}^{1} \frac{d\tau_1}{\tau_1} \int_{w\tau}^{\tau_1} \frac{d\tau_2}{\tau_2} \Delta(1, \tau_1) R'(\tau_1) \Delta(\tau_1, \tau_2) R'(\tau_2) \Delta(\tau_2, w\tau)$$

$$+ \dots \Bigg) \Theta\Big(\tau - \sum_{i=1}^{n} \tau_i\Big),$$

where I traded the $1/n!$ for a τ_i ordering.

This equation can be solved with the usual shower algorithm (see Höche's lectures).

In the rest of this lecture we show how to solve the above equation both numerically and analytically, and compare the two solutions.

Small comment on angular ordering:

In the derivation we noticed that we can integrate over the azimuthal angles inclusively. This ensures that the radiation dynamics follows an angular-ordered evolution. Therefore, a parton shower built within the coherent (angular ordered) branching algorithm is at least LL accurate in this observable.

The above shower equation can be solved as is using the usual parton-shower (simplified) algorithm.
However, due to the presence of the Θ function many events are lost and this leads to a quite inefficient numerical evaluation.

It is useful to make a few operations in order to make the computation more efficient (faster).

First of all we can simplify the integrand and the Sudakov radiator by noticing that

$$R(w\tau) = R(\tau) + R'(\tau)\ln\frac{1}{w} + \underbrace{\frac{1}{2!}R''(\tau)\ln^2 1/w}_{\text{NNLL}} + \dots$$

$$R'(\tau_i) = R'(\tau) + \overbrace{R''(\tau)\ln\frac{\tau}{\tau_i}}^{\text{NNLL}} + \dots$$

at the accuracy we are working with, one can neglect terms proportional to $R''(\tau)$. Therefore the cross section becomes

$$\Sigma(\tau) = e^{-R(\tau)}\, w^{R'(\tau)} \sum_{n=0}^{\infty} \frac{1}{M!} \prod_{i=1}^{M} \int_{w\tau}^{1} \frac{d\tau_i}{\tau_i} R'(\tau)\, \Theta\Big(\tau - \sum_{i=1}^{M} \tau_i\Big) \qquad \text{(B)}$$

We start by showing how to solve the above equation in closed form in the case of thrust.

ANALYTIC SOLUTION

The shower equation can be evaluated analytically if we can find a factorized expression for the observable, i.e. expressed as a product of contributions coming from different emissions. This gives rise to a factorization formula for the resummed cross section

In the case of Thrust, however, the measurement function is additive (instead of multiplicative)

$$\Theta\left(\tau - \sum_{i=1}^{n} \tau_i\right)$$

Nevertheless, this can be transformed into a product by taking a Laplace Transform

$$\Theta\left(\tau - \sum_{i=1}^{n} \tau_i\right) = \frac{1}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\tau}\left(\prod_{i=1}^{n} e^{-\nu\tau_i}\right)$$

which allows me to write the cross section as

$$\Sigma(\tau) = \frac{e^{-R(\tau)}}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\tau}\, \omega^{R'(\tau)} \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int_{\omega\tau}^{1}\frac{d\tau_i}{\tau_i} R'(\tau)\, e^{-\nu\tau_i}$$

$$= \frac{e^{-R(\tau)}}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\tau}\, e^{\int_{\boxed{\omega\tau}}^{1}\frac{d\tau'}{\tau'} R'(\tau)\underbrace{\left(e^{-\nu\tau'}-1\right)}_{-\Theta\left(\tau - \frac{\nu_0}{\nu}\right),\ \nu_0 = e^{-\gamma_E}}}$$

now I can take the $\omega \to 0$ limit

$$= \frac{e^{-R(\tau)}}{2\pi i}\int_{-i\infty}^{i\infty} \frac{d\nu}{\nu}\, e^{\nu\tau}\, e^{-R'(\tau)\ln\frac{\nu}{\nu_0}} = \boxed{e^{-R(\tau)}\,\frac{e^{-\gamma_E R'(\tau)}}{\Gamma(1+R'(\tau))}}$$

NUMERICAL SOLUTION

In order to solve eq (B) numerically, we write it as follows

$$\Sigma(\tau) = e^{-R(\tau)} \mathcal{F}(\tau)$$

where $\mathcal{F}(\tau) = \omega^{R'(\tau)} \sum_{n=0}^{\infty} \frac{1}{M!} \prod_{i=1}^{M} \int_{\omega\tau}^{1} \frac{d\tau_i}{\tau_i} R'(\tau)\, \Theta(\tau - \sum_{i=1}^{m} \tau_i)$

R is computable analytically:

$$R(\tau) = \sum_{\ell=1}^{2} \int_0^Q \frac{dk_t}{k_t} \int_{k_t/Q}^{1} \frac{dz^{(\ell)}}{z^{(\ell)}}\, 2C_F \frac{\alpha_s(k_t)}{\pi}\, \Theta\left(\frac{k_t^2}{Q^2 z^{(\ell)}} - \tau\right)$$

(exercise)

$$= 2C_F \left\{ \frac{(1-2\lambda)\ln(1-2\lambda) - 2(1-\lambda)\ln(1-\lambda)}{2\pi\beta_0\lambda} \right\}$$

use $\alpha_s(k_t) = \dfrac{\alpha_s(Q)}{1-\alpha_s(Q)\beta_0 \ln \frac{Q^2}{k_t^2}}$ with $\lambda = \alpha_s(Q)\beta_0 \ln \frac{1}{\tau}$.

notice that $R(\tau)$ has a singularity at $\lambda = 1/2 \rightarrow$ Landau Pole

$\mathcal{F}(\tau)$ can be obtained using the simple MC algorithm

1) start from $\tau_0 = \tau$
2) generate first emission with weight $R'(\tau)\, e^{-R'(\tau)\ln 1/\tau_1}$
3) generate subsequent emission by solving $e^{-R'(\tau)\ln\frac{\tau_i}{\tau_{i+1}}}$ = random for τ_{i+1}
4) if $\tau_i < \omega\tau$ exit
5) accept the event if $\sum_i \tau_i < \tau$

(see mathematica notebook)

Point 5) of the above algorithm, however, is quite inefficient. A lot of MC events will be lost since they have $\sum_i \tau_i > \tau$.

It is possible to modify the starting equation in order to make the numerical evaluation more efficient, i.e. in order to accept all events.

Since the origin of the problem is the Θ function, we want to find a formulation where it disappears.

One possible way to do it is to notice that in our master formula, events with <u>no emissions</u> are infinitely suppressed

$$e^{-R(\omega\tau)} \xrightarrow[\omega \to 0]{} 0$$

Therefore, we can repeat the derivation of the master formula by starting from an event with a single emission k_1. The slicing scale is therefore replaced as follows

$$\omega\tau \longrightarrow \omega\tau_1$$

i.e. the first emission is always resolved.

The rest follows by generating all subsequent emissions with $\tau_i < \tau_1$ and obtain the new equation

$$\Sigma(\tau) = \int_0 \frac{d\tau_1}{\tau_1} e^{-R(\omega\tau_1)} R'(\tau_1) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_{\omega\tau_1}^{\tau_1} \frac{d\tau_i}{\tau_i} R'(\tau_i)$$
$$\times \Theta\left(\tau - \sum_{i=1}^{n+1} \tau_i\right)$$

as before we can expand R and R' about τ, neglecting NNLL corrections

$$R(\omega\tau_1) = R(\tau) + R'(\tau) \ln \frac{\tau}{\omega\tau_1} + \cdots$$

$$R'(\tau_i) = R'(\tau) + R''(\tau) \ln \tau/\tau_i + \cdots$$

which results in

$$\Sigma(\tau) = \int \frac{d\tau_1}{\tau_1} e^{-R(\tau)} \left(\frac{\omega \tau_1}{\tau}\right)^{R'(\tau)} R'(\tau) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=2}^{n+1} \int_{\omega\tau_1}^{\tau_1} \frac{d\tau_i}{\tau_i} R'(\tau) \times$$
$$\times\, \Theta\left(\tau - \sum_{i=1}^{n+1} \tau_i\right)$$

I define $\zeta_i = \tau_i / \tau_1$

$$\Sigma(\tau) = \int \frac{d\tau_1}{\tau_1} e^{-R(\tau)} \left(\frac{\omega \tau_1}{\tau}\right)^{R'(\tau)} R'(\tau) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=2}^{n+1} \int_{\omega}^{1} \frac{d\zeta_i}{\zeta_i} R'(\tau)$$
$$\times\, \Theta\left(\tau - \tau_1 \left(1 + \sum_{i=2}^{n+1} \zeta_i\right)\right)$$

and integrate analytically over τ_1 to find

$$\Sigma(\tau) = e^{-R(\tau)}\, \omega^{R'(\tau)} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=2}^{n+1} \int_{\omega}^{1} \frac{d\zeta_i}{\zeta_i} R'(\tau)$$
$$\times\, e^{-R'(\tau) \ln\left(1 + \sum_{i=2}^{n+1} \zeta_i\right)}$$

which can be solved with the following, more efficient, MC

1) start from $\zeta_1 = 1$
2) generate ζ_{i+1} by solving $e^{-R'(\tau) \ln \zeta_i / \zeta_{i+1}} = \text{random}$ for ζ_{i+1}
3) if $\zeta_{i+1} < \omega$ exit
4) reweight the final event by $e^{-R'(\tau) \ln\left(1 + \sum_{i=2}^{n+1} \zeta_i\right)}$

(see mathematica notebook for comparisons)

Therefore, concluding, if we can find a factorized expression for the observable then we can aim at an analytic solution, while we have to rely on the numerical simulation in the general case.

The above formulation can be extended at all logarithmic orders for "well behaved" observables that fulfill the recursive IRC safety criteria.

For observables of the factorizing type, it is often possible to build an effective-field theory approach to separate the singular modes (i.e. soft and collinear) from the hard ones at the Lagrangian level.

This will be the subject of the next lecture.