A new method to calculate Feynman integrals

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Based on:

X. Liu, Y.Q. Ma, C. Y. Wang, Phys.Lett. B779 (2018) 353

X. Liu, Y.Q. Ma, arXiv:1801.10523

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Outline

I. Introduction

II. Series representation

III. Reduction

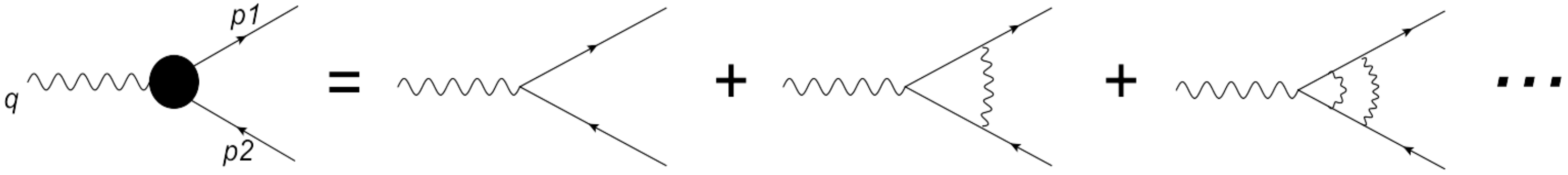
III. Analytical continuation for MI

IV. Summary

High precision calculation

- Test for SM
- Search for signals of new physics

Feynman loop integrals need to be considered



Methods for calculating Feynman integrals

➢ Direct Calculation:

- Sector decomposition Binoth, Heinrich, 0004013
- Mellin-Barnes representation Usyukina (1975) Smirnov, 9905323

➢ Indirect Calculation: 1) reducing to master integrals; 2) calculating MIs

1) integration-by-parts (IBP) reduction, unitarity-based reduction, …

2) difference equation, differential equation (DE), …

Chetyrkin, Tkachov, NPB (1981)
Laporta, 0102033
Kotikov, PLB (1991)
…

Where is the prospect?

➢ Examples

- For two-loop $p + p \to H + H$: complete reduction cannot be achieved within tolerable time Borowka et. al., 1604.06447
- For four-loop nonplanar cusp anomalous dimension, achieved numerical result with 10% uncertainty within tolerable computational expense Boels, Huber, Yang, 1705.03444
- …

New ideas are badly needed to give a better solution!!!

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Feynman integrals with an auxiliary variable

➢ **Dimensionally regularized Feynman integral**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

- Think of it as an analytical function of η
- Physical result is defined by

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

➢ Rescaling $\ell \to \eta^{1/2}\tilde{\ell}$

$$\frac{1}{(\ell+p)^2 - m^2 + \mathrm{i}\eta} = \eta^{-1} \times \frac{1}{(\tilde{\ell}+\eta^{-1/2}p)^2 - \eta^{-1}m^2 + \mathrm{i}}$$

- **Small quantities:** $\frac{\tilde{\ell}\cdot p}{\sqrt{\eta}}, \frac{p^2}{\eta}, \frac{m^2}{\eta}$

➢ Taylor Expansion

$$\frac{1}{(\tilde{\ell}+\eta^{-1/2}p)^2 - \eta^{-1}m^2 + \mathrm{i}} = \frac{1}{\tilde{\ell}^2 + \mathrm{i}} \sum_{j=0}^{\infty} \left\{ -\frac{2\eta^{-1/2}\tilde{\ell}\cdot p + \eta^{-1}(p^2 - m^2)}{\tilde{\ell}^2 + \mathrm{i}} \right\}^j$$

Vacuum propagators with equal squared masses.

Series representation

➢ **Asymptotic series near $\eta = \infty$**

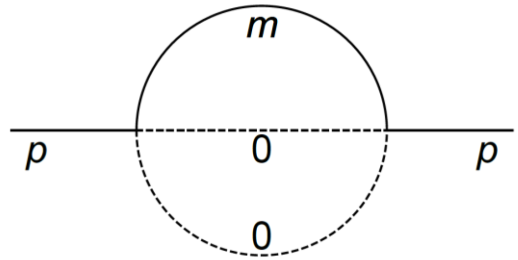
$$\mathcal{M}(D, \vec{s}, \eta) = \eta^{LD/2-\nu} \sum_{k=0}^{\infty} \eta^{-k} \mathcal{M}_k^{\text{vac}}(D, \vec{s})$$

- **The coefficients contain only vacuum integrals, easier to calculate**

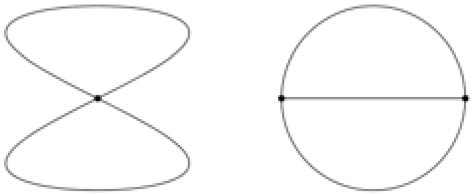
➢ **Obtain Feynman integrals by analytical continuation (next two sections)**

Example: sunrise diagram

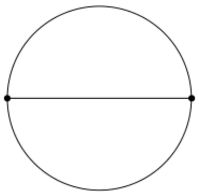
$$\mathcal{M}(D, \{p^2, m^2\}, \eta) =$$



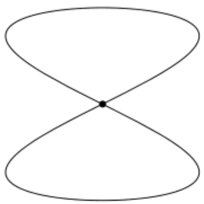
Vacuum MIs:



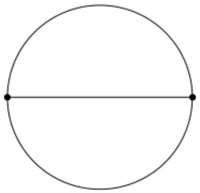
$$\mathcal{M}_0^{\text{vac}} =$$



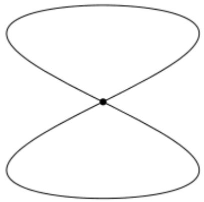
$$\mathcal{M}_1^{\text{vac}} = -\frac{(D-2)^2 p^2}{3D} \times$$



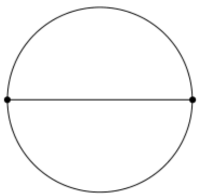
$$+\frac{\mathrm{i}(D-3)\left(3Dm^2 - Dp^2 - 4p^2\right)}{9D} \times$$



$$\mathcal{M}_2^{\text{vac}} = \cdots$$



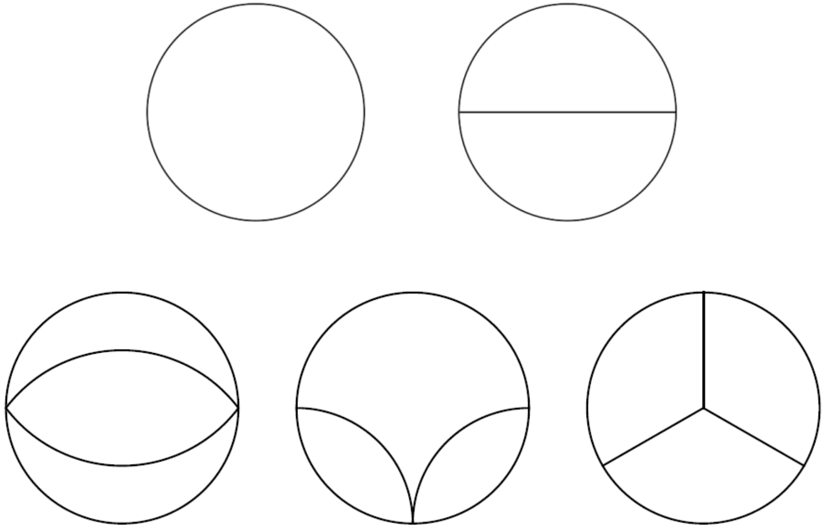
$$+ \cdots$$



Vacuum master integrals

- Vacuum MIs up to 3-loop, analytical results are known

Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136



Numerical results are known up to 5-loop order!!!

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

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Reduction relation

➢ Example: a simple differential equation

$$\mathcal{M}(D, m^2, \eta) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{\ell^2 - m^2 + \mathrm{i}\eta}$$

$$\frac{\partial}{\partial \eta} \mathcal{M}(D, m^2, \eta) \overset{\text{IBP}}{=} \frac{(D-2)}{2\,(\eta + \mathrm{i}m^2)} \mathcal{M}(D, m^2, \eta)$$

- **Series Rep:** $\mathcal{M}(D, m^2, \eta) = \eta^{D/2-1} \sum_{k=0}^{\infty} \eta^{-k} \mathcal{M}_k^{\text{vac}}$

➢ Recurrence relation for M_k^{vac}

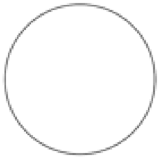
$$\mathcal{M}_{k+1}^{\text{vac}} = \frac{\mathrm{i}m^2(D-2k-2)}{2(k+1)} \mathcal{M}_k^{\text{vac}}, \quad k = 0, 1, 2, \ldots.$$

- **Construct M_k^{vac} from M_0^{vac}**

Reduction relation

- M_k^{vac} from series representation

$$\mathcal{M}_k^{\mathrm{vac}} = \int \frac{\mathrm{d}^D\tilde{\ell}}{\mathrm{i}\pi^{D/2}} \frac{(m^2)^k}{(\tilde{\ell}^2+\mathrm{i})^{k+1}} = (-\mathrm{i}m^2)^k \frac{(1-D/2)_k}{k!} \times$$



- The results should be self-consistent

$$\mathcal{M}_0^{\mathrm{vac}} \quad \stackrel{\mathrm{DE}}{\Longrightarrow} \quad \mathcal{M}_k^{\mathrm{vac}}$$

$$\mathcal{M}(D, m^2, \eta) \quad \stackrel{\mathrm{SR}}{\Longrightarrow} \quad \mathcal{M}_k^{\mathrm{vac}}$$

The information of DE is hidden in SR!

Obtain reduction relation from SR

➢ Unknown DE

$$\frac{\partial}{\partial \eta}\mathcal{M}(D,m^2,\eta) = \frac{Q_1^{00}}{Q_2^{10}\eta + Q_2^{01}m^2}\mathcal{M}(D,m^2,\eta)$$

- $Q_1^{00}, Q_2^{10}, Q_2^{01}$ are unknown functions of D

➢ Linear equations for Qs

$$\left[Q_2^{10}(D-2) - 2Q_1^{00}\right]\mathcal{M}_0^{\text{vac}} = 0$$

$$\left[Q_2^{10}(D-4-2k) - 2Q_1^{00}\right]\mathcal{M}_{k+1}^{\text{vac}} + Q_2^{01}m^2(D-2k-2)\mathcal{M}_k^{\text{vac}} = 0, \quad k = 0,1,2,\ldots$$

$$\boxed{\mathcal{M}_k^{\text{vac}} = (-\mathrm{i}m^2)^k \frac{(1-D/2)_k}{k!} \times \bigcirc}$$

- **Solution:** $Q_2^{10} = \frac{2Q_1^{00}}{D-2}, Q_2^{01} = \frac{2iQ_1^{00}}{D-2}$

General Discussions

- **The Number of Master Integrals is Finite**

Smirnov, Petukhov, 1004.4199
Georgoudis, Larsen, Zhang, 1612.04252

  - Feynman integrals form a finite dimensional linear space

- **For n-dim space, $\{M_1, M_2, \dots, M_{n+1}\}$ must be linear dependent**

$$\sum_{i=1}^{n+1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

  - Q_i: unknown polynomials of $\vec{s}, \eta$

General Discussions

➢ **Decompose Q_i**

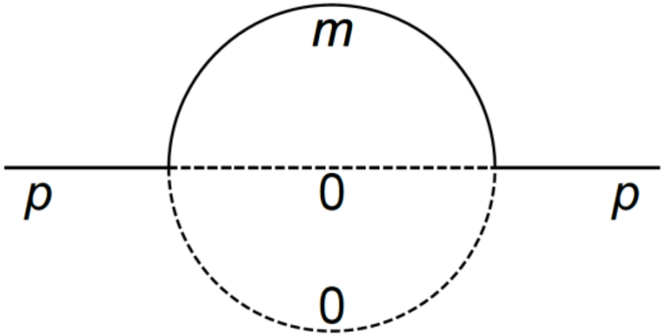
$$Q_i(D,\vec{s},\eta) = \sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\ldots\lambda_r}(D)\,\eta^{\lambda_0}\,s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

➢ **Obtain enough linear equations through SR to solve** $Q_i^{\lambda_0\ldots\lambda_r}(D)$

➢ **Analytical relation valid on the whole η plane**

$$\sum_{i=1}^{n+1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

➢ Reduction of $I_{\nu 11}$



- ν is the power of massive propagator

	Our reduction		FIRE5	
ν	Time/s	# of relations	Time/s	# of relations
5	0.14	14	12	203
10	0.33	34	42	1313
15	0.48	54	346	4073
20	0.75	74	2169	9233
100	5.43	394	-	-

- The new method needs much less relations than IBP method (FIRE5), and thus much faster

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Analytical continuation for MIs

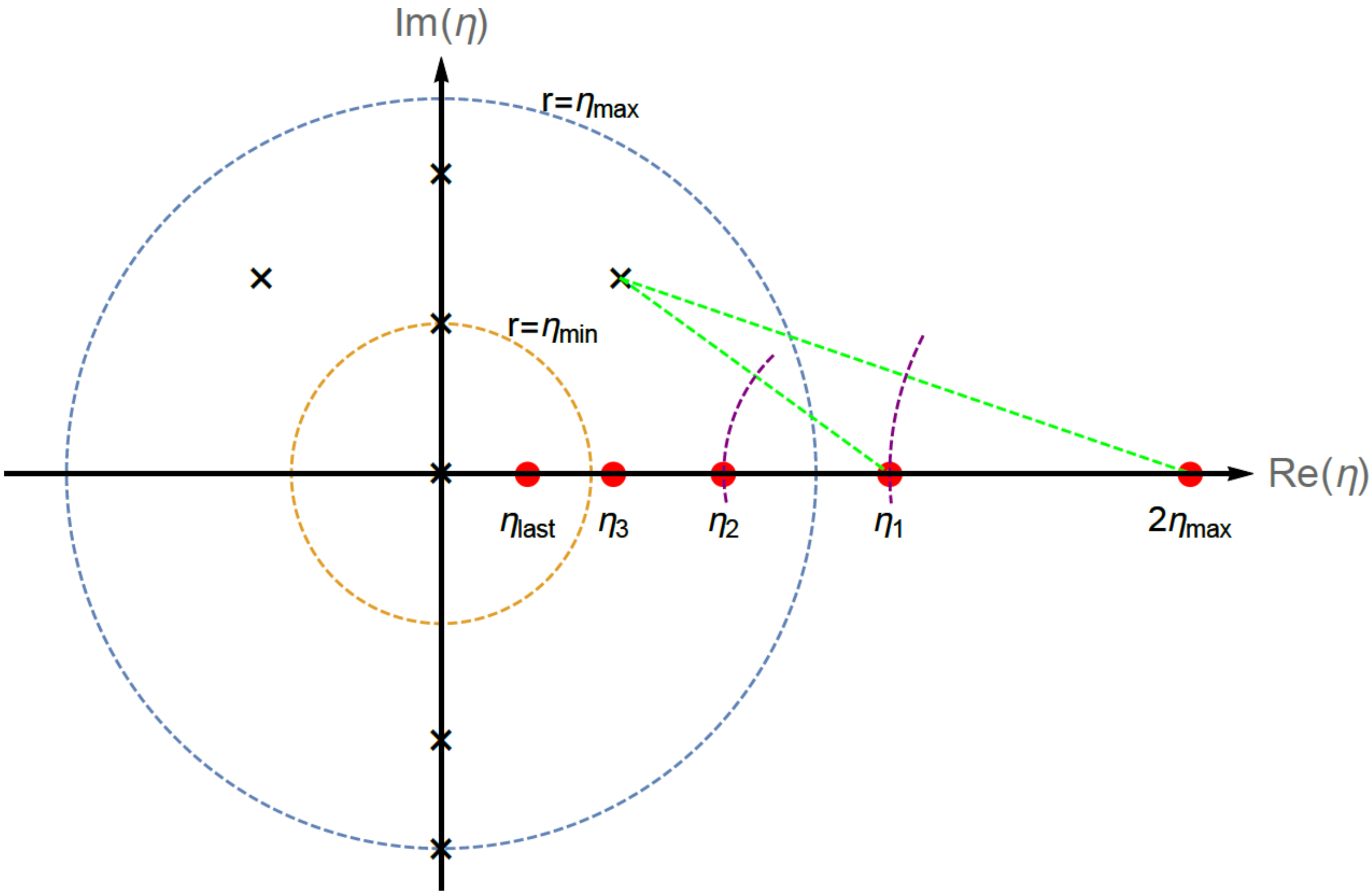
- **ODE for MIs**

$$\frac{\partial}{\partial\eta}\vec{I}(D,\vec{s},\eta) = A(D,\vec{s},\eta)\vec{I}(D,\vec{s},\eta)$$

- **Boundary conditions at $\eta = \infty$: leading term of the series representation, known**

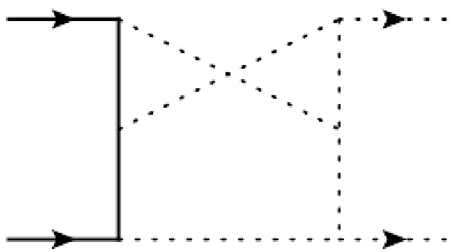
Numerically solving ODE – a well-studied problem

Solving ODE



2-Loop Test

- **Test for 2-loop non-planar integral**



with $\mathrm{m}^2 = 1, \mathrm{s} = 4, \mathrm{t} = -1$

- **ODE with 168 master integrals**

$$
\begin{aligned}
I_{\rm np}(4-2\epsilon) &= 0.0520833\epsilon^{-4} - (0.131616 - 0.147262\mathrm{i})\epsilon^{-3} \\
&- (0.741857 + 0.185602\mathrm{i})\epsilon^{-2} + (3.73984 - 4.15756\mathrm{i})\epsilon^{-1} \\
&- (4.75677 - 12.0749\mathrm{i}) + (23.9674 - 55.4214\mathrm{i})\epsilon + \cdots ,
\end{aligned}
$$

- **It takes a few minutes**
- **To compare with, FIESTA: $O(10^4)$ CPU core-hour**

Summary

- The series representation of Feynman integrals: a calculable series, which can be analytical continued to the origin

1) Construct the series representation
2) Set up reduction relations
3) Analytical continuation by solving DEs

Thank you!

Analytic structure at infinity

➢ Feynman parametric rep.

$$I(\eta)=(-1)^{\nu}\frac{\Gamma\left(\nu-LD/2\right)}{\prod_i\Gamma(\nu_i)}\int\prod_{\alpha}(x_{\alpha}^{\nu_{\alpha}-1}\mathrm{d}x_{\alpha})\,\delta\left(1-\sum_j x_j\right)\frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U}-\mathrm{i}\eta)^{\nu-LD/2}}$$

- $\mathcal{U}$: graph polynomial of 1-tree
- $\mathcal{F}$: graph polynomial of 2-tree

➢ Observation: $|\mathcal{F}/\mathcal{U}|$ is bounded in the Feynman parameter space!

$$|\mathcal{F}_i|<|t_i||\mathcal{U}_i|<|t_i||\mathcal{U}|\ \text{and}\ |\mathcal{F}|<\textstyle\sum_i|t_i||\mathcal{U}|$$

➢ Thus: $J(D;\eta)\equiv\eta^{\nu-LD/2}I(D;\eta)$ is analytic at $\eta=\infty$

Vacuum Reduction at 2-loop level

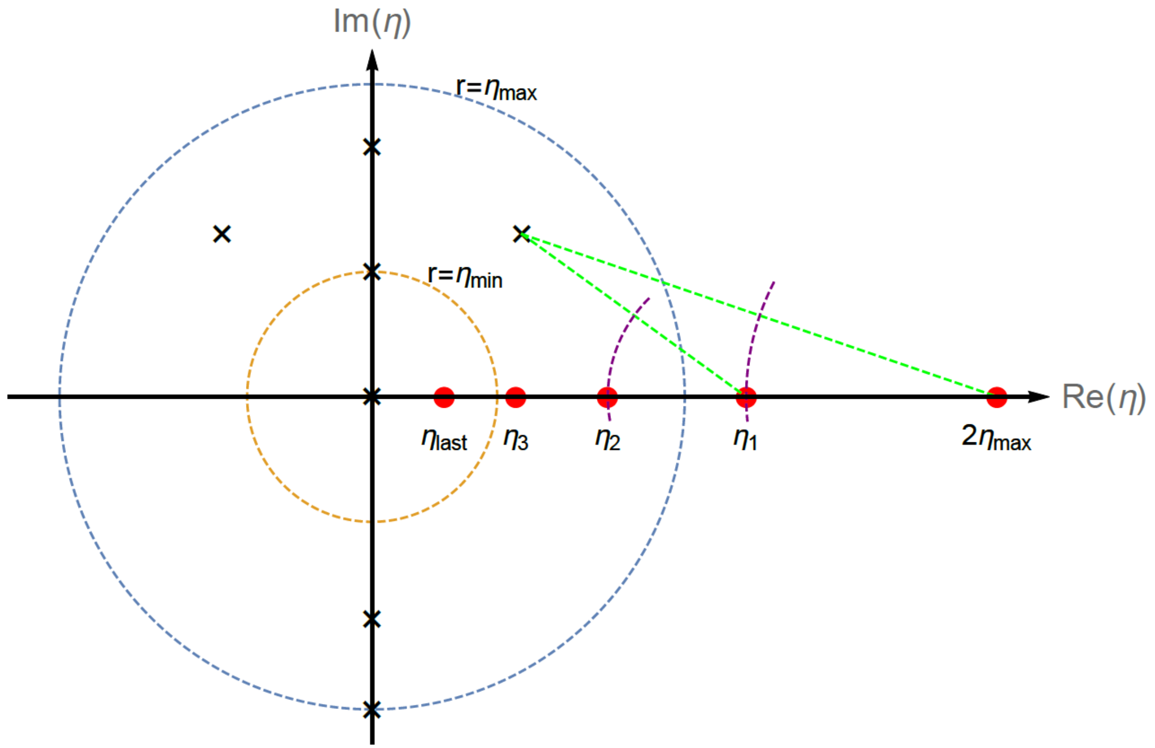
$$I^{\text{bub}}_{n_1,n_2,n_3} = \frac{2n_3}{3(n_1-1)\mathrm{i}} I^{\text{bub}}_{n_1-2,n_2,n_3+1} - \frac{2n_3}{3(n_1-1)\mathrm{i}} I^{\text{bub}}_{n_1-1,n_2-1,n_3+1} + \frac{1}{3\mathrm{i}} I^{\text{bub}}_{n_1,n_2-1,n_3} - \frac{1}{3\mathrm{i}} I^{\text{bub}}_{n_1,n_2,n_3-1} + \frac{3n_1-3-D}{3(n_1-1)\mathrm{i}} I^{\text{bub}}_{n_1-1,n_2,n_3}.$$

General Structure of DEs

- **DEs**

$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$

- **Pole Structure**



Step1: Expansion at the infinity

- **Transformation**

$$J(D;\eta) = \eta^{\nu - LD/2} I(D;\eta)$$

- $\eta \to x^{-1}$

$$x \frac{\partial}{\partial x} \vec{J}(x) = B_1(x) \vec{J}(x)$$

- **"Outside of the large circle"**

$$\vec{J}(x) = \sum_{n=0}^{\infty} \vec{J}_n x^n, \quad B_1(x) = \sum_{n=0}^{\infty} B_{1n} x^n$$

Step1: Expansion at the infinity

- Recurrence relations

$$(n - B_{10})\vec{J}_n = \sum_{k=0}^{n-1} B_{1n-k}\vec{J}_k$$

- Can be used to determine any order of $\vec{J}_n$

- Estimation of $\vec{J}(x)$

$$\vec{J}(x) \sim \sum_{n=0}^{n_0} \vec{J}_n x^n$$

e.g. at $x = \frac{1}{2}r, \vec{J}\left(\frac{\mathrm{r}}{2}\right) \Rightarrow \vec{I}\left(\frac{2}{r}\right)$

Step2: Expansion at analytical points

- At $\eta = \eta_k$:
  - Expand the differential equation and obtain the recurrence relations
  - Solve for high-order expansion coefficients
  - Estimate the value of $\vec{I}(\eta)$ at $\eta = \eta_{k+1}$

- End if we have entered the small circle

Step3: Expansion at $\eta = 0$

- $\vec{I}(\eta_0)$ is known. How to determine $\vec{I}(0)$?
- DE

$$\eta \frac{\partial}{\partial \eta} \vec{I} = \tilde{A} \vec{I}$$

- Asymptotic behavior

$$\vec{I}(\eta) \sim \eta^{\tilde{A}(0)} \vec{v}_0$$

with $\vec{v}_0$ being constant

- In general

$$\vec{I}(\eta) \equiv P(\eta) \eta^{\tilde{A}(0)} \vec{v}_0$$

Step3: Expansion at $\eta = 0$

- Expand and obtain recurrence relations

$$nP_n + [P_n, \tilde{A}_0] = \sum_{k=0}^{n-1} \tilde{A}_{n-k} P_k$$

- Can be used to determine any order of P_n
- $\vec{v}_0$ contains all information of boundary
- Determine $\vec{v}_0$ via matching

$$\vec{I}(\eta_0) = P(\eta_0)\eta_0^{\tilde{A}(0)}\vec{v}_0$$

then

$$\vec{I}(0) = \lim_{\eta\to 0} \eta^{A(0)}\vec{v}_0$$