INTEGRAND REDUCTION FOR PARTICLES WITH SPINS

Based on arXiv: 1710.10208 and 1802.06761

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March 30, 2018 @ZJU, Hangzhou

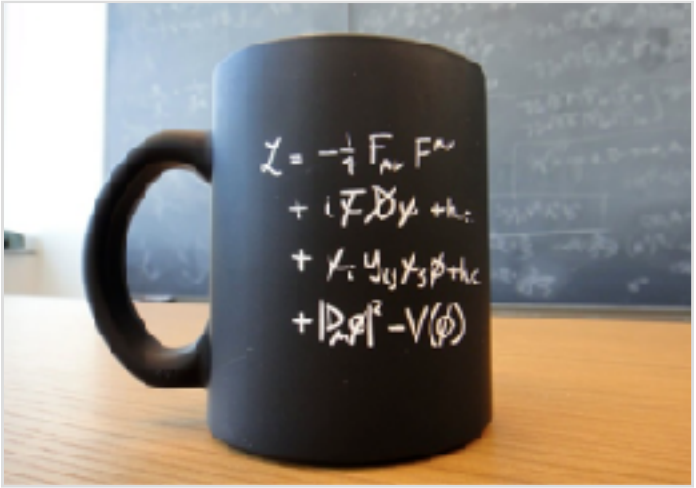
Outline

- Motivations
- Kinematics basis for particles with spin
- Examples: 5pt basis, for planar 2-loop calculation

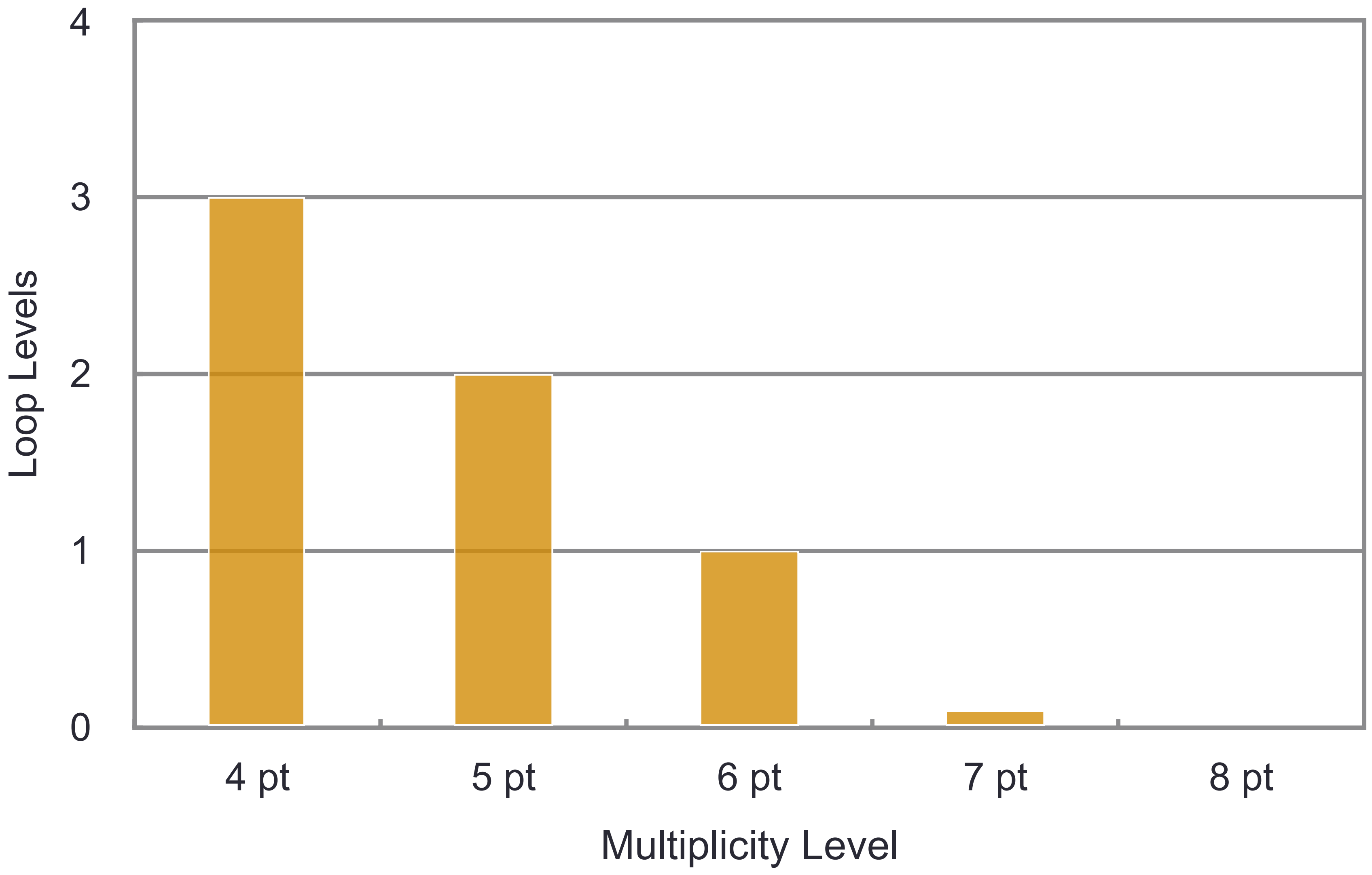
  [Badger et al. , 17'; Abreu et al., 17']
- Conclusions

TH v.s. EX

- Cross section: $\sigma = \int |\mathcal{M}|^2 \, d\Omega$

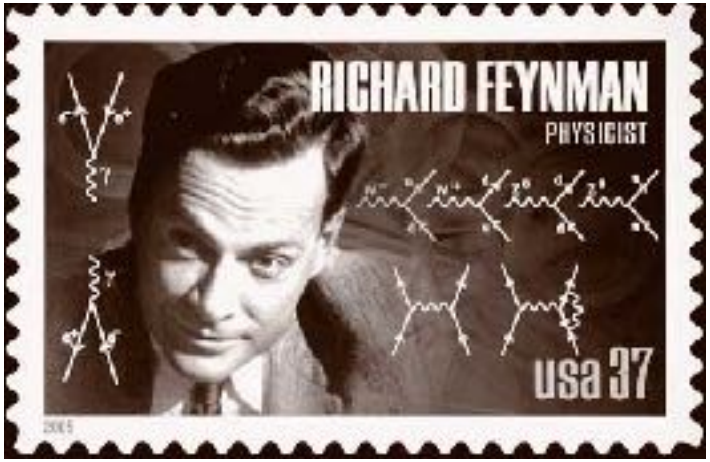


- Calculation of amplitudes as the first step


4
3
2
1
0
Loop Levels
4 pt
5 pt
6 pt
7 pt
8 pt
Multiplicity Level

Feynman diagrams

- **Advantages**:
- Start from Lagrangian, reflect the interactions intuitively
- Already successful in many cases: eg. g-2 up to 6 loops

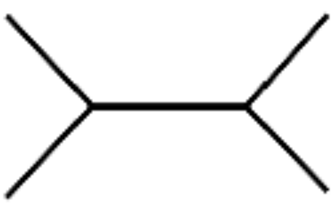


- **Limitations**:
- Huge number of diagrams in calculations
- Each diagram is NOT gauge invariant
- Significant cancellations of gauge-variant while summing over all diagrams
- Results often turn out to be very simple

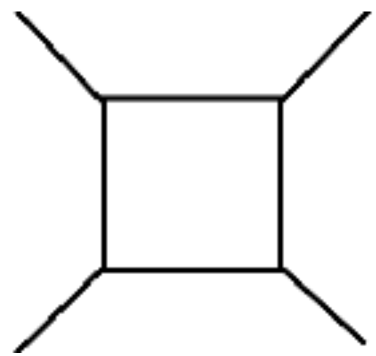
Modern Methods

- Improve the efficiency of calculation compared to Feyn. Diag.

- Key idea:
  - ★ chops problem into on-shell gauge invariant smaller pieces, (recursively) constructing scattering process
  - ★ Unitarity $S^\dagger S = 1$ & physical singularities



$$\sim \frac{1}{(p_1 + p_2)^2}$$



$$\sim \log^2(s/t)$$

- eg. BCFW for tree level, unitarity cuts for loop level, etc.

A critical point of modern methods:
on-shell gauge invariant inputs

Inherit spirit of modern methods and develop an efficient way for integrand reductions of spinning particles

- Aim: attack high-multiplicity high-loop analytical amplitude
- Preserve physical properties during intermediate steps A.M.A.P
- Unitarity cuts imposed for loop computations
- Illustrate in pure-YM, easy to generalize to other matter content
- Compute in CDR scheme

Physics properties

- Scattering amplitude: Lorentz scalar and little group tensor

$$\mathcal{A}_n(\{\xi_i, p_i\}) = \mathcal{A}_n(\{\xi_i \cdot p_j, \xi_i \cdot \xi_j\}) = \xi_1^{\mu_1} \xi_2^{\mu_2} \cdots \xi_n^{\mu_n} \hat{A}_n(\{\eta_{\mu_i \mu_j}, p_k\})$$

- Physical constraints:
  - Momentum conservation $\sum_i p_i^\mu = 0$
  - Transversality $p_i^\mu \xi_{i,\mu} = 0$
  - On-shell gauge invariance $\mathcal{A}_n(\xi_i \to p_i) = 0$
  - Unitarity & Physical Singularities

Amplitude constructions: general

- Step 1: Construct independent kinematic bases by requiring [Glover et al. , 03'; Glover et al., 12'; Z. Bern et al., 17']
  - A local little group tensor $B_i = \xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_n} f_B(\{\eta_{\mu_j\mu_k}, p_{\mu_l}\})$
  - $\sum_i p_i^\mu = 0 \qquad p_i^\mu \xi_{i,\mu} = 0 \qquad B_i(\xi_j \to p_j) = 0$
- Step 2: Construct Amplitude $\mathcal{A}_n = \sum \alpha_i B_i$
  - $\alpha_i(\{\, p_j \cdot p_k, \int f[l \cdot p]\,\})$ functions of LSPs from in-& external mom.
  - Given any form of $\mathcal{A}_n$, eg. derived from unitarity cuts

$$\sum_{\text{helicities}} B_j \mathcal{A}_n = \sum_i \alpha_i \left(\sum_{\text{helicities}} B_j B_i \right) \equiv \sum_i P_{ji}\alpha_i \qquad \sum_{\text{helicities}} \xi_\mu \xi_\nu = \eta_{\mu\nu} - \left(\frac{p_\mu q_\nu + p_\nu q_\mu}{q \cdot p} \right)$$

  - Merge all cuts and Imposing IBP $\mathcal{A}_n = \sum_i \left(\sum_j c_{ij}\, \mathrm{MI}_j \right) B_i$

Kinematic basis

- Brute-force construction by solving physical constraints
  [R. Boels & R. Medina, 16'; R. Boels & HL, 17']
  - ✦ Application: up to 6-pt tree; 4-pt 2-loop pure-YM
  - ✦ Shortcomings: complicated for (>=) 5-pt, ie. $P_{ij} = \sum_{\text{helicity}} B_i B_j$

    eg. 5pt {142,142} full matrix, super hard to inverse

    6pt {2364, 2364} full matrix, impossible to inverse
  - ✦ This construction way is kind of arbitrary, linear combinations of bases are still on-shell gauge invariant kinematic bases.

Kinematic basis

- "Canonical" kinematic basis construction [R. Boels, Q. Jin and HL,18']

  - A-type building block: $A_i(j,k) = (p_k \cdot p_i)\, p_j \cdot \xi_i - (p_j \cdot p_i)\, p_k \cdot \xi_i$

$$\{A_i(j) = A_i(i+j, i+j+1) | j \in \{1, \ldots, n-3\}\}$$

    - ➡ Solutions for 1 gluon (n-1) scalar scattering [R. Boels and HL,17']
    - ➡ For m-gluon scattering, m copies A form a basis

  - C-type building block: $C_{i,j} = (\xi_i \cdot \xi_j)(p_i \cdot p_j) - (p_i \cdot \xi_j)(p_j \cdot \xi_i)$

    - ➡ One solution for 2-gluon (n-2)-scalar (Another from 2-copies of A-type building blocks) [R. Boels and HL,17']
    - ➡ Proportional to two contracted linearized field strength tensor $F_{\mu\nu}(\xi_1) F^{\mu\nu}(\xi_2)$

A &C-type building blocks: on-shell gauge invariant

Kinematic basis

- "Canonical" kinematic basis construction [R. Boels, Q. Jin and HL,18']

  - D-type building block: $$D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} X_{ij}(k,l) A_i(k) A_j(l)$$

    Require orthogonality $$\sum_{h_i} A_i(k) D_{i,j} = 0 = \sum_{h_j} A_j(k) D_{i,j}, \ \forall k$$

    Fix the constructions with $$P_i^A(k,l) = \sum_{h_i} A_i(k) A_i(l)$$

$$A^i(k) \equiv \sum_l (P_i^A)^{-1}(k,l) A_i(l) \qquad A^i(k) A_i(l) \equiv \sum_{\text{helicities},i} A^i(k) A_i(l) = \delta(k,l)$$

$$D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} A_i(k) A_j(l) \left(A^m(k) A^n(l) C_{m,n}\right)$$

$$\sum_{\text{helicities}} D_{i,j} D_{i,j} = (p_i \cdot p_j)^2 (d-n+1) \qquad \sum_{\text{helicities},i} D_{i,j} D_{i,k} = \frac{(p_i \cdot p_j)(p_i \cdot p_k)}{(p_j \cdot p_k)} D_{j,k}$$

Kinematic basis

- “Canonical” kinematic basis construction [R. Boels, Q. Jin and HL,18’]

  - Given >=3 gluon particles in the process, kinematic basis can be constructed from multi-copies of all possible A and C/D types

    Conjecture: linearly independent and complete in general dimensions

  - The total number of basis elements with n gluons and no scalars is

$$N_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(n-2)^{(n-2k)}}{2^k k!(n-2k)!}$$

Construct Loop Amplitudes

- Using “Canonical” kinematic basis: $\mathcal{A}_n = \sum_i \left(\sum_j c_{ij}\, \mathrm{MI}_j \right) B_i$
- Coefficient for cut amplitude: $\sum_k c_{ik}^{\mathrm{cut}}\, \mathrm{MI}_k^{\mathrm{cut}} = \sum_j P_{ij}^{-1} \left(\sum_{\mathrm{helicities}} B_j\, \mathcal{A}_n^{\mathrm{cut}} \right)$
  - $\left(\sum_{\mathrm{helicities}} B_j\, \mathcal{A}_n^{\mathrm{cut}} \right)$ gives integrands for a particular cut
  - Run IBP reductions for cut integrands and preserve cut MIs part
  - Inverse inner product of kinematic bases to get cut coeffs.
- Merge and cross check for different cuts

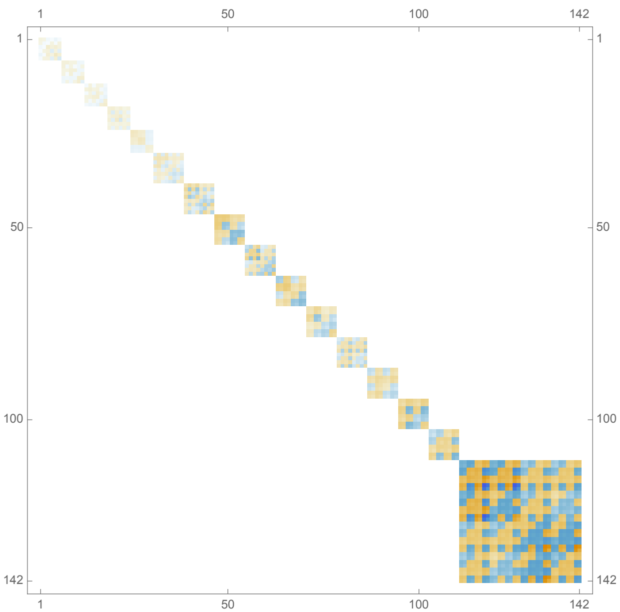
Example: 5pt kinematic basis

- "Canonical" kinematic basis construction [R. Boels, Q. Jin and HL,18']

  - 1 A + 2 D s: in total $5 \times 2 \times C_4^2/2! = 30$, eg. $A_1(2)D_{2,3}D_{4,5}$
  - 3 A s + 1 D: in total $2^3 \times C_5^2 = 80$, eg. $A_1(2)A_2(3)A_3(4)D_{4,5}$
  - 5 A s: in total $2^5 = 32$, eg. $A_1(2)A_2(3)A_3(4)A_4(5)A_5(1)$

  - Inner product matrix and its inverse can be derived by direct products of inner product matrices of A and D type building blocks

    About 20 s for P_{ij} and $(P_{ij})^{-1}$

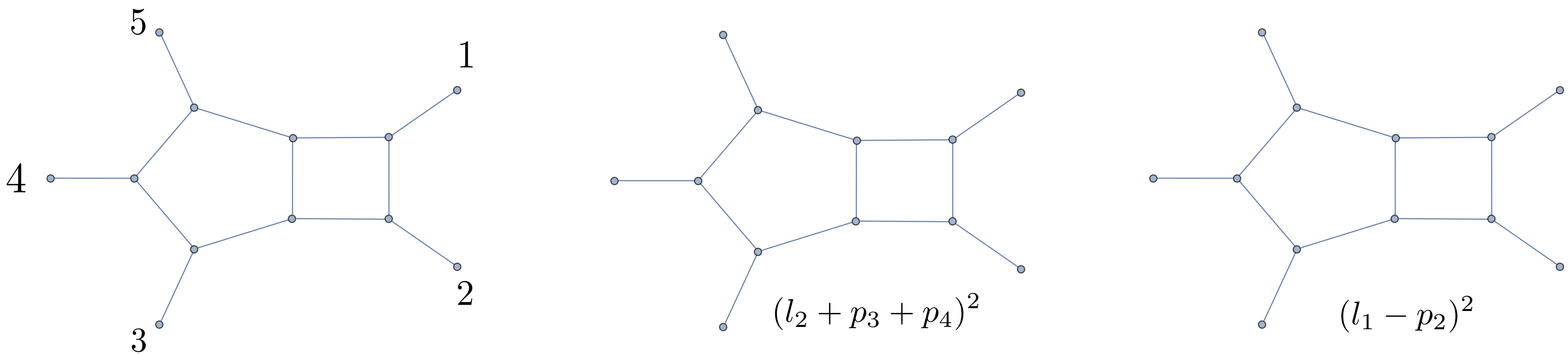


Example: 5pt planar 2-loop

- Choose full propagators $\{(l_1-l_2)^2,(l_1)^2,(l_1-p_2)^2,(l_1-p_1-p_2)^2,$ $(l_1+p_3+p_4)^2,(l_1+p_3)^2,(l_2)^2,(l_2-p_2)^2,$ $(l_2-p_1-p_2)^2,(l_2+p_3+p_4)^2,(l_2+p_3)^2\}$

  IBP for all integrands done

- Maximal cuts for 5pt planar 2-loop



$$\{(l_1-l_2)^2,(l_1)^2,(l_1-p_1-p_2)^2,(l_1+p_3+p_4)^2,(l_1+p_3)^2,(l_2)^2,(l_2-p_2)^2,(l_2-p_1-p_2)^2\}$$

Coefficients of highest MIs about 300M with unphysical singularities

- Integrand reductions for other cuts done, without substituting IBPs

Conclusions

- “Canonical” kinematic basis constructions for external particles
- Amplitudes as linear combinations of kinematic bases
- High-multiplicity high-loop amplitudes:
  Kinematic bases + unitarity cuts + IBP

- Implementations for 5pt planar 2-loop
  - Done: integrands for different cuts || IBP reductions to MIs
  - Obstacles : (1) IBP reductions huge ~ 30G
    (2) Unphysical poles
  - Todo: (1) Merge into a readable results
    (2) Compare with numerical results [Badger et al., 17’; Abreu et al., 17’]
    (3) Unphysical poles as conditions for better MIs basis choice?

Thank you for your attention !

Example: Three Gluons

- All possible tensors:

$$\vec{T} = \{(\xi_1 \cdot \xi_2)(p_2 \cdot \xi_3), (\xi_1 \cdot \xi_3)(p_1 \cdot \xi_2), (\xi_2 \cdot \xi_3)(p_2 \cdot \xi_1), (p_1 \cdot \xi_1)(p_1 \cdot \xi_2)(p_2 \cdot \xi_3)\}$$

- On-shell gauge invariance:

$$\vec{T}|_{\xi_3 \to p_3} = \{0, -(p_2 \cdot \xi_1)(p_1 \cdot \xi_2), -(p_1 \cdot \xi_2)(p_2 \cdot \xi_1), 0\}$$

three constraints from on-shell gauge invariance

$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \vec{\alpha} = 0$$

two independent basis
1st: 3pt tree amplitude
2nd: F^3

$$\vec{\alpha} = \{-1, -1, 1, 0\} \quad \text{or} \quad \vec{\alpha} = \{0, 0, 0, 1\}$$

Kinematic basis

- Brute-force construction by solving physical constraints
  - 4-gluon with metric # ≥1: 27 tensor Ansatz

Metric #=2

```
{ss[ξ1R, ξ4R] ss[ξ2R, ξ3R], ss[ξ1R, ξ3R] ss[ξ2R, ξ4R], ss[ξ1R, ξ2R] ss[ξ3R, ξ4R],
 ss[p1, ξ3R] ss[p2, ξ4R] ss[ξ1R, ξ2R], ss[p2, ξ3R] ss[p2, ξ4R] ss[ξ1R, ξ2R],
 ss[p1, ξ3R] ss[p3, ξ4R] ss[ξ1R, ξ2R], ss[p2, ξ3R] ss[p3, ξ4R] ss[ξ1R, ξ2R],
 ss[p1, ξ2R] ss[p2, ξ4R] ss[ξ1R, ξ3R], ss[p2, ξ4R] ss[p3, ξ2R] ss[ξ1R, ξ3R],
 ss[p1, ξ2R] ss[p3, ξ4R] ss[ξ1R, ξ3R], ss[p3, ξ2R] ss[p3, ξ4R] ss[ξ1R, ξ3R],
 ss[p1, ξ2R] ss[p1, ξ3R] ss[ξ1R, ξ4R], ss[p1, ξ2R] ss[p2, ξ3R] ss[ξ1R, ξ4R],
 ss[p1, ξ3R] ss[p3, ξ2R] ss[ξ1R, ξ4R], ss[p2, ξ3R] ss[p3, ξ2R] ss[ξ1R, ξ4R],
 ss[p2, ξ1R] ss[p2, ξ4R] ss[ξ2R, ξ3R], ss[p2, ξ4R] ss[p3, ξ1R] ss[ξ2R, ξ3R],
 ss[p2, ξ1R] ss[p3, ξ4R] ss[ξ2R, ξ3R], ss[p3, ξ1R] ss[p3, ξ4R] ss[ξ2R, ξ3R],
 ss[p1, ξ3R] ss[p2, ξ1R] ss[ξ2R, ξ4R], ss[p2, ξ1R] ss[p2, ξ3R] ss[ξ2R, ξ4R],
 ss[p1, ξ3R] ss[p3, ξ1R] ss[ξ2R, ξ4R], ss[p2, ξ3R] ss[p3, ξ1R] ss[ξ2R, ξ4R],
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 ss[p2, ξ1R] ss[p3, ξ2R] ss[ξ3R, ξ4R], ss[p3, ξ1R] ss[p3, ξ2R] ss[ξ3R, ξ4R]}
```

Metric #=1

Kinematic basis

- Brute-force construction by solving physical constraints
  - 4-gluon with metric # ≥1, unique solution

```
2 t ss[p1, ξ3] ss[p2, ξ4] ss[ξ1, ξ2] + 2 s ss[p2, ξ3] ss[p2, ξ4] ss[ξ1, ξ2] +
 2 t ss[p2, ξ3] ss[p2, ξ4] ss[ξ1, ξ2] + 2 s ss[p2, ξ3] ss[p3, ξ4] ss[ξ1, ξ2] +
 2 t ss[p2, ξ3] ss[p3, ξ4] ss[ξ1, ξ2] - 2 s ss[p2, ξ4] ss[p3, ξ2] ss[ξ1, ξ3] +
 2 t ss[p1, ξ2] ss[p3, ξ4] ss[ξ1, ξ3] - 2 s ss[p3, ξ2] ss[p3, ξ4] ss[ξ1, ξ3] +
 2 t ss[p1, ξ2] ss[p1, ξ3] ss[ξ1, ξ4] + 2 s ss[p1, ξ2] ss[p2, ξ3] ss[ξ1, ξ4] +
 2 t ss[p1, ξ2] ss[p2, ξ3] ss[ξ1, ξ4] - 2 s ss[p1, ξ3] ss[p3, ξ2] ss[ξ1, ξ4] +
 2 s ss[p2, ξ4] ss[p3, ξ1] ss[ξ2, ξ3] - 2 s ss[p2, ξ1] ss[p3, ξ4] ss[ξ2, ξ3] -
 2 t ss[p2, ξ1] ss[p3, ξ4] ss[ξ2, ξ3] - s^2 ss[ξ1, ξ4] ss[ξ2, ξ3] -
 s t ss[ξ1, ξ4] ss[ξ2, ξ3] - 2 t ss[p1, ξ3] ss[p2, ξ1] ss[ξ2, ξ4] -
 2 s ss[p2, ξ1] ss[p2, ξ3] ss[ξ2, ξ4] - 2 t ss[p2, ξ1] ss[p2, ξ3] ss[ξ2, ξ4] -
 2 s ss[p2, ξ3] ss[p3, ξ1] ss[ξ2, ξ4] + s t ss[ξ1, ξ3] ss[ξ2, ξ4] -
 2 t ss[p1, ξ2] ss[p3, ξ1] ss[ξ3, ξ4] + 2 s ss[p2, ξ1] ss[p3, ξ2] ss[ξ3, ξ4] +
 2 t ss[p2, ξ1] ss[p3, ξ2] ss[ξ3, ξ4] + 2 s ss[p3, ξ1] ss[p3, ξ2] ss[ξ3, ξ4] -
 s t ss[ξ1, ξ2] ss[ξ3, ξ4] - t^2 ss[ξ1, ξ2] ss[ξ3, ξ4]
```

Kinematic basis

- Brute-force construction by solving physical constraints
  - 4-gluon with metric # ≥0: 43 tensor Ansatz

Metric #=2

```
{ss[ξ1R, ξ4R] ss[ξ2R, ξ3R], ss[ξ1R, ξ3R] ss[ξ2R, ξ4R], ss[ξ1R, ξ2R] ss[ξ3R, ξ4R],
```

Metric #=1

```
ss[p1, ξ3R] ss[p2, ξ4R] ss[ξ1R, ξ2R], ss[p2, ξ3R] ss[p2, ξ4R] ss[ξ1R, ξ2R],
ss[p1, ξ3R] ss[p3, ξ4R] ss[ξ1R, ξ2R], ss[p2, ξ3R] ss[p3, ξ4R] ss[ξ1R, ξ2R],
ss[p1, ξ2R] ss[p2, ξ4R] ss[ξ1R, ξ3R], ss[p2, ξ4R] ss[p3, ξ2R] ss[ξ1R, ξ3R],
ss[p1, ξ2R] ss[p3, ξ4R] ss[ξ1R, ξ3R], ss[p3, ξ2R] ss[p3, ξ4R] ss[ξ1R, ξ3R],
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ss[p2, ξ1R] ss[p3, ξ2R] ss[ξ3R, ξ4R], ss[p3, ξ1R] ss[p3, ξ2R] ss[ξ3R, ξ4R],
```

Metric #=0

```
ss[p1, ξ2R] ss[p1, ξ3R] ss[p2, ξ1R] ss[p2, ξ4R], ss[p1, ξ2R] ss[p2, ξ1R] ss[p2, ξ3R] ss[p2, ξ4R],
ss[p1, ξ2R] ss[p1, ξ3R] ss[p2, ξ4R] ss[p3, ξ1R], ss[p1, ξ2R] ss[p2, ξ3R] ss[p2, ξ4R] ss[p3, ξ1R],
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ss[p1, ξ2R] ss[p1, ξ3R] ss[p2, ξ1R] ss[p3, ξ4R], ss[p1, ξ2R] ss[p2, ξ1R] ss[p2, ξ3R] ss[p3, ξ4R],
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ss[p1, ξ3R] ss[p3, ξ1R] ss[p3, ξ2R] ss[p3, ξ4R], ss[p2, ξ3R] ss[p3, ξ1R] ss[p3, ξ2R] ss[p3, ξ4R]}
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Example: Four Gluons

- Ten solutions for on-shell constraints
  7 (symmetric) + 3(partial anti-symmetric)

- Projector as diagonal block matrix $P = \begin{pmatrix} P_7 & 0 \\ 0 & P_3 \end{pmatrix}$

- Complete symmetric polynomial in Mandelstams

$$\det(P) \propto (-4+D)^2(-3+D)^9(-1+D)\\ (s\,t\,u)^{16}\left[(s-t)(s-u)(t-u)\right]^8$$