

Summary so far:

→ spinor-helicity method $A(\langle i; \rangle, [i; j])$

→ BCFW on-shell recursion

QCD @ tree-level was extremely simple!

$$A(1^- 2^- \dots n^-) = 0$$

$$A(1^- 2^- \dots (n-1)^- n^+) = 0$$

$$A(1^- 2^- 3^+ \dots n^+) =$$

$$\frac{\langle 12 \rangle^+}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \quad (\text{Parke, Taylor})$$

Maximal Helicity Violating (MHV)

$$d\sigma \sim \sum_{\text{helicities}} |A|^2 \quad \begin{matrix} \text{use parity symmetry} \\ < > \leftrightarrow [J] \\ + \leftrightarrow - \end{matrix}$$

$$A(1^+ 2^+ 3^- \dots n^-) = \frac{[12]^q}{[12][23] \dots [n1]}$$

but adding more -ve helicities makes the amplitudes more complicated.

- counting complexity

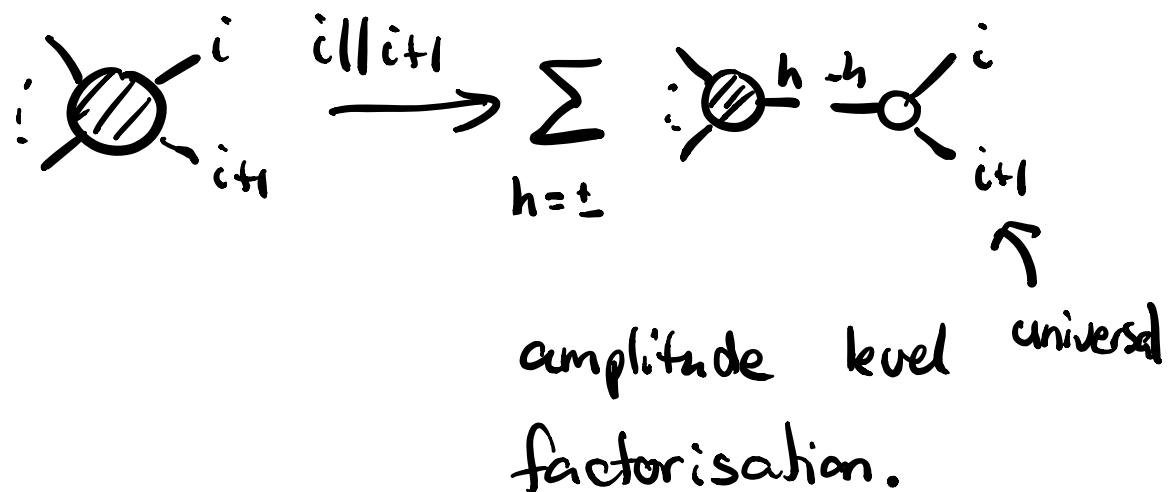
| n | 3 | 4 | 5 | 6 | 7 | 8 |
|--------------------|---|---|----|-----|------|-------|
| N_n | 1 | 4 | 25 | 220 | 2485 | 34300 |
| N_n^{ord} | 1 | 3 | 10 | 38 | 154 | 654 |
| MHV | 1 | 1 | 1 | 1 | 1 | 1 |
| MULM | - | - | - | 2 | 6 | 18 |

$$N^2MHV = - - - - \quad 20$$

$NMHV$ 3^{-ve} hel.

N^2MHV 9^{-ve} hel.

- Parke Taylor denominator has a special cyclic form:
satisfies all collinear limits



unitarity $S^+S = I \Rightarrow$

$$\text{Disc}_{\rho \rightarrow \rho'} \left(\rho \begin{array}{c} \diagup \\ \diagdown \end{array} \rho' \right) = \sum_{\alpha} \rho \begin{array}{c} \diagup \\ \diagdown \end{array} \stackrel{\alpha}{\dots} \begin{array}{c} \diagup \\ \diagdown \end{array} \rho' \quad (*)$$

$$\rho \begin{array}{c} \diagup \\ \diagdown \end{array} \rho' = \underbrace{g_{L0}}_{\text{leading order coupling}} \left(\rho \begin{array}{c} \diagup \\ \diagdown \end{array} \rho' + g^2 \begin{array}{c} \diagup \\ \diagdown \end{array} \circ \circ \right. \\ \left. + g^4 \begin{array}{c} \diagup \\ \diagdown \end{array} \circ \circ \right. \\ \left. + \dots \right)$$

expand pert. on both sides
of (*)

$$g_{L0} \left(\text{Disc} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) + g^2 \text{Disc} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \right. \\ \left. + g^4 \text{Disc} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \dots \right)$$

$$\begin{aligned}
 &= g_{L_0} \left(g^2 \int_{\ell_1, \ell_2} \text{Diagram } + g^4 \int_{\ell_1, \ell_2} \text{Diagram} \right. \\
 &\quad \left. + g^4 \int_{\ell_1, \ell_2} \text{Diagram} + g^4 \int_{\ell_1, \ell_2, \ell_3} \text{Diagram} \right. \\
 &\quad \left. + \dots \right)
 \end{aligned}$$

$$\Rightarrow \text{Disc}_{p \rightarrow p'} (\text{Diagram}) = 0$$

$$\text{Disc}_{p \rightarrow p'} (\text{Diagram}) =$$

$$\int_{\ell_1, \ell_2} \text{Diagram}$$

$$\begin{aligned}
 \text{Disc}_{p \rightarrow p'} (\text{Diagram}) &= \int_{\ell_1, \ell_2} \text{Diagram} \\
 &+ \text{Diagram} + \int_{\ell_1, \ell_2, \ell_3} \text{Diagram}
 \end{aligned}$$

more explicitly

$$\int_{\ell_1, \ell_2} \rho \langle \cdots \rangle \rho'$$

$$= \int dLIPS(\ell_1, \ell_2; \rho')$$

$$\sum_{\substack{h_1, h_2 \\ = \pm}} \rho \langle \cdots \rangle \rho'$$

$$dLIPS(\ell_1, \ell_2, \rho') = \frac{1}{(2\pi)^2}$$

$$\times d^4 \ell_1 d^4 \ell_2 \delta^{(+)}(\ell_1) \delta^{(+)}(\ell_2)$$

$$\times \delta^{(\alpha)}(-\ell_1 + \ell_2 + \rho')$$

Reconstruct amplitude by summing over all cuts (i.e. different channels $i \rightarrow p'$)

everything is on-shell.

One-loop amplitudes for 4g Scattering

$A^{(1)}(1^- 2^- 3^+ 4^+)$ in $\mathcal{N}=4$ SYM

(super-symmetric
yang-mills theory)

$\mathcal{N}=4$ is a toy model for QCD particle spectrum

(gluons, \pm), (gluinos, \pm), (scalars,
complex)

$$A^{(1)}[N=4] = A^{(1)}[\text{gluon}] + 4 A^{(1)}[\text{gluino}] + 3 A^{(1)}[\text{scalar}]$$

See $N=4$ as part of QCD
at 1-loop.

$$\begin{aligned} A^{(1), \text{QCD}} &= A^{(1)}[\text{gluon}] \\ &\quad + n_f A^{(1)}[\text{quark}] \\ &\quad + n_s A^{(1)}[\text{scalar}] \\ &= (A^{(1)}[g] + 4 A^{(1)}[q] + 3 A^{(1)}[s]) \\ &\quad + (4 - n_f) (-A^{(1)}[q] - A^{(1)}[s]) \end{aligned}$$

$$+ (1 - n_f + n_s) A^{(1)}[s]$$

1st term $\mathcal{N} = 4$

2nd term $\mathcal{N} = 1$ chiral

3rd term $\mathcal{N} = 0$ scalar

now compute the S_{12} discontinuity

$$\text{Disc}_{S_{12}} \left(\begin{array}{c} q^+ \\ \gamma \\ s^+ \end{array} \begin{array}{c} \omega^- \\ 0 \\ e_2^- \end{array} \right)$$

$$= \int_{\ell_1, \ell_2} \sum_q \begin{array}{c} q^+ \\ \gamma \\ s^+ \end{array} \begin{array}{c} \ell_1^- \\ e_1^- \\ \ell_2^- \end{array} \begin{array}{c} \omega^- \\ 0 \\ e_2^- \end{array}$$

sum is over all states

$$g^\pm, g^\pm, s(s^*)$$

don't contribute due

to current conservation.

$$= \int_{\ell_1 \ell_2} q^+ \text{ (outgoing)} \quad \text{incoming } l^-$$

The Feynman diagram shows two incoming particles, labeled ℓ_1 and ℓ_2 , represented by wavy lines. They interact to produce one outgoing particle, labeled ℓ_3 , also represented by a wavy line. The outgoing particle ℓ_3 has a plus sign above it, indicating it is moving away from the interaction point. The incoming particles have minus signs below them, indicating they are moving towards the interaction point.

$$= \int_{\ell_1 \ell_2} \frac{\langle \ell_1 (-\ell_2) \rangle^3}{\langle (-\ell_2) 3 \rangle \langle 34 \rangle \langle 4 \ell_1 \rangle} \times \frac{\langle 12 \rangle^3}{\langle 2 \ell_2 \rangle \langle \ell_2 (\ell_1) \rangle \langle -\ell_1 1 \rangle}$$

$$= \int_{\ell_1 \ell_2} \frac{\langle 12 \rangle^3}{\langle 34 \rangle} \frac{\langle \ell_1 \ell_2 \rangle^2}{\langle \ell_2 3 \rangle \langle 4 \ell_1 \rangle \langle 2 \ell_2 \times \ell_1 \rangle}$$

$$= \int_{\ell_1 \ell_2} \frac{-\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{S_{12} S_{14}}{(p_2 - p_3)^2 (p_2 + p_1)^2}$$

(use spinor-helicity algebra)

$$= - \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} S_{12} S_{14} \int_{\ell_1, \ell_2} \begin{array}{c} 4 \\ | \\ \text{---} \\ | \\ 3 \end{array} \begin{array}{c} 1 \\ | \\ \text{---} \\ | \\ 2 \end{array}$$

$$= - \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} S_{12} S_{14} \int_{\ell_1, \ell_2} \begin{array}{c} 4 \\ | \\ \text{---} \\ | \\ 3 \end{array} \begin{array}{c} 1 \\ | \\ \text{---} \\ | \\ 2 \end{array}$$

This channel shows

$$A = - (\text{tree}) \times S_{12} S_{14} \times (\text{box integral})$$

+ anything in S_{23} cut.

reconstruct part of the amplitude.

S_{23} channel is more complicated.

$$\text{Disc}_{S_{23}} \left(\begin{array}{c} 1^- \\ \text{---} \\ \text{qt} \end{array} \circlearrowleft \begin{array}{c} 2^+ \\ \text{---} \\ \text{un}_3 + \end{array} \right)$$

$$= \int_{e_1, e_2} \left(\begin{array}{c} 1^- \\ \text{---} \\ \text{qt} \end{array} \circlearrowleft \begin{array}{c} 2^+ \\ \text{---} \\ \text{un}_3 + \end{array} - \right. \\ \left. + 4 \begin{array}{c} 1^- \\ \text{---} \\ \text{qt} \end{array} \circlearrowleft \begin{array}{c} 2^+ \\ \text{---} \\ \text{un}_3 + \end{array} - \right. \\ \left. + 3 \begin{array}{c} 1^- \\ \text{---} \\ \text{qt} \end{array} \circlearrowleft \begin{array}{c} 2^+ \\ \text{---} \\ \text{un}_3 + \end{array} \right)$$

final result is simple (as a consequence of SUSY)

$$\text{Disc}_{S_{23}} \left(\begin{array}{c} \text{---} \\ \text{qt} \end{array} \circlearrowleft \begin{array}{c} \text{---} \\ \text{un}_3 + \end{array} \right) = \frac{-\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

$$\times S_{12} S_{14} \quad \begin{array}{c} 4 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ | \quad / \quad \backslash \\ 3 \quad 2 \end{array}$$

putting both channels together

$$A^{(1)[N=4]}(1^- 2^- 3^+ 4^+)$$

$$= - A^{(0)}(1^- 2^- 3^+ 4^+) S_{12} S_{14} \quad \begin{array}{c} 4 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ | \quad / \quad \backslash \\ 3 \quad 2 \end{array}$$

this amplitude must have
the same universal IR poles
as the real radiation (Rreal)

$$\begin{array}{c} 4 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ | \quad / \quad \backslash \\ 1 \quad 2 \end{array} = C_F \frac{2}{S_{12} S_{14}} \left(\frac{1}{\epsilon^2} \left(\frac{\mu_R}{-S_{12}} \right)^\epsilon \right)$$

$$+ \frac{1}{\epsilon^2} \left(\frac{\mu_R^2}{-S_{14}} \right)^\epsilon - \frac{1}{2} \log^2 \left(\frac{S_{12}}{S_{14}} \right) - \frac{\pi^2}{2}$$

$$A^{(1)} = I^{(1)} A^{(0)} + \text{finite}$$

$$I^{(1)} = -2 \left(\left(\frac{\mu_R^2}{-S_{14}} \right)^\epsilon + \left(\frac{\mu_R^2}{-S_{23}} \right)^\epsilon \right) \frac{1}{\epsilon^2}$$

$$C_R = \frac{\Gamma^2(1-\epsilon) \Gamma(1+\epsilon)}{(4\pi)^\epsilon \Gamma(1-2\epsilon)}$$

NB: we only computed
to coefficient in $d=4$
but put integral in $4-2\epsilon$

what about corrections to
the coeff. at $O(\epsilon)$?

$$A^{(1)} \sim C \cdot I$$

$$= (C^{(0)} + \epsilon C^{(1)} + \dots)$$

$$\times \left(\frac{1}{\epsilon^2} I^{(-2)} + \frac{1}{\epsilon} I^{(-1)} + \dots \right)$$

higher order terms in
the coefficient can contribute
to the amplitude $\sim \frac{\epsilon}{\epsilon}$

I need $d=4-2\epsilon$ cuts to

reconstruct them.

$$A^{(1)} = C^{(0)} \cdot I + R \\ + O(\varepsilon)$$

R is a rational term

\rightarrow all $\frac{\varepsilon}{\varepsilon}$ or IR origin cancel

R is of UV origin.

- in the $SV=4$ amplitude

$$R = 0 .$$

quark current conservation

$$\langle 11 \sigma^M 12 \rangle ?$$

$$\bar{u}_{(1)} \gamma^\mu v_{(2)}$$

$$(0 \langle 1 |) \begin{pmatrix} 0 & \sigma' \\ \bar{\sigma}' & 0 \end{pmatrix} (0 | \langle 2 |)$$

$$= (\langle 1 | \bar{\sigma} \quad 0) \begin{pmatrix} 0 \\ \langle 2 | \end{pmatrix} = 0$$

$$\bar{u}_{(1)} \gamma^\mu v_{(2)}$$

$$= (\langle 1 | \ 0) \begin{pmatrix} 0 & \sigma' \\ \bar{\sigma}' & 0 \end{pmatrix} (0 | \langle 2 |)$$

$$= \langle 1 | \bar{\sigma}^\mu | 2 \rangle$$

A one-loop integral basis

In 4 dim ≤ 4 propagators
that are linearly independent.

But in $4-2\epsilon$ dim we can
find 1 more independent prop.

\Rightarrow pentagon integral would
be most complicated.

however we can arrange
such that in $d \rightarrow 4$ the
pentagon drops out i.e. is
 $O(\epsilon)$

[NB: external momenta fixed]

in 4 dimensions \mathbb{J}

Integral basis

$$I = \left\{ \begin{array}{c} \text{:} \begin{array}{|c|c|} \hline \diagup & \diagdown \\ \hline \end{array} : \\ \text{:} \begin{array}{|c|c|} \hline \diagdown & \diagup \\ \hline \end{array} : \\ \text{:} \begin{array}{|c|c|} \hline \diagup & \diagup \\ \hline \diagdown & \diagdown \\ \hline \end{array} : \end{array}, \begin{array}{c} \text{:} \begin{array}{|c|c|} \hline \diagup & \diagdown \\ \hline \diagdown & \diagup \\ \hline \end{array} : \\ \text{:} \begin{array}{|c|c|} \hline \diagdown & \diagup \\ \hline \diagup & \diagdown \\ \hline \end{array} : \\ \text{:} \begin{array}{|c|c|} \hline \diagup & \diagup \\ \hline \diagdown & \diagdown \\ \hline \end{array} : \end{array} \right\}$$

in a massless theory we don't include wave-function bubbles or tadpoles (sealless)



Q: the double cut isolated a subset of the basis I
 \rightarrow many coefficient appear at the same time.

Can I have just 1 coeff?

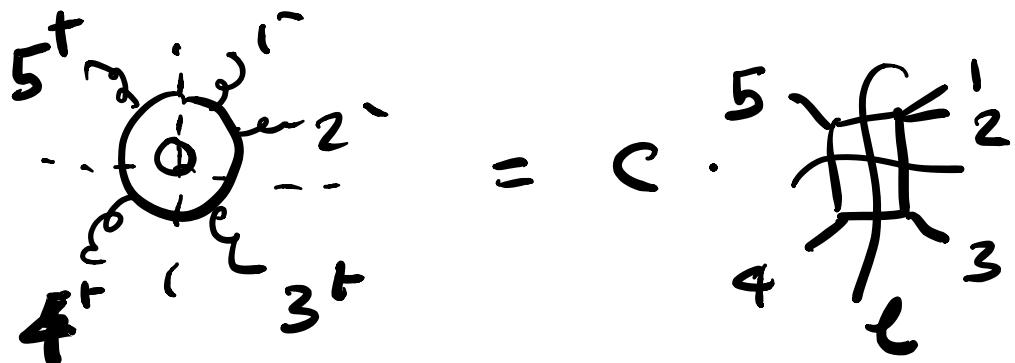
A: Yes, look at multiple cuts.

Generalised Unitarity

The connection with $S\delta^t = 1$ is lost, but we can put more propagators on-shell and factorise into tree-level amplitudes.

Old idea ('60s) but new insights (~ 2004) told us how to use complex momenta to define the cuts on-shell.

Let's look at the quadruple cut :



cut amplitude factorizes into
frees evaluated at the
solutions to

$$\left\{ e^2 = 0, (\ell - p_4)^2 = 0, \right.$$

$$\left. (\ell - p_4 - p_5)^2 = 0, (\ell + p_3)^2 = 0 \right\}$$

using a spinor basis we
find 2 solutions:

$$\ell^{\mu} = a p_3^{\mu} + b p_4^{\mu} + \frac{c}{2} \langle 3 \bar{\sigma}^{\mu} \gamma_4] \\ [+ \frac{d}{2} \langle 4 \bar{\sigma}^{\mu} \gamma_3]$$

fix a, b, c, d

$$(\ell + p_3)^2, (\ell - p_4)^2$$

$$\Rightarrow a = b = 0$$

$$\ell^2 = 0$$

$$\Rightarrow c d = 0$$

either $c = 0$ or $d = 0$

$$\ell^{(1)} \propto \langle 3 \bar{\sigma}^{\mu} \gamma_4]$$

$$\ell^{(2)} \propto \langle 4 \bar{\sigma}^{\mu} \gamma_3]$$

run through the evaluation
of the two cuts

$$\begin{array}{c}
 5^+ \\
 + \text{---} \\
 4^+ \quad \text{---} \\
 \ell^{(1)} \quad \ell^{(2)}
 \end{array}
 \begin{array}{c}
 l^- \\
 \text{---} \\
 l^- \\
 3^+
 \end{array}
 \begin{array}{l}
 \text{but} \\
 (\ell^{(1)} - p_4) \\
 \alpha < 5\bar{\sigma}^M q
 \end{array}$$

so this solution gives
zero.

on solution 2)

$$\begin{array}{c}
 + \text{---} \\
 + \text{---} \\
 + \text{---} \\
 + \text{---}
 \end{array}
 \begin{array}{c}
 l^- \\
 \text{---} \\
 l^- \\
 3^+ \\
 \ell^{(2)} \\
 + \text{---} \\
 + \text{---}
 \end{array}
 = -S_{34} S_{45} \\
 \times A^{(6)}(1^- 2^- 3^+ \ell^+ 5^+)$$

(can check)

value of box coeff.

$$c = \frac{1}{2} \left(\text{cut}(\ell^{(1)}) + \text{cut}(\ell^{(2)}) \right)$$

$$= -\frac{1}{2} S_{3q} S_{4s} A^{(c)} (1 - z^{-3} q^+ s^+)$$

again this relates to the
IR pole structure.