Summary so far:

★ spinor-helicity method $A(\langle ij \rangle, [ij])$

★ BCFW on-shell recursion

QCD @ tree-level was extremely simple!

$$A(1^- 2^- \cdots n^-) = 0$$

$$A(1^- 2^- \cdots (n-1)^- n^+) = 0$$

$$A(1^- 2^- 3^+ \cdots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

(Parke, Taylor)

Maximal Helicity Violating (MHV)

$$d\sigma \sim \sum_{\text{helicities}} |A|^2$$

use parity symmetry

$$\langle\ \rangle \leftrightarrow [\ ]$$

$$+ \leftrightarrow -$$

$$A(1^+ 2^+ 3^- \cdots n^-) = \frac{[12]^4}{[12][23]\cdots[n1]}$$

but adding more -ve helicities makes the amplitudes more complicated.

- counting complexity

n	3	4	5	6	7	8
N_n	1	4	25	220	2485	34300
N_n^{ord}	1	3	10	38	154	654
MHV	1	1	1	1	1	1
NMHV	-	-	-	3	6	10

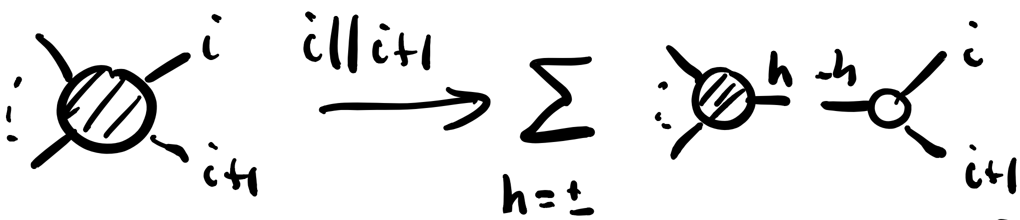
NMHV - - - - - 10

N^2MHV - - - - - 20

NMHV 3^{-ve} hel.

N^2MHV 4^{-ve} hel.

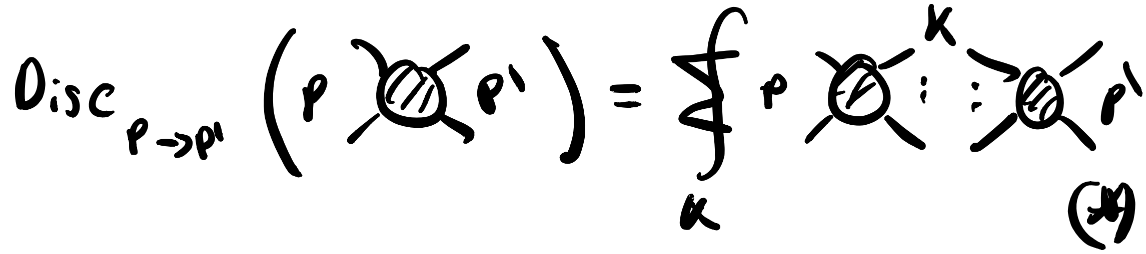
- Parke Taylor denominator has a special cyclic form: satisfies all collinear limits

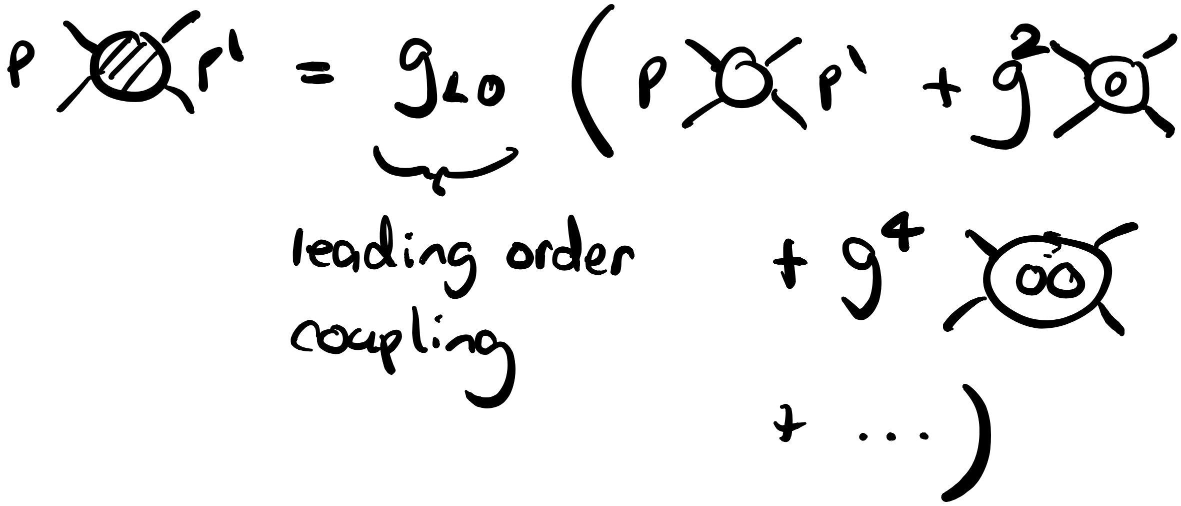


universal

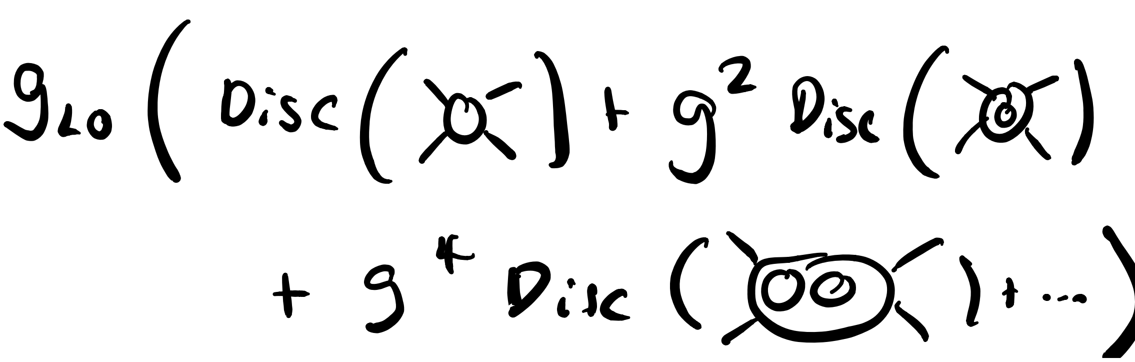
amplitude level factorisation.

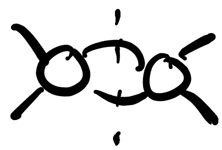
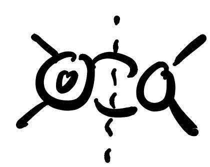
unitarity $S^\dagger S = 1 \Rightarrow$

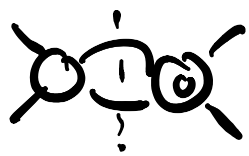
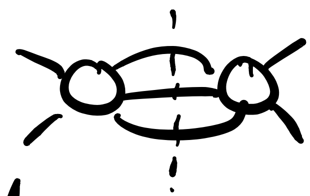




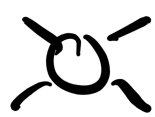
expand pert. on both sides of (*)

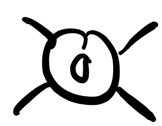


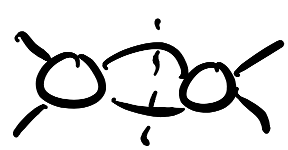
$= g_{LO} \Big(g^2 \int_{\ell_1 \ell_2}$  $+ g^4 \int_{\ell_1 \ell_2}$ 

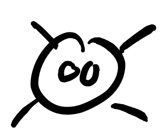
$+ g^4 \int_{\ell_1 \ell_2}$  $+ g^4 \int_{\ell_1 \ell_2 \ell_3}$ 

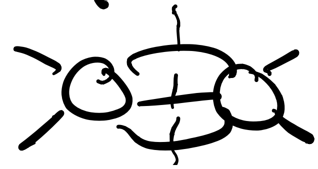
$+ \cdots \Big)$

$\Rightarrow$ $\mathrm{Disc}_{p \to p'} \Big($  $\Big) = 0$

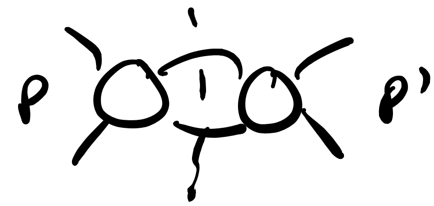
$\mathrm{Disc}_{p \to p'} \Big($  $\Big) =$

$\int_{\ell_1 \ell_2}$ 

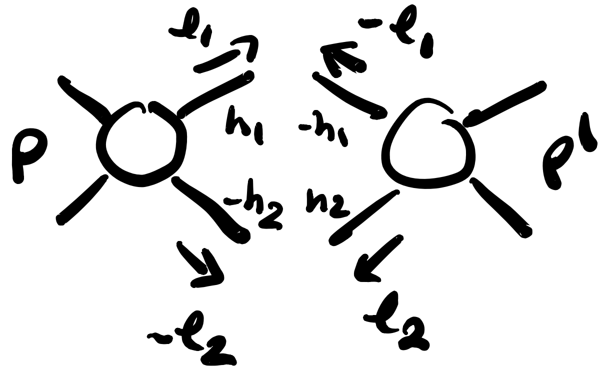
$\mathrm{Disc}_{p \to p'} \Big($  $\Big) = \int_{\ell_1 \ell_2} \Big($ 

$+$  $\Big) + \int_{\ell_1 \ell_2 \ell_3}$ 

more explicity

$$\int_{\ell_1 \ell_2}$$ 

$$= \int dLIPS(\ell_1, \ell_2 ; p')$$

$$\sum_{\substack{h_1, h_2 \\ = \pm}}$$ 

$$dLIPS(\ell_1, \ell_2, p') = \frac{1}{(2\pi)^2}$$

$$\times\, d^4\ell_1\, d^4\ell_2\ \delta^{(+)}(\ell_1^2)\,\delta^{(+)}(\ell_2^2)$$

$$\times\ \delta^{(4)}(-\ell_1 + \ell_2 + p')$$

Reconstruct amplitude by summing over all cuts (i.e. different channels $\gamma \to p'$)

everything is <u>on-shell</u>.

<u>One-loop amplitudes for 4g Scattering</u>

$A^{(1)}(1^- 2^- 3^+ 4^+)$ in $\mathcal{N}=4$ SYM (super-symmetric yang-mills theory)

$\mathcal{N}=4$ is a toy model for QCD

particle spectrum

(gluons, $\pm$), (gluinos, $\pm$), (scalars, complex)

$$A^{(1)[\mathcal{N}=4]} = A^{(1)[\text{gluon}]} + 4A^{(1)[\text{gluino}]} + 3A^{(1)[\text{scalar}]}$$

See $\mathcal{N}=4$ as part of QCD at 1-loop.

$$A^{(1),\text{QCD}} = A^{(1)[\text{gluon}]} + n_f A^{(1)[\text{quark}]} + n_s A^{(1)[\text{scalar}]}$$

$$= \left(A^{(1)[g]} + 4A^{(1)[q]} + 3A^{(1)[s]}\right) + (4-n_f)\left(-A^{(1)[q]} - A^{(1)[s]}\right)$$

$$+\,(1-n_f+n_s)\,A^{(1)[s]}$$

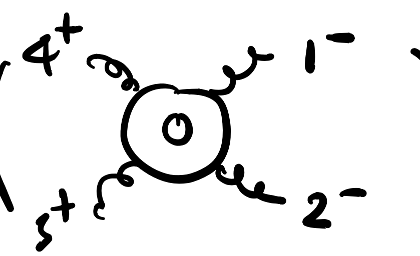
1^{st} term $\quad \mathcal{N}=4$

2^{nd} term $\quad \mathcal{N}=1$ chiral

3^{rd} term $\quad \mathcal{N}=0$ scalar

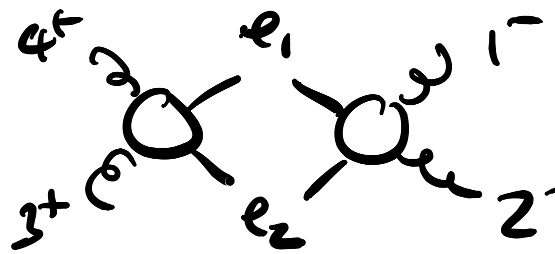
now compute the s_{12} discontinuity

$\mathrm{Disc}_{s_{12}}\Bigg($



$\Bigg)$

$= \int_{\ell_1 \ell_2} \sum$

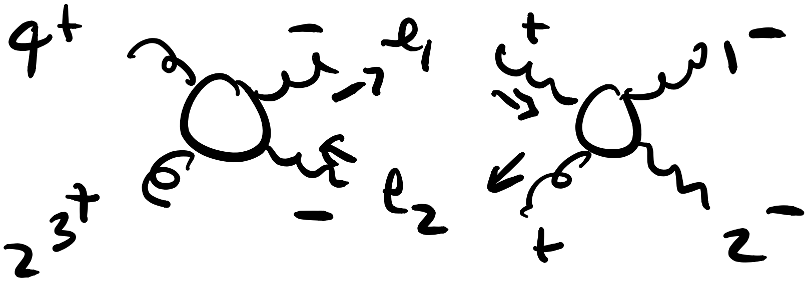


↑

sum is over all states

$$g^{\pm},\ \underbrace{q^{\pm},\ s\,(s^*)}_{\text{don't contribute due}}$$

to current conservation.

$$= \int_{\ell_1 \ell_2}$$ 

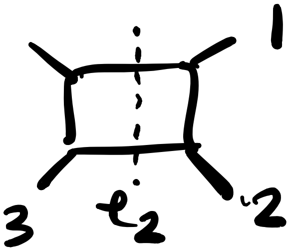
$$= \int_{\ell_1 \ell_2} \frac{\langle \ell_1 (-\ell_2)\rangle^3}{\langle (-\ell_2) 3\rangle \langle 34\rangle \langle 4 \ell_1\rangle}$$

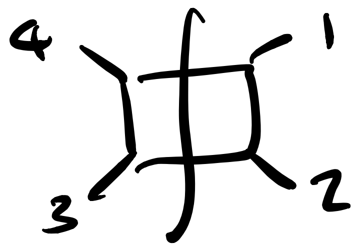
$$\times \frac{\langle 12\rangle^3}{\langle 2 \ell_2\rangle \langle \ell_2 (-\ell_1)\rangle \langle (-\ell_1) 1\rangle}$$

$$= \int_{\ell_1 \ell_2} \frac{\langle 12\rangle^3}{\langle 34\rangle} \frac{\langle \ell_1 \ell_2\rangle^2}{\langle \ell_2 3\rangle \langle 4 \ell_1\rangle \langle 2 \ell_2\rangle \langle \ell_1 1\rangle}$$

$$= \int_{\ell_1 \ell_2} -\frac{\langle 12\rangle^3}{\langle 23\rangle \langle 34\rangle \langle 41\rangle} \frac{s_{12} s_{14}}{(\ell_2 - p_3)^2 (\ell_2 + p_2)^2}$$

(use spinor-helicity algebra)

$$= -\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} s_{12} s_{14} \int_{\ell_1 \ell_2}$$ 

$$= \frac{-\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} s_{12} s_{14}$$ 

this channel show

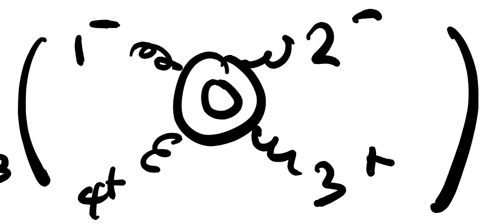
$A = -$ (tree) $\times\, s_{12} s_{14} \times$ (box integral)

+ anything in s_{23} cut.

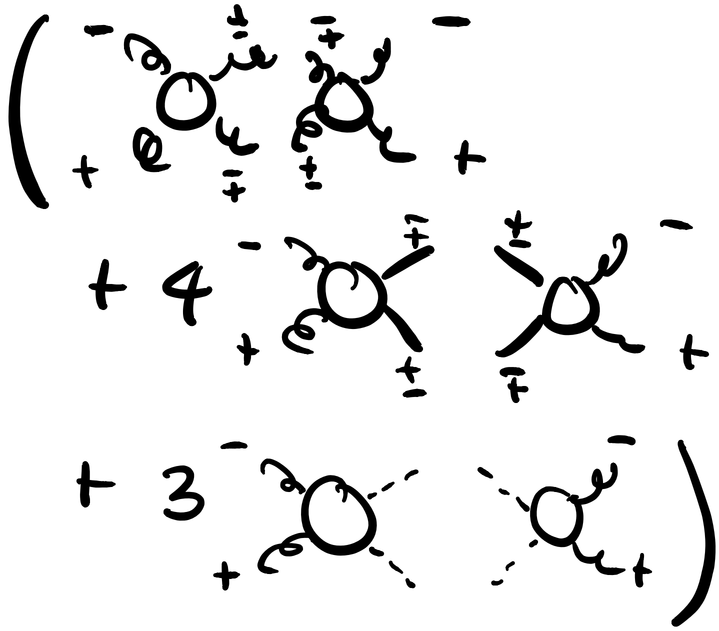
reconstruct part of the amplitude.

s_{23} channel is more complicated.

$\mathrm{Disc}_{s_{23}}$

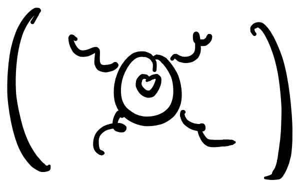


$= \int_{\ell_1 \ell_2}$

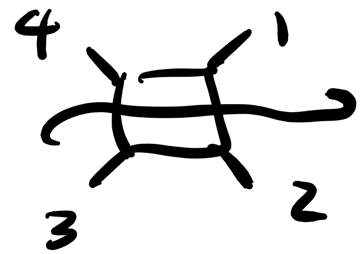


final result is simple (as a consequence of susy)

$\mathrm{Disc}_{s_{23}}$

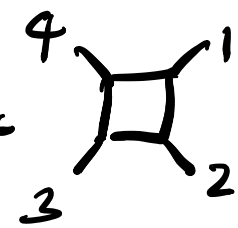


$$= \frac{-\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

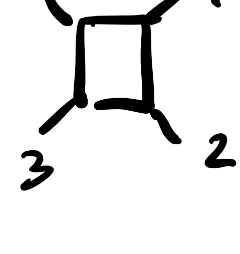
$\times\ s_{12}\, s_{14}$ 

putting both channels together

$$A^{(1)[N=4]}(1^-2^-3^+4^+)$$

$$= -A^{(0)}(1^-2^-3^+4^+)\, s_{12} s_{14}$$ 

this amplitude must have the same universal IR poles as the real radiation (Raoul)

 $$= c_\Gamma \frac{2}{s_{12} s_{14}} \left(\frac{1}{\epsilon^2} \left(\frac{\mu_R^2}{-s_{12}} \right)^{\epsilon} \right.$$

$$+ \frac{1}{\varepsilon^2}\left(\frac{\mu_R^2}{-s_{14}}\right)^{\varepsilon} - \frac{1}{2}\log^2\left(\frac{s_{12}}{s_{14}}\right) - \frac{\pi^2}{2}\Bigg)$$

$$A^{(1)} = I^{(1)} A^{(0)} + \text{finite}$$

$$I^{(1)} = -2\left(\left(\frac{\mu_R^2}{-s_{14}}\right)^{\varepsilon} + \left(\frac{\mu_R^2}{-s_{23}}\right)^{\varepsilon}\right)\frac{1}{\varepsilon^2}$$

$$C_\Gamma = \frac{\Gamma^2(1-\varepsilon)\,\Gamma(1+\varepsilon)}{(4\pi)^{\varepsilon}\,\Gamma(1-2\varepsilon)}$$

NB: we only computed to coefficient in $d=4$ but put integral in $4-2\varepsilon$

what about corrections to the coeff. at $O(\varepsilon)$?

$$A^{(1)} \sim c \cdot I$$

$$= \left(c^{(0)} + \varepsilon c^{(1)} + \cdots\right) \times \left(\frac{1}{\varepsilon^2} I^{(-2)} + \frac{1}{\varepsilon} I^{(-1)} + \cdots\right)$$

higher order terms in the coefficient can contribute the the amplitude $\sim \frac{\varepsilon}{\varepsilon}$

I need $d = 4 - 2\varepsilon$ cuts to

reconstruct them.

$$A^{(1)} = c^{(0)} \cdot I + R$$
$$+ O(\varepsilon)$$

R is a rational term

→ all $\frac{\varepsilon}{\varepsilon}$ or IR origin cancel

R is of UV origin.

- in the $\mathcal{N}=4$ amplitude

$$R = 0 .$$

quark current conservation

$$\langle 1 | \sigma^{\mu} | 2 \rangle \; ?$$

$\bar{u}_-(1)\gamma^\mu v_-(2)$

$$(0 \quad \langle 1|)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \langle 2| \end{pmatrix}$$

$$= (\langle 1|\bar{\sigma} \quad 0)\begin{pmatrix} 0 \\ \langle 2| \end{pmatrix} = 0$$

$\bar{u}_+(1)\gamma^\mu v_-(2)$

$$= ([1| \quad 0)\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \langle 2| \end{pmatrix}$$

$$= [1|\bar{\sigma}^\mu|2\rangle$$

A one-loop integral basis

In 4 dim ≤ 4 propagators that are linearly independent.

But in $4-2\varepsilon$ dim we can find 1 more independent prop.

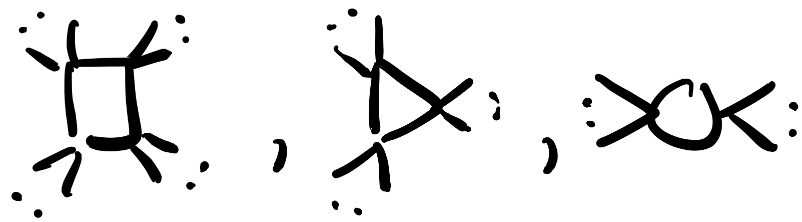
$\Rightarrow$ pentagon integral would be most complicated.

<u>however</u> we can arrange such that in $d \to 4$ the pentagon drops out i.e. is $O(\varepsilon)$

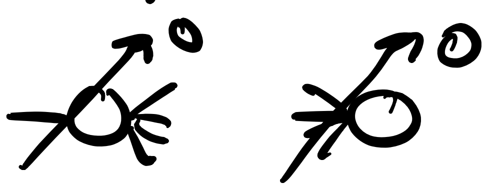
[NB: external momenta fixed

in 4 dimensions]

Integral basis

$I = \{$  $\}$

in a massless theory we don't include wave-function bubbles or tadpoles (scaleless)



in dim reg.

<u>Q</u>: the double cut isolated a subset of the basis I

→ many coefficient appear at the same time.

Can I have just 1 coeff?

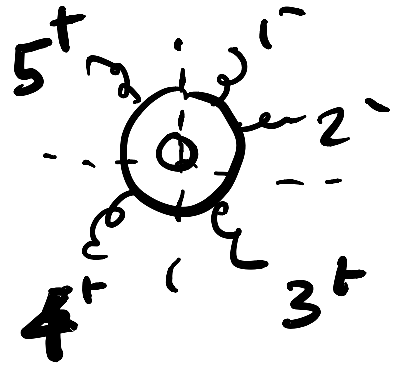
A: Yes, look at multiple cuts.

Generalised Unitarity

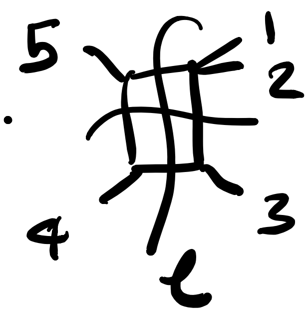
The connection with $SS^{\dagger} = 1$ is lost, but we can put more propagators on-shell and factorise into tree-level amplitudes.

Old idea ('60s) but new insights (~2004) told us how to use <u>complex</u> momenta to define the cuts on-shell.

Let's look at the quadruple cut:



$= c \cdot$



cut amplitude factorises into trees evaluated at the solutions to

$$\left\{ \ell^2 = 0 \,,\; (\ell - p_4)^2 = 0 \,,\; (\ell - p_4 - p_5)^2 = 0 \,,\; (\ell + p_3)^2 = 0 \right\}$$

using a spinor basis we find 2 solutions:

$$\ell^\mu = a\, p_3^\mu + b\, p_4^\mu + \frac{c}{2}\langle 3\bar{\sigma}^\mu 4] + \frac{d}{2}\langle 4\bar{\sigma}^\mu 3]$$

fix a, b, c, d

$(\ell+p_3)^2, \; (\ell-p_4)^2$

$$\Rightarrow \quad a = b = 0$$

$\ell^2 = 0$

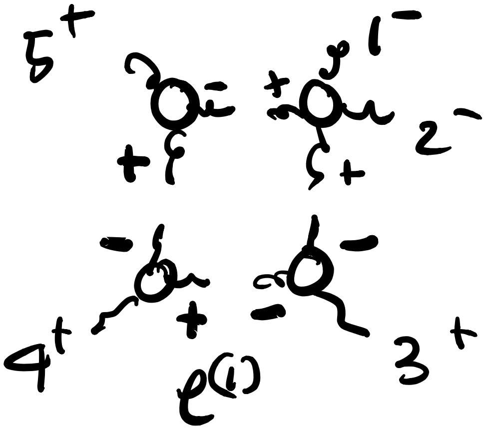
$$\Rightarrow \quad c\, d = 0$$

either $c = 0$ <u>or</u> $d = 0$

$$\ell^{(1)} \propto \langle 3\bar{\sigma}^\mu 4]$$

$$\ell^{(2)} \propto \langle 4\bar{\sigma}^\mu 3]$$

run through the evaluation
of the two cuts



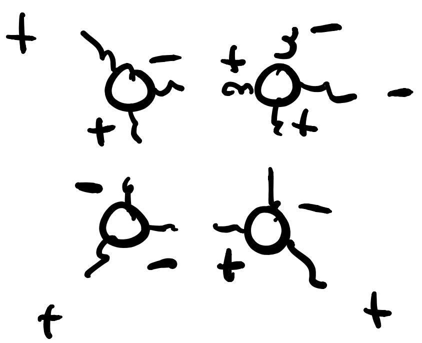
but

$(\ell^{(1)} - p_4)$

$\propto \langle 5 \bar{\sigma}^\mu 4]$

so this solution gives
zero.

on solution 2)



$$= -s_{34} s_{45} \times A^{(0)}(1^- 2^- 3^+ 4^+ 5^+)$$

(can check)

value of box coeff.

$$C = \frac{1}{2}\left(\text{cut}(\ell^{(1)}) + \text{cut}(\ell^{(2)})\right)$$

$$= -\frac{1}{2} s_{34} s_{45} A^{(0)}(1^- 2^- 3^+ 4^+ 5^+)$$

again this relates to the IR pole structure.