Infrared subtractions in perturbative quantum chromodynamics

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Lecture 3: Infrared subtractions at next-to-next-to-leading order

Recap (1)

In the previous lecture, we constructed the next-to-leading order **FKS subtraction scheme**, which can be summarized in the following expression:

$$\langle F_{LM}(1,2,3)\rangle = \langle (I - C_{31} - C_{32})(I - S_3) F_{LM}(1,2,3)\rangle + \langle (C_{31} + C_{32})(I - S_3) F_{LM}(1,2,3)\rangle + \langle S_3 F_{LM}(1,2,3)\rangle,$$

for a real emission correction $q\bar{q} \to V + g$.

- In the second and third terms, we **extract** the poles of the real emission corrections by using **the universal factorizations** in the soft and collinear limits (discussed in lecture 1) and integrating over the ***d*-dimensional phase space of the *unresolved* parton**.
- These poles in $1/\epsilon$ completely describe the singular behavior of the real emission corrections.
- We **cancel the poles** from the real emission corrections against poles from the *virtual corrections* and absorb remaining collinear poles in *universal renormalizations of the parton distribution functions*.
- We take the $\epsilon \to 0$ limit and compute the finite remainder in four space-time dimensions.

Recap (2)

$$\langle F_{LM}(1,2,3)\rangle = \langle (I - C_{31} - C_{32})(I - S_3) F_{LM}(1,2,3)\rangle$$
$$+ \langle (C_{31} + C_{32})(I - S_3) F_{LM}(1,2,3)\rangle + \langle S_3 F_{LM}(1,2,3)\rangle.$$

- In the first term, we **subtract** the leading singularities originating from the soft and collinear emissions from the full real emission amplitude-squared.
- This term is then manifestly **finite** when integrated over the **full phase space** of the emitted parton, and can be evaluated in four space-time dimensions.

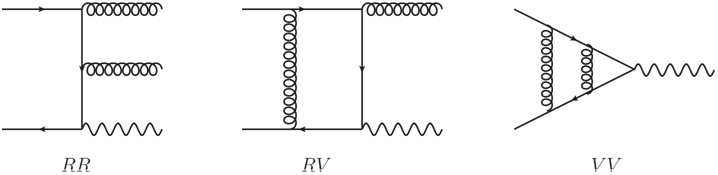
- The result is a **finite** expression for the **fully differential** NLO corrections.
- Although we demonstrated this for color singlet production, this subtraction scheme can be formulated for *arbitrary processes* at a hadron or lepton collider.
- Other NLO subtraction schemes exist for arbitrary processes, the most commonly used of which is the **Catani-Seymour dipole** method.

Subtraction schemes at NNLO

In this lecture we will consider the infrared singularities that appear in next-to-next-to-leading order (NNLO) corrections.

NNLO corrections have three contributions:

- **Real-real (RR) corrections**: two additional real partons are emitted at tree level;
- **Real-virtual (RV) corrections**: one additional real parton is emitted at one-loop;
- **Virtual-virtual (VV) corrections**: two-loop corrections.



Infrad singularities at NNLO

As at NLO, the loop corrections give rise to **explicit** $1/\epsilon$ singularities, while the real emissions lead to singularities arising from the **unresolved phase space** of the emitted partons.

- The **virtual-virtual** corrections have explicit $1/\epsilon^4$ (and lower) singularities which can be calculated using the *Catani formula*, as at NLO.
- The *finite contributions* of the **VV** corrections can be tremendously difficult to calculate (see lectures by Simon Badger and Lorenzo Tancredi).
- The **real-virtual** corrections have explicit $1/\epsilon^2$ (and lower) poles from the one-loop integral, which can also be calculated using the Catani formula.
- The **RV** corrections also have singular regions associated with the single real emission. Since these have the same singular structure as the real corrections at NLO, the NLO subtraction strategy can be employed here with minor adjustments. Poles of $\mathcal{O}(1/\epsilon^2)$ (and lower) are extracted, so the **RV** corrections also have poles at $\mathcal{O}(1/\epsilon^4)$.
- The **real-real** corrections have a much more complicated singular structure than the real corrections at NLO, and consequently the extraction of the singularities of real-real corrections is much more difficult. This is the main challenge in the treatment of IR singularities at NNLO.

The challenge of double-real emissions (1)

To understand *why* real-real corrections have such complicated singular behavior, consider the NLO correction $q\bar{q} \to V + g$. We identified the following singular regions:

- the radiated gluon is **soft**;
- the radiated gluon is **collinear** to one of the initial state partons.

The **overlap** of these two singular regions is the **soft-collinear** configuration, which is quite simple.

Consider the emission of *two* real gluons in color singlet production

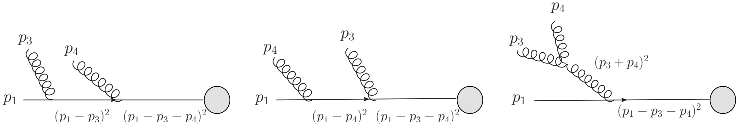
$$q(p_1)\bar{q}(p_2) \to V + g(p_3) + g(p_4).$$

Singularities will arise if:

- either radiated gluon is soft;
- either radiated gluon is collinear to either initial state parton;
- the radiated gluons are collinear to each other.

Overlapping singularities (1)

Consider the emission of gluons $g(p_3)$ and $g(p_4)$ from $q(p_1)$:



We have the following propagators:

$$D_{13} \equiv (p_1-p_3)^2 = -2E_1E_3\rho_{13}$$

$$D_{14} \equiv (p_1-p_4)^2 = -2E_1E_4\rho_{14}$$

$$D_{34} \equiv (p_3+p_4)^2 = 2E_3E_4\rho_{34}$$

$$D_{134} \equiv (p_1-p_3-p_4)^2 = -2E_1E_3\rho_{13} - 2E_1E_4\rho_{14} + 2E_3E_4\rho_{34}$$

Consider $E_3 \to 0$:

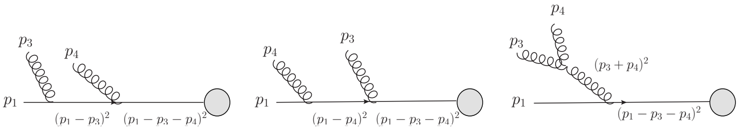
- $D_{13} \to 0$ and $D_{34} \to 0$ – *single-soft* singularities.
- $D_{134} \to -2E_1E_4\rho_{14}$:
- $\rho_{14} \to 0 \Rightarrow D_{134} \to 0$ – *overlap* between soft and collinear limits.

Now consider additional limit $E_4 \to 0$:

- $D_{14} \to 0$ – "single-soft^2"
- $D_{134} \to 0$ – *double-soft* singularity – **more complicated** than "single-soft^2".
- *Overlap* between double-soft and single-soft limits.

Overlapping singularities (2)

Consider the emission of gluons $g(p_3)$ and $g(p_4)$ from $q(p_1)$:



$$D_{13} = -2E_1E_3\rho_{13}$$
$$D_{14} = -2E_1E_4\rho_{14}$$
$$D_{34} = 2E_3E_4\rho_{34}$$
$$D_{134} = -2E_1E_3\rho_{13} - 2E_1E_4\rho_{14} + 2E_3E_4\rho_{34}$$

Consider $\rho_{13} \to 0$:

- $D_{13} \to 0$ – *double-collinear* singularities.
- $D_{34} \to 2E_3E_4\rho_{14}$
- $D_{134} \to 2E_4\rho_{14}(-E_1 + E_3)$.

Now consider additional limit $\rho_{14} \to 0$:

- $D_{14} \to 0$ – "double-collinear^2"
- $D_{134} \to 0$ – *triple-collinear* singularities $\vec{p}_1||\vec{p}_3||\vec{p}_4$.
- These are **more complicated** than "double-collinear^2".
- *Overlap* between double-collinear and triple-collinear limits.

Overlapping singularities (3)

The **overlap** of the above regions is now **very convoluted**; we may have, e.g.:

- Gluons 3 and 4 becoming collinear to an initial state parton and gluon 4 becoming soft;

- Gluons 3 and 4 becoming soft *and* collinear to each other *and* to one of the initial state partons

- ...

We can also have singularities from radiation from *both* initial state partons, e.g.:

- Gluon 3 becoming soft and collinear to one initial state parton and gluon 4 becoming collinear to the other;

- ...

A subtraction scheme would need to integrate over the **unresolved** phase space of *each* parton while also avoiding *overlapping* singular regions leading to non-convergent integrals.

- The presence of **multiple overlapping** singular regions makes handling the IR divergences at NNLO non-trivial.
- This is an active field of research today.
- Two **fundamentally different** approaches: **subtraction schemes** and **phase space slicing methods.**

Phase space slicing (1)

From lecture 1: non-converging integrals over energy and angles of the gluon

$$\int_0^{E_{\max}} \frac{\mathrm{d}E_3}{E_3}; \qquad \int_0^{\pi} \frac{\mathrm{d}\theta_{13}}{\theta_{13}}; \qquad \int_0^{\pi} \frac{\mathrm{d}\theta_{23}}{\theta_{23}}.$$

The subtraction approach is to regulate these singularities by working in $d = 4 - 2\epsilon$ dimensions. One could also introduce *cut-offs* which remove the singular regions

$$\int_{E_{\min}}^{E_{\max}} \frac{\mathrm{d}E_3}{E_3}; \qquad \int_{\theta_{\min}}^{\pi} \frac{\mathrm{d}\theta_{13}}{\theta_{13}}; \qquad \int_{\theta_{\min}}^{\pi} \frac{\mathrm{d}\theta_{23}}{\theta_{23}},$$

which **converge**.

- Slicing is intuitive because the soft and collinear radiated partons **will not** be resolved into a jet, so it makes sense to *slice* the phase space into a *resolved* and an *unresolved* region. The above integrals denote the **resolved region.**
- Slicing removes an entire region of phase space which includes the singularities; subtraction schemes remove the singularities point-by-point. Thus slicing methods are *non-local*, while subtraction schemes are (in principle) *local*.

Phase space slicing (2)

On the other hand, it is clear from the subtraction methods that *unresolved* partons **do** contribute to the cross sections and distributions. So we also need a way to calculate the unresolved regions

$$\int_0^{E_{\min}} \frac{\mathrm{d}E_3}{E_3}; \qquad \int_0^{\theta_{\min}} \frac{\mathrm{d}\theta_{13}}{\theta_{13}}; \qquad \int_0^{\theta_{\min}} \frac{\mathrm{d}\theta_{23}}{\theta_{23}}.$$

It is also clear that introducing a cutoff for every singular region of phase space is not practical. We should find a **single variable** which can be used to control all the singular limits of the phase space. This is called the *slicing parameter*.

The choice of slicing parameter defines the slicing method.

The two (related) slicing methods being studied and employed for NNLO calculations are:

- q_T **slicing** (Catani, Grazzini) – uses more specialized observable q_T;
- N**-jettiness slicing** (Boughezal, Focke, Liu, Petriello, Gaunt, Stahlhofen, Tackmann, Walsh) – uses less restricted observable N-jettiness.

Phase space slicing (3)

Then we can write

$$\int [\mathrm{d}\phi_d]\, |\mathcal{M}|^2\, F_J = \int_0^\delta [\mathrm{d}\phi_d]\, |\mathcal{M}|^2\, F_J + \int_\delta [\mathrm{d}\phi_4]\, |\mathcal{M}|^2\, F_J + \mathcal{O}(\delta),$$

where

- $\mathcal{M}$ is the amplitude for the RR emission,
- $[\mathrm{d}\phi]$ is the RR phase space, in either d or four space-time dimensions,
- F_J defines an infrared-safe observable,
- δ is the cutoff for the slicing parameter.

The first term has only **soft and collinear** radiation, and can be computed in Soft Collinear Effective Theory (SCET) – see lecture by Pier Francesco Monni.

The second term has no singular regions and can be integrated in four space-time dimensions.

We can regard it as an NLO calculation with an additional hard jet – we can use the sophisticated NLO techniques developed over the last two decades to compute it.

Phase space slicing (4)

Thus we have

$$\int [\mathrm{d}\phi_d]\, |\mathcal{M}|^2\, F_J = \int_0^{\delta} \left[[\mathrm{d}\phi_d]\, |\mathcal{M}|^2\, F_J\right]_{\mathrm{SCET}} + \int_{\delta} [\mathrm{d}\phi_4]\, |\mathcal{M}|^2\, F_J + \mathcal{O}(\delta).$$

The difficulty in these methods arises from choosing the parameter δ.

- *A priori*, there is no way of choosing δ.
- It needs to be sufficiently small to prevent the power corrections $\mathcal{O}(\delta)$ from becoming large
  (which would reflect that SCET is not applicable in the whole *unresolved region*).
- One simply picks a small value for δ and then checks that the results are stable under variations of δ.
  - If the results are not stable, then the power corrections are large and δ must be decreased.
  - If the results are stable, δ is assumed to be small enough that the power corrections are negligible.
- Lower values of δ require big increases in computation time.
- There has been recent progress in computing the leading power corrections, which improves the speed and stability of slicing methods.

Subtraction schemes at NNLO (1)

We write the correction arising from $q(p_1)\bar{q}(p_2) \to V + g(p_3) + g(p_4)$ emissions as

$$\mathrm{d}\sigma^{\mathrm{RR}} = \frac{1}{2s}\frac{1}{2!}\int [\mathrm{d}p_3][\mathrm{d}p_4] F_{LM}(1,2,3,4),$$

with the factor $1/2!$ accounting for the symmetric final state, and $F_{LM}(1,2,3,4)$ is defined analogously to $F_{LM}(1,2,3)$

$$F_{LM}(1,2,3,4) = \mathrm{dLips}_V \; |\mathcal{M}(1,2,3,4,V)|^2 \; \mathcal{F}_{\mathrm{kin}}(1,2,3,4,V).$$

Then a NNLO subtraction would be

$$\int [\mathrm{d}p_3][\mathrm{d}p_4] F_{LM}(1,2,3,4) = \int [\mathrm{d}p_3][\mathrm{d}p_4]\big(F_{LM}(1,2,3,4) - \mathcal{S}\big) + \int [\mathrm{d}p_3][\mathrm{d}p_4]\mathcal{S},$$

where

- $\mathcal{S}$ reproduces the leading singular behavior of $F_{LM}(1,2,3,4)$ in all **overlapping** singular regions.
- The phase space integration over $[\mathrm{d}p_3]$ and $[\mathrm{d}p_4]$ integrates over **only the unresolved phase space** of *each* parton *while also avoiding* overlapping singular regions leading to **non-convergent integrals.**

NNLO subtraction schemes (1)

The main subtraction methods being developed and used are:

- **Antenna subtractions** (Gehrmann-De Ridder, Gehrmann, Glover, Daleo, Maître, Luisoni, Monni, Boughezal, Ritzmann, Currie, Wells).
- **Residue-improved sector decomposition (STRIPPER)** (Czakon, Heymes).
- **Nested soft-collinear subtraction** (Caola, Melnikov, R.R.).
- **CoLoRFulNNLO** (Del Duca, Duhr, Kardos, Somogyi, Szőr,Trócsányi, Tulipánt).
- **Projection-to-Born** (Cacciari, Dreyer, Karlberg, Salam, Zanderighi). Requires an analytic expression for the NLO result integrated over the additional radiated parton.

These methods are at different states of maturity and each have advantages and disadvantages.

They differ both in how they **construct the subtraction term** $\mathcal{S}$ and in how they deal with **overlapping singular regions.**

NNLO subtraction schemes (2)

None of the NNLO subtraction schemes are at the level of NLO subtractions, i.e. a method:

- which is *fully local*,
- in which the cancellation of the IR poles is shown *explicitly* and *analytically*,
- which is completely general and may be applied to *arbitrary processes* at a hadron or lepton collider,
- which allows all amplitudes to be computed in *four dimensions*.

Nested soft-collinear subtractions (1)

It is not possible to study any of the NNLO subtraction or slicing methods in the same detail as we studied FKS subtraction at NLO in the previous lecture.

I will therefore describe the *basic idea* behind of one of the subtraction schemes – **the nested soft-collinear subtraction scheme** – without worrying about the details.

This scheme is a natural continuation of the FKS subtraction method to NNLO, so this should be possible (hopefully!)

I will consider the real-real corrections to color singlet production

$$\mathrm{d}\sigma^{\mathrm{RR}} = \frac{1}{2s}\frac{1}{2!}\int [\mathrm{d}p_3][\mathrm{d}p_4] F_{LM}(1,2,3,4).$$

from the partonic channel $q(p_1)\bar{q}(p_2) \to V + g(p_3) + g(p_4)$.

Other partonic channels, e.g. $q(p_1)g(p_2) \to V + q(p_3) + g(p_4)$, would also contribute. However, quarks emitted in the final state have a simpler singularity structure than gluons. Therefore the partonic channel $q\bar{q} \to V + g + g$ has the most complicated singularity structure for color singlet production.

Nested soft-collinear subtractions (2)

As for the FKS subtraction scheme at NLO, we will construct the subtraction terms $\mathcal{S}$ directly from the soft and collinear limits, using the universal factorizations of the amplitudes in these limits.

We also need a way to handle the overlapping divergences in the phase space.

The distinguishing feature of the nested soft-collinear subtraction scheme is the use of color coherence to separate the soft and collinear singularities **from the beginning**.

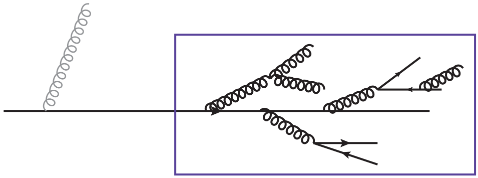
We then use:

- energy ordering to separate the soft regions, and
- phase-space partitioning and sector decomposition (as done in the **residue-improved sector decomposition subtraction method**) to separate the collinear regions.

Color coherence

Onshell, gauge invariant QCD amplitudes display a property known as **color coherence**, which is most commonly considered in the context of parton showers (see yesterday's lecture by Stefan Höche).

Consider the emission of a soft gluon from a quark line which then undergoes multiple splittings, which may or may not be collinear:



The wavelength of the soft gluon is **too large** to resolve the details of the splittings, including any potential collinear singularities – it only depends on the **total color charge** of all the radiated partons.

Therefore the factorization of the amplitude as a result of the soft radiation is insensitive to any other radiation. As a result, **soft and collinear emissions can be treated independently of one another.**

This allows us to extend part of our NLO strategy to NNLO: **we subtract the soft singularities first, then subtract the collinear singularities.**

Subtracting soft singularities (1)

We begin by defining an energy-ordering: $E_4 < E_3 < E_{\rm max}$, where the energies are taken in the center-of-mass frame.
As at NLO, the energies of both gluons are also bounded from above by $E_{\rm max}$.

We define the RR corrections as

$$\mathrm{d}\sigma^{\rm RR} = \frac{1}{2s}\int [\mathrm{d}p_3][\mathrm{d}p_4] F_{LM}(1,2,3,4)\theta(E_3 - E_4) \equiv \langle F_{LM}(1,2,3,4)\rangle .$$

The $1/2!$ factor for symmetric final states is removed by the energy ordering $E_4 < E_3$.

We now recall the soft operator S_i that we introduced last lecture and introduce a *double-soft operator* $\mathbb{S}$

$$S_i A = \lim_{E_i\to 0} A, \qquad \mathbb{S} A = \lim_{E_3,E_4\to 0} A \text{ at fixed } E_4/E_3.$$

The energy ordering ensures that p_3 can only become soft if p_4 is also soft – i.e. in the double soft limit. Thus the S_3 limit **does not** occur, and the only soft limits are S_4 and $\mathbb{S}$.

Subtracting soft singularities (2)

We can now subtract the soft singularities:

$$\langle F_{LM}(1,2,3,4)\rangle = \langle \not{S} F_{LM}(1,2,3,4)\rangle + \langle S_4(I-\not{S})F_{LM}(1,2,3,4)\rangle + \langle (I-S_4)(I-\not{S})F_{LM}(1,2,3,4)\rangle.$$

- The first term on the left-hand side corresponds to the double-soft limit, in which both gluons **decouple completely.**
- The second term captures the limit where g_4 is soft but singularities from S_3 are removed.
- The final term has all soft singularities removed. However, it still contains collinear singularities and is thus **not integrable**. The different singular collinear limits overlap, and these must be disentangled.

Soft limits

For the double-soft limit $\langle \mathbb{S} F_{LM}(1,2,3,4) \rangle$:

- Both radiated gluons decouple completely, and the amplitude factorizes into an amplitude-squared with no gluon emissions and the double-soft eikonal function:

$$\mathbb{S} F_{LM}(1,2,3,4) = \mathrm{Eik}_2(1,2,3,4) F_{LM}(1,2).$$

- We can integrate the double-soft eikonal function over the energies and angles of the decoupled gluons, to obtain poles at $\mathcal{O}(1/\epsilon^4)$ and lower.

For the single-soft limit $\left\langle S_4(I - \mathbb{S}) F_{LM}(1,2,3,4) \right\rangle$:

- $g(p_4)$ decouples completely and we integrate over its energies and angles.
- This leaves us with an NLO-like expression $F_{LM}(1,2,3)$ which still has singularities related to $g(p_3)$.
- These can be treated as at NLO, resulting in poles at $\mathcal{O}(1/\epsilon^3)$ and lower.

Phase space partitions (1)

Both the double-soft limit $\langle \mathbb{S} F_{LM}(1,2,3,4)\rangle$ and soft-subtracted term $\langle (I-S_4)(I-\mathbb{S})F_{LM}(1,2,3,4)\rangle$ contain **overlapping** collinear singularities, which must be disentangled before they can be extracted.

We do this in two steps. First, we define *phase space partition functions*:

$$w^{13,14} = \frac{\rho_{23}\rho_{24}}{d_3 d_4}\left(1+\frac{\rho_{13}}{d_{3421}}+\frac{\rho_{14}}{d_{3412}}\right), \qquad w^{13,24} = \frac{\rho_{23}\rho_{14}\rho_{34}}{d_3 d_4 d_{3412}},$$
$$w^{23,24} = \frac{\rho_{13}\rho_{14}}{d_3 d_4}\left(1+\frac{\rho_{24}}{d_{3421}}+\frac{\rho_{23}}{d_{3412}}\right), \qquad w^{23,14} = \frac{\rho_{13}\rho_{24}\rho_{34}}{d_3 d_4 d_{3421}}.$$

where

$$d_{i=3,4} = \rho_{1i}+\rho_{2i}, \quad d_{3421} = \rho_{34}+\rho_{23}+\rho_{14}, \quad d_{3412} = \rho_{34}+\rho_{13}+\rho_{24}.$$

The phase space partition functions have the following **collinear limits**:

$$C_{32}w^{13,14} = C_{42}w^{13,14} = 0 \qquad C_{32}w^{13,24} = C_{41}w^{13,24} = C_{34}w^{13,24} = 0$$
$$C_{31}w^{23,24} = C_{41}w^{23,24} = 0 \qquad C_{31}w^{23,14} = C_{42}w^{23,14} = C_{34}w^{23,14} = 0,$$

Phase space partitions (2)

The behavior of the partition functions in the collinear limits

$$C_{32}w^{13,14} = C_{42}w^{13,14} = 0 \qquad C_{32}w^{13,24} = C_{41}w^{13,24} = C_{34}w^{13,24} = 0$$
$$C_{31}w^{23,24} = C_{41}w^{23,24} = 0 \qquad C_{31}w^{23,14} = C_{42}w^{23,14} = C_{34}w^{23,14} = 0,$$

means that only certain collinear limits are relevant in each partition:

$$w^{13,14} : C_{31},\ C_{41},\ C_{34} \qquad w^{13,24} : C_{31},\ C_{42}$$
$$w^{23,24} : C_{32},\ C_{42},\ C_{34} \qquad w^{23,14} : C_{32},\ C_{41}$$

Moreover,

$$w^{13,14} + w^{23,24} + w^{13,24} + w^{14,23} = 1,$$

so we can insert the sum over all partitions into a phase space integral,

$$\int [\mathrm{d}p_3][\mathrm{d}p_4] = \int [\mathrm{d}p_3][\mathrm{d}p_4]\left(w^{13,14} + w^{13,24} + w^{23,24} + w^{23,14}\right)$$

thus dividing the phase space into four partitions $w^{13,14}$, $w^{13,24}$, $w^{23,24}$ and $w^{23,14}$.

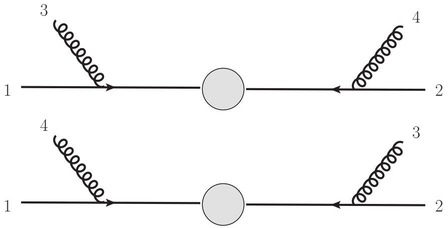
Double-collinear partitions

The phase space partitions have a **physical interpretation**.

The partitions $w^{13,24}$ and $w^{23,14}$ only have singularities if one of the emitted gluons is collinear to one initial state parton and the other emitted gluon is collinear to the other initial state parton – i.e. the gluons are well-separated in rapidity.

$w^{13,24}$

$w^{23,14}$



The configurations can be thought of as "NLO × NLO" – there are **no overlapping singularities!**

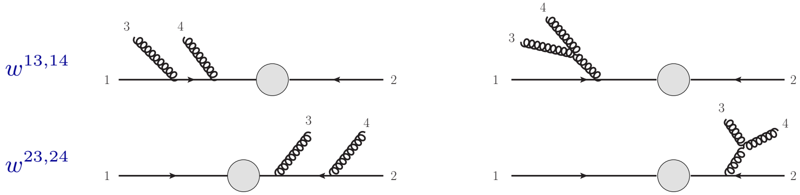
These are called the *double-collinear partitions*, since only two partons can be collinear to one another in these partitions.

Triple-collinear partitions

The other two partitions have gluons close in rapidity, which allows three partons to be simultaneously collinear to one another:

$$w^{13,14} : \vec{p}_1||\vec{p}_3||\vec{p}_4 \qquad\qquad w^{23,24} : \vec{p}_2||\vec{p}_3||\vec{p}_4$$

and are thus called *triple collinear partitions*.



These still contain **overlapping singularities**.

We need to do one more thing to separate the singularities: perform a *sector decomposition*.

Sector decomposition

Let's consider partition $w^{13,14}$.

The angular integration space containing the singularities is the square $0 < \eta_{13}, \eta_{14} < 1$. We decompose this into four sectors using an angular ordering in η_{13} and η_{14}:

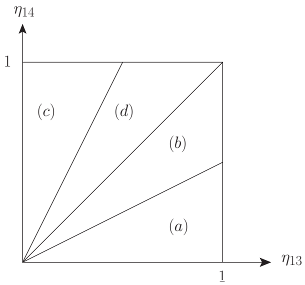
$$1 = \theta\left(\eta_{14} < \frac{\eta_{13}}{2}\right) + \theta\left(\frac{\eta_{13}}{2} < \eta_{14} < \eta_{31}\right) + \theta\left(\eta_{13} < \frac{\eta_{14}}{2}\right) + \theta\left(\frac{\eta_{14}}{2} < \eta_{13} < \eta_{41}\right)$$
$$\equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}.$$

Thus the limits are:

- $\theta^{(a)} : C_{14}$
- $\theta^{(b)} : C_{34}$
- $\theta^{(c)} : C_{13}$
- $\theta^{(d)} : C_{34}$

There is only **one** collinear limit in each *sector* of the $w^{13,14}$ *partition*, so we have **separated the overlapping singularities** for this partition.

We can perform an analogous decomposition for $w^{23,24}$ by ordering η_{23} and η_{24}.



Removing collinear singularities

We can apply the **phase space partitioning** and **sector decomposition** to separate the overlapping collinear limits in the soft-subtracted term

$$\left\langle (I - S_5)(I - \mathbb{S}) F_{\mathrm{LM}}(1,2,4,5) \right\rangle = \left\langle F_{\mathrm{LM}}^{s_r c_s}(1,2,4,5) \right\rangle + \left\langle F_{\mathrm{LM}}^{s_r c_t}(1,2,4,5) \right\rangle + \left\langle F_{\mathrm{LM}}^{s_r c_r}(1,2,4,5) \right\rangle,$$

where

- $\left\langle F_{\mathrm{LM}}^{s_r c_s}(1,2,4,5) \right\rangle$ gives the single-collinear limits, with poles at $\mathcal{O}(1/\epsilon^2)$ and lower.
- $\left\langle F_{\mathrm{LM}}^{s_r c_t}(1,2,4,5) \right\rangle$ gives the triple-collinear limits, with poles at $\mathcal{O}(1/\epsilon)$.
- $\left\langle F_{\mathrm{LM}}^{s_r c_r}(1,2,4,5) \right\rangle$ has all singularities removed through the *nested subtractions* and is therefore **finite**, and may be evaluated in four space-time dimensions and integrated numerically.
- In each of these three terms, the phase space is separated into partitions and sectors, each of which contains only one collinear singularity.

Recap (1)

Thus we have replicated our NLO procedure at NNLO:

- We have **subtracted** all the singular regions, resulting in an integral

$$\left\langle F_{\rm LM}^{s_r c_r}(1,2,4,5)\right\rangle,$$

  that is **finite** and can be evaluated in four space-time dimensions.
- We have **subtraction terms**

$$\begin{array}{ll} \left\langle \mathbb{S} F_{LM}(1,2,3,4)\right\rangle & \left\langle S_4(I-\mathbb{S}) F_{LM}(1,2,3,4)\right\rangle \\ \left\langle F_{\rm LM}^{s_r c_s}(1,2,4,5)\right\rangle & \left\langle F_{\rm LM}^{s_r c_t}(1,2,4,5)\right\rangle, \end{array}$$

  in which the gluons decouple either partially (in the case of collinear limits) or completely (in the case of soft limits).
- Only **one** singularity is present in each phase space partition and sector, allowing us to integrate the subtraction terms over the *unresolved* phase space of the gluons without encountering additional singularities that would prevent this integral from converging.
- We obtain **lower multiplicity processes** ($\hat{\mathcal{O}}_{\rm NLO} F_{LM}(1,2,3)$, $F_{LM}(z\cdot 1,2)$, $F_{LM}(1,2)$ etc.) multiplied with **explicit poles** in $1/\epsilon$ which capture the singular behavior.

Recap (2)

- After including the RV and VV corrections and renormalizing the pdfs, we observe that the **poles cancel** (not shown here...).
- **Relatively compact expressions** are found for the finite remainders, which may be calculated in four dimensions.
- Thus the d-dimensional calculation is only needed for the **extraction and cancellation** of the poles. Once this is done, *all contributions* may be computed in four space-time dimensions.
- We have only performed an analytic integration over the *unresolved* phase space of the radiated gluons, never over the *resolved* phase space, meaning that our results are again **fully differential.**

Summary

- The presence of **soft** and **collinear** emissions means that evaluating real radiative corrections in a fully differential way is not straightforward.
- We need to regularize the singularities by extending the phase space integration to d-dimensions.
- We then construct an NLO subtraction procedure which:
  - removes the singular regions, allowing the integration over the full phase space of the emitted partons to be performed, and
  - allows the singularities to be identified as poles in $1/\epsilon$ after integrating over the d-dimensional phase space of the unresolved partons.
- The infrared poles from the subtraction procedure cancel against the poles from the virtual corrections and remaining collinear poles are absorbed into the pdf renormalization.
- At NNLO, the singularities of the real-real corrections become more complicated, with overlapping collinear singularities.
- Several subtraction and slicing approaches are being investigated and employed for NNLO calculation.