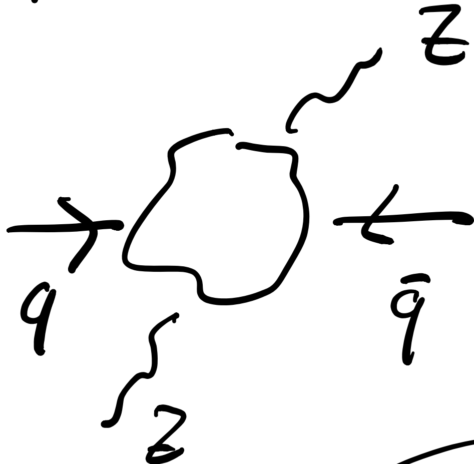
Feynman Integrals

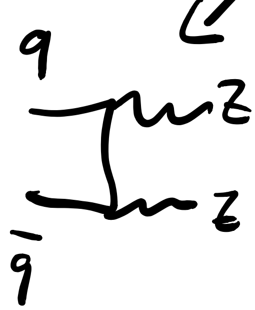
arXiv:0707.4037

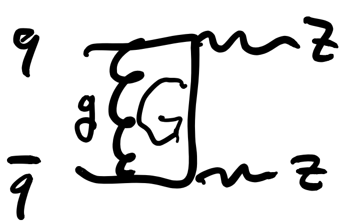
$$q\bar{q} \to Z Z$$

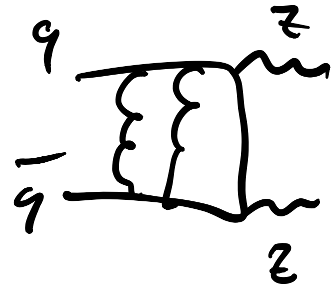


$$\sigma = \int dPS\, |M|^2$$

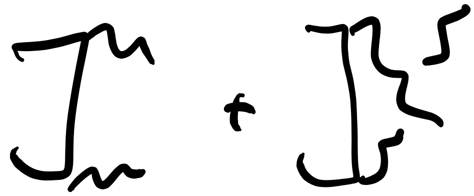
$$M = M^{(0)} + \left(\frac{\alpha_s}{2\pi}\right) M^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 M^{(2)} + \dots$$







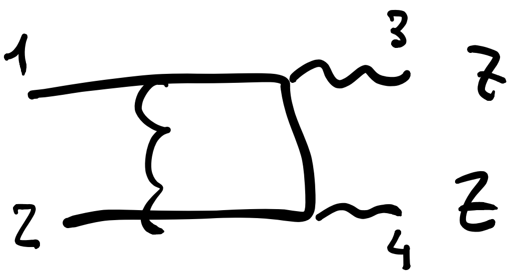
$q\bar{q} \to ZZ$ (QCD!)

$\mathcal{M}^{(0)} = 2$ digr 

$\mathcal{M}^{(1)} = 10$ digr

$\left[\mathcal{M}^{(2)} = 143 \text{ diagr}\right]$!

$\mathcal{M}^{(3)} = 2922$ digr



$= \mathcal{F}_1^{(1)}(p_1, p_2, p_3, p_4)$

$$\mathcal{F}^{(1)} = \varepsilon_3^{\mu} \varepsilon_4^{\nu}\, \bar{u}(p_2) \left[\underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{T^{\mu\nu}(p_i,k)}{D_1 D_2 D_3 D_4}} \right] u(p_1)$$

$$D_j = q_j^2 - m_j^2$$

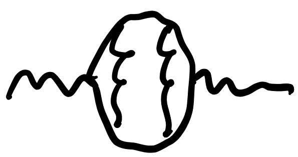
$$T^{\mu\nu}(p_i, k) = \text{rank 2 tensor}$$
$$= \{p_i^\mu, k_i^\mu, \gamma^\mu, g^{\mu\nu}, \dots\}$$

$$F_f^{(l)}(p_1 \dots p_4) = \varepsilon_3^\mu \varepsilon_4^\nu \bar{u}(p_2)\left[\int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{T^{\mu\nu}(p_i,k_j)}{D_1^{b_1}\cdots D_t^{b_t}}\right] u(p_1)$$

① Tensor Reduction

$$= \int D^d k \frac{k^\mu k^\nu}{\underbrace{(k^2+m^2)}_{D_1}\underbrace{((k-p)^2+m^2)}_{D_2}}$$

$$\Pi^{\mu\nu}(p,m) = C_1(m^2,p^2)\, g^{\mu\nu} + C_2(m^2,p^2)\,\frac{p^\mu p^\nu}{p^2}$$

 $$= C_1(m^2,p^2)\, g^{\mu\nu} + C_2(m^2,p^2)\,\frac{p^\mu p^\nu}{p^2}$$

$$(*)\ \int D^d k \,\frac{k^\mu k^\nu}{D_1 D_2} = C_1(m^2,p^2)\, g^{\mu\nu} + C_2(m^2,p^2)\,\frac{p^\mu p^\nu}{p^2}$$

$$g^{\mu\nu}(*) \to \underbrace{\int D^d k \,\frac{k^2}{D_1 D_2}}_{} = d\, C_1 + C_2$$

$$p^\mu p^\nu (*) \to \underbrace{\int D^d k \,\frac{(k\cdot p)^2}{D_1 D_2}}_{} = p^2 \left[C_1 + C_2 \right]$$

$$C_1 = \frac{1}{d-1}\left(\int D^d k \, \frac{k^2}{D_1 D_2} - \frac{1}{p^2}\int D^d k \, \frac{(k\cdot p)^2}{D_1 D_2}\right)$$

$$C_2 = \frac{1}{d-1}\left(\frac{d}{p^2}\int D^d k \, \frac{(k\cdot p)^2}{D_1 D_2} - \int D^d k \, \frac{k^2}{D_1 D_2}\right)$$

$$F^{(\ell)} = \varepsilon_3^{\mu}\varepsilon_4^{\nu}\, \bar{u}(p_2)\left[\underbrace{\int \prod_j D^d k_j \, \frac{T^{\mu\nu}(p_i, k_j)}{D_1 \cdots D_t}}\right] u(p_1)$$

$$\sum_{i=1}^{m} C_i(p_1 \ldots p_4, m_i^2)\, \underline{\tilde{T}^{\mu\nu}(p_j)}$$

Contain dependence on Loop Integrals

Feynman Integrals

$$I(p_j, m_j) = \int \prod_{i=1}^{\ell} D^d k_j \, \frac{S_1^{a_1} \cdots S_\rho^{a_\rho}}{D_1^{b_1} \cdots D_\tau^{b_\tau}}$$

Integer powers

$$S_j = \begin{cases} p_i \cdot k_j \\ k_i \cdot k_j \end{cases}$$

τ propagators

ρ scalar products

How many scalar products are there?

- N external momenta
- ℓ loops

$$\rho = \ell\left(N + \frac{\ell}{2} - \frac{1}{2}\right)$$

$$\int D^d k \frac{k\cdot p}{(k^2+m^2)((k-p)^2+m^2)} = \int \frac{k\cdot p}{D_1 D_2} \quad *$$

$$k\cdot p = \frac{1}{2}\left[(k^2+m^2) - ((k-p)^2+m^2) + p^2\right]$$

$$= \frac{1}{2}\left[D_1 - D_2 + p^2\right]$$

$$\int D^d k \frac{k\cdot p}{D_1 D_2} = \frac{1}{2}\left[\underset{*}{\cancel{\int \frac{D^d k}{D_2}}} - \underset{*}{\cancel{\int \frac{D^d k}{D_1}}} + p^2 \int \frac{D^d k}{D_1 D_2}\right]$$

$$\int \frac{D^d k}{D_2} = \int \frac{D^d k}{(k-p)^2+m^2} \xrightarrow{k \to k+p} \int \frac{D^d k}{k^2+m^2}$$

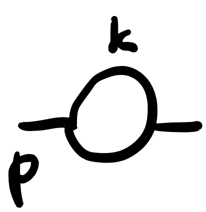
$$= \int \frac{D^d k}{D_1}$$

$$\int D^d k \, \frac{k \cdot p}{D_1 D_2} = \frac{p^2}{2} \underbrace{\int D^d k \, \frac{1}{D_1 D_2}}$$

REDUCIBLE SCALAR PRODUCTS

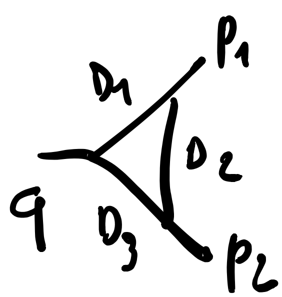
@ 1 loop ALL SCALAR PRODUCTS ARE REDUCIBLE

1. 2 legs



$k \cdot k$	D_1
$k \cdot p$	D_2

2. 3 legs

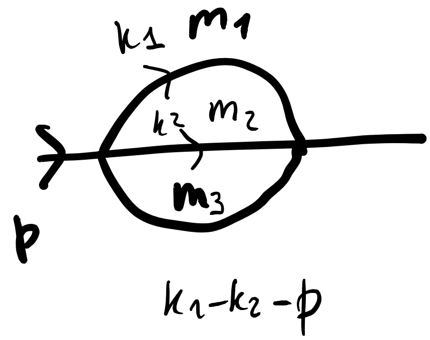


$k \cdot k$	D_1
$k \cdot p_1$	D_2
$k \cdot p_2$	D_3

4 LEGS ... etc. --

@ 2 loops

2 loop SUNRISE SUNSET



$$D_1 = k_1^2 + m_1^2$$

$$D_2 = k_2^2 + m_2^2$$

$$D_3 = (k_1 - k_2 - p)^2 + m_3^2$$

Scalar products

$$\left.\begin{matrix} k_1 \cdot k_1 & k_1 \cdot p \\ k_1 \cdot k_2 & k_2 \cdot p \\ k_2 \cdot k_2 & \end{matrix}\right]$$

5 scalar products

3 propagators

irreducible SCALAR PRODUCTS

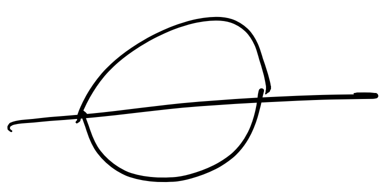
$$I(n_1, n_2, n_3, n_4, n_5) = \int D^d k_1 D^d k_2 \frac{S_4^{n_4} S_5^{n_5}}{D_1^{n_1} D_2^{n_2} D_3^{n_3}}$$

$$S_4 = k_1 \cdot p \qquad S_5 = k_2 \cdot p$$

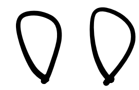
$$\underbrace{I(p_i) = \int \prod_{j=1}^{\ell} D^d k_j \, \frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{\underbrace{D_1^{b_1} \cdots D_\tau^{b_\tau}}_{}}}_{} \longleftarrow \text{IRR.}$$

FEYNMAN INTEGRAL

- FAMILY OF INTEGRALS
- TOPOLOGY = set of propagators

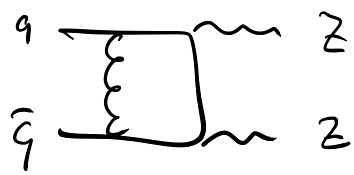
 $= \int D^d k_1 D^d k_2 \, \frac{S_4^{n_4} S_5^{n_5}}{D_1^{n_1} D_2^{n_2} \cancel{D_3^{n_3}}}$

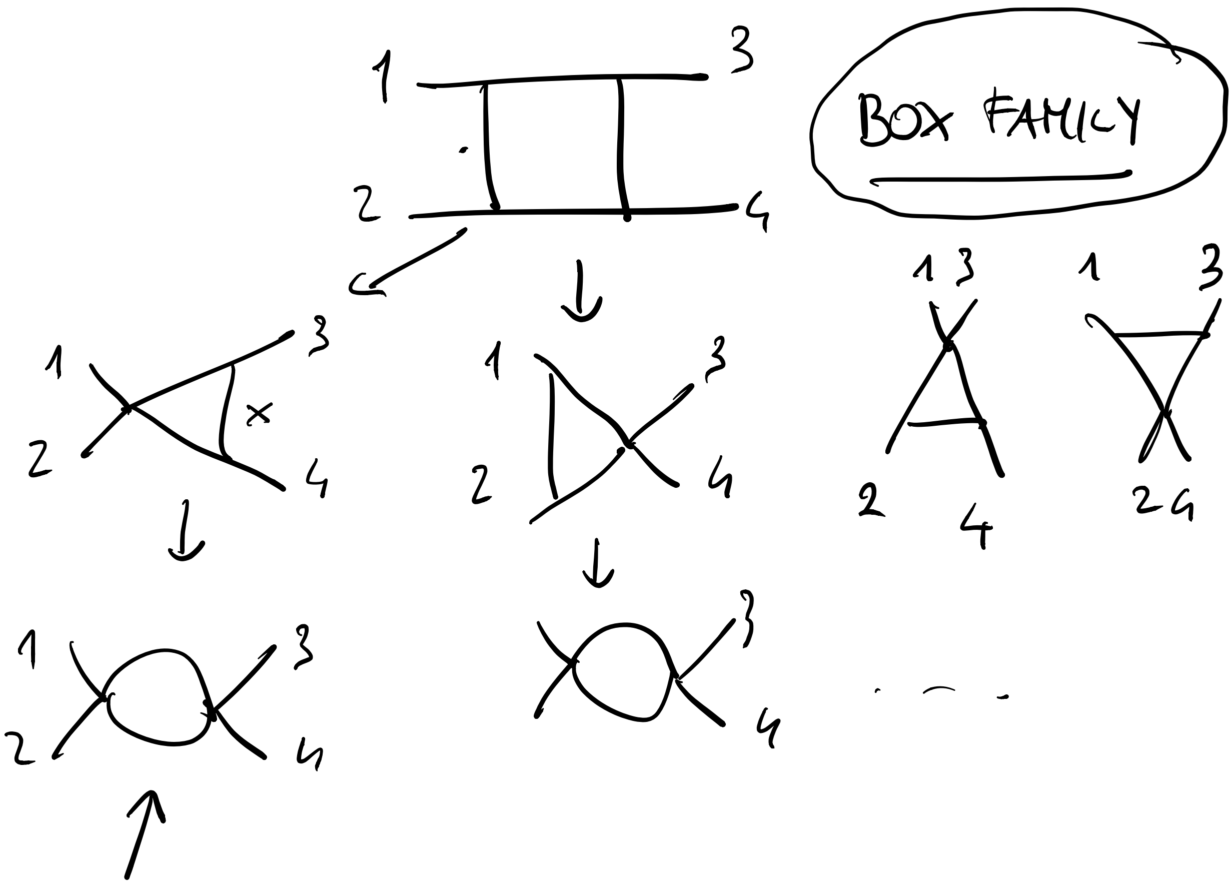
$n_3 = 0$ $\quad \int D^d k_1 D^d k_2 \, \frac{S_4^{n_4} S_5^{n_5} D_3^{n_3>0}}{\underbrace{D_1^{n_1} D_2^{n_2}}_{}}$

$\int D^d k_1 D^d k_2 \, \frac{1}{D_1 D_2} = \frac{1}{(k_1^2+m_1^2)(k_2^2+m_2^2)} =$ 

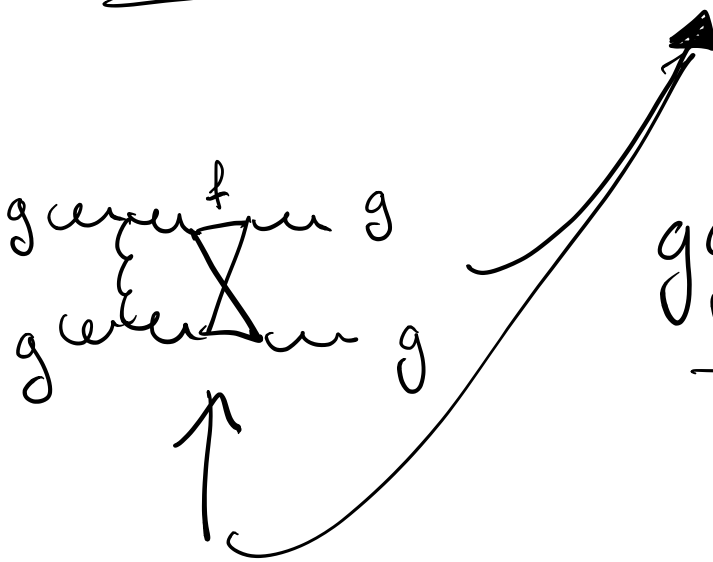
In general, I need to compute ALL INTEGRALS in the <u>SUBTOPOLOGY TREE</u>

$q\bar{q} \to ZZ$ @ 1 Loop

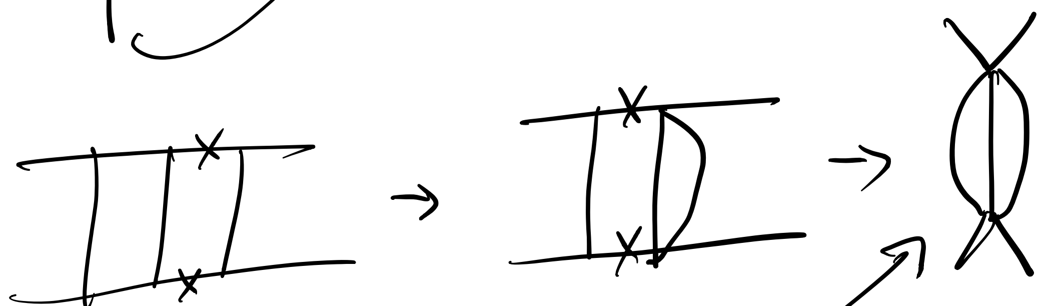




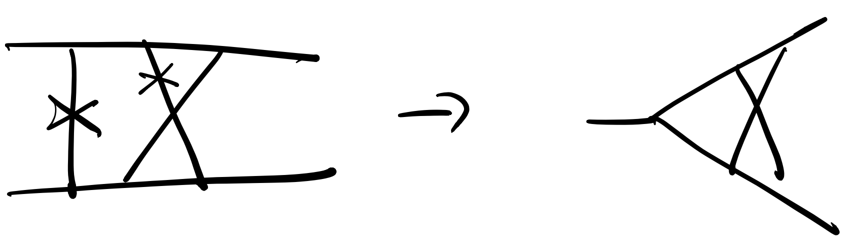
NON PLANAR GRAPH



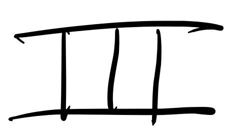
gg → gg @ 2 loops

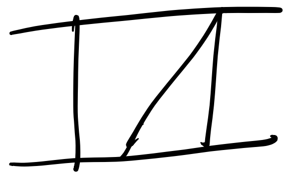


SUNRISE



integral family



 PLANAR BOX

INTEGRATION BY PARTS (IBPs)

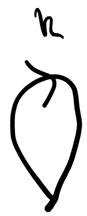
Reduce number of Feynman Integrals of many orders of magnitude !!

Dim-Reg

$$\int \prod_j D^d k_j \; \frac{\partial}{\partial k_\ell^{\mu}} \left[v^{\mu} \frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}} \right] = 0$$

$$v^\mu = (p_i^\mu \,;\, k_j^\mu) \qquad \frac{\partial}{\partial k_\ell^\mu}$$

TADPOLE @ 1 loop



$$= \int D^d k \, \frac{1}{(k^2+m^2)^n} = T(n)$$

1 loop momentum, no external momenta

$$\int D^d k \, \frac{\partial}{\partial k^\mu}\left[k^\mu \frac{1}{(k^2+m^2)^n}\right] = 0$$

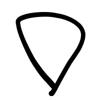
$$\frac{\partial}{\partial k_\mu} \frac{k^\mu}{(k^2+m^2)^n} = \frac{d}{(k^2+m^2)^n} - \frac{n\, k^\mu}{(k^2+m^2)^{n+1}} \, \underline{\underline{2k_\mu}}$$

$$= \frac{d}{(k^2+m^2)^n} - 2n\,\frac{k^2}{(k^2+m^2)^{n+1}} \qquad k^2 = \underbrace{k^2+m^2}_{D_1} - m^2$$

$$= \frac{(d-2n)}{(k^2+m^2)^n} + 2n\,m^2\,\frac{1}{(k^2+m^2)^{n+1}} = 0$$

$$(d-2n)\,T(n) + 2n\,m^2\,T(n+1) = 0$$

$$T(n+1) = -\frac{(d-2n)}{2n\,m^2}\,T(n)$$

$$T(2) = -\frac{(d-2)}{2m^2}\,T(1) = -\frac{(d-2)}{2m^2}$$  $=$ 

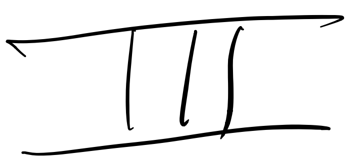
$$T(3) = -\frac{d-4}{4m^2}\,T(2) = \frac{(d-4)(d-2)}{8m^4}\,T(1)$$

TADPOLE HAS 1 MASTER INTEGRAL

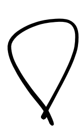
$T(2)$; $T(3)$

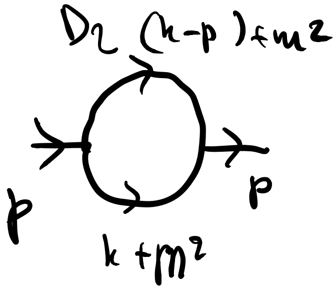
$$T(1) = \int \frac{D^d k}{(k^2+m^2)}$$

MI of TADPOLE graph

 ?

IMPORTANT @1 loop $\exists$ 1 MI for every TOPOLOGY

 1 MI

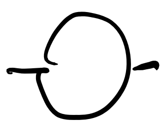


$$= \int D^d k \, \frac{1}{D_1^{n_1} D_2^{n_2}} = I(n_1, n_2)$$

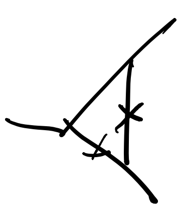
D1

$$\left[\begin{array}{l} \displaystyle\int \frac{\partial}{\partial k^\mu} \cdot k_\mu \left(\frac{1}{D_1^{n_1} D_2^{n_2}} \right) = 0 \qquad 2 \text{ IBPs} \\ \displaystyle\int \frac{\partial}{\partial k^\mu} \, p_\mu \left(\frac{1}{D_1^{n_1} D_2^{n_2}} \right) = 0 \end{array}\right]$$

PLEASE !

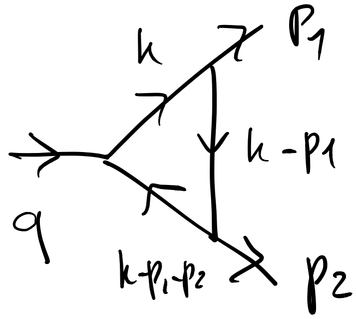


$$\rightarrow \int \frac{D^d k}{D_1 D_2} + \int \frac{D^d k}{D_1}$$



$$= \int \frac{D^d k}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} \rightarrow \int \frac{D^d k}{D_1 D_2 D_3}$$

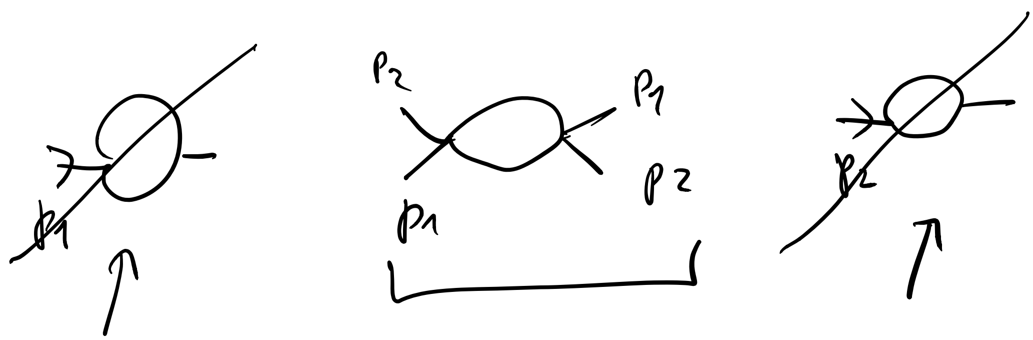
@ 1 loop AT MOST 1 MI per <u>TOPOLOGY</u>



massless ; $p_1^2=p_2^2=0$

$q^2=(p_1+p_2)^2=s$

$$I(n_1,n_2,n_3)=\int \frac{D^d k}{D_1^{n_1}\,D_2^{n_2}\,D_3^{n_3}}$$



$\int D^d k \, \frac{1}{k^2(k-p_1)^2}$; $p_1^2=0$ <u>DIM REG</u>

3 IBPs

$$\int \mathcal{D}^d k \, \frac{\partial}{\partial k_\mu} \left\{ \begin{array}{c} k_\mu \\ p_{1\mu} \\ p_{2\mu} \end{array} \right\} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = 0$$

LAPORTA ALGORITH :

explicit values of n_1, n_2, n_3

$$I(1,1,1) = \int \mathcal{D}^d k \, \frac{1}{D_1 D_2 D_3}$$

EXERCISE

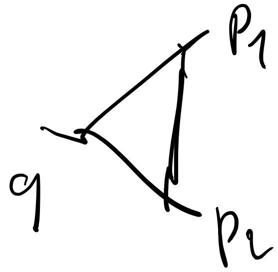
$$\begin{cases} s I(1,1,2) + (d-4) I(1,1,1) = 0 & (1) \\ s I(1,1,2) + I(1,0,2) + I(2,0,1) = 0 & (2) \\ s I(2,1,1) + I(1,0,2) + I(2,0,1) = 0 \end{cases}$$

1) $$I(1,1,2) = \frac{d-4}{s} I(1,1,1)$$

$$(d-4)\, I(1,1,1) = -I(1,0,2) - I(2,0,1)$$

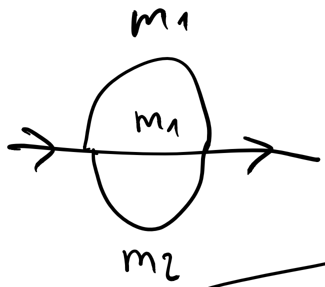
$$I(1,1,1) = -\frac{1}{(d-4)} \left[I(1,0,2) + I(2,0,1) \right]$$

$$I(n_1, 0, n_3) = \int D^d k \, \frac{1}{D_1^{n_1} D_3^{n_3}}$$

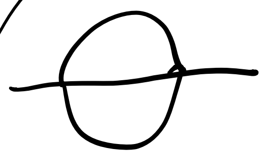


REDUCIBLE TO BUBBLES

$=$ 

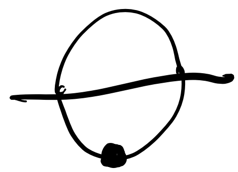


= 2 MIs + simpler



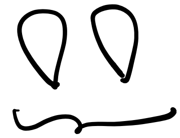
$$\int \frac{D^d k}{D_1 D_2 D_3}$$

+



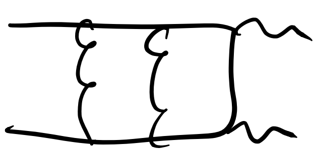
$$\int \frac{D^d k}{D_1 D_2 D_3^2}$$

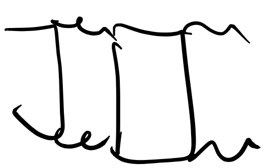
+



$$\int \frac{D^d k}{D_1 D_2}$$

$q\bar{q} \to ZZ$ @ 2loops





+ - - - .

143

Scalar integrals $\simeq \mathcal{O}(5000)$

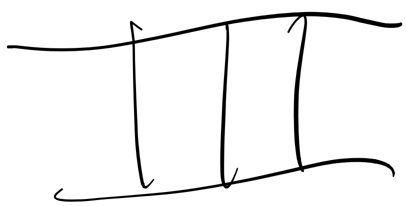
MIs $\simeq \mathcal{O}(50)$

2 orders of magnitude

Codes

Reduze
AIR
FIRE
KIRA

AUTOMATED CODES



$q\bar{q} \to ZZ$ @ 2 loop → QGRAF →

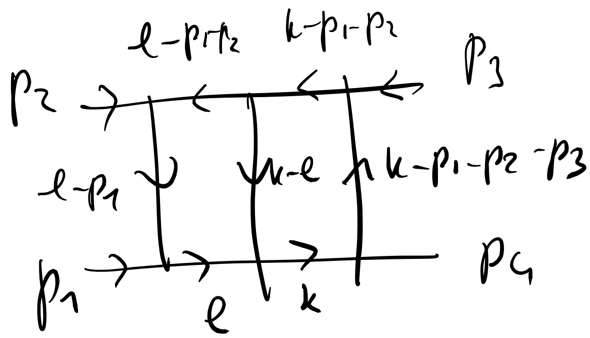
s MAX
t DEN

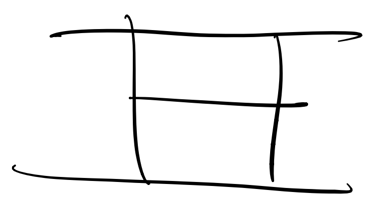
7 den , 4 scal pr

Seeds 7 den 4 scal prod

Hi

$$\underbrace{I(1\,1\,1\,1\,1\,1\,1\;\; n_1\;\; n_2)}_{I(1\,0\,\cdots\; n_1,\, n_2)}$$





$$\left\{\begin{array}{l} k^2 \\ l^2 \\ (k-l)^2 \\ (l-p_1)^2 \\ (k-p_1-p_2)^2 \\ (l-p_1-p_2)^2 \\ (k-p_1-p_2-p_3)^2 \\ \left.\begin{array}{l}(l-p_1-p_2-p_3)^2 \\ (k-p_1)^2\end{array}\right\} \end{array}\right.$$