Infrared subtractions in perturbative quantum chromodynamics

Raoul Röntsch

Karlsruhe Institute of Technology

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Outline of lectures

- **Lecture 1: The origin and universality of infrared divergences in QCD:**
  - Infrared singularities due to soft and collinear radiation.
  - Universal factorization of infrared radiation.
  - Interpretation of infrared singularities.
  - Regularization in d-dimensions.
- **Lecture 2: Infrared subtractions at next-to-leading order:**
  - Introduction to the Frixione-Kunszt-Signer subtraction scheme.
  - Extraction of soft and collinear poles in color singlet production.
  - Partial cancellation of poles between real and virtual corrections.
  - Pdf renormalization and cancellation of remaining poles.
  - Final result for fully finite NLO correction.
- **Lecture 3: Infrared subtractions at next-to-next-to-leading order:**
  - Obstacles to treating singularities at NNLO.
  - Overview of different NNLO subtraction schemes.
  - Extension of FKS subtraction scheme – Nested soft-collinear subtraction. Independence of soft and collinear radiation, phase space partitioning and sector decomposition.
  - (Sketch of) Extraction and cancellation of poles in NNLO corrections to color singlet production.

Useful references

- The calculation of the soft and collinear limit:
  S. Catani and M. Grazzini, Nucl. Phys. **B570** (2000) 287. [hep-ph/9908523]
- The expression for infrared poles appearing in loop amplitudes:
  S. Catani, Phys. Lett. **B427** (1998) 161 [hep-ph/9802439]
- *"QCD and Collider Physics"*, R. K. Ellis, W. J. Stirling and B. R. Webber, Cambridge University Press (1996).
  The IR behavior of QCD is discussed in Chapter 4.3.
- F. Caola, K. Melnikov, R. Röntsch, Eur. Phys. J. **C77** (2017) 248 [hep-ph/1702.01352].
  The second lecture will follow section 3 of this paper quite closely, and the third lecture will summarize sections 4 and 8.
- The original references for FKS subtraction:
  S. Frixione, Z. Kunszt, A. Signer Nucl. Phys. **B467** (1996) 399 [hep-ph/9512328]
  S. Frixione, Nucl. Phys. B507 (1997) 295 [hep-ph/9706545]
- Sector decomposition for NNLO subtractions:
  M. Czakon, Phys. Lett. **B693** (2010) 259 [hep-ph/1005.0274]
  M. Czakon, Nucl. Phys. **B849** (2011) 250 [hep-ph/1101.0642]

Lecture 1: The origin and universality of infrared divergences in QCD

Preliminary remarks (1)

Perturbative QCD corrections can be divided into two types: real radiation and virtual (loop) corrections.

- Different **final state multiplicity**: virtual corrections have n particles, real corrections have $n+1$.
- Real and virtual corrections have **no physical meaning by themselves**:
  - We cannot speak of "virtual" or "real radiation" cross sections – only a cross section (or differential distribution) at LO, NLO, NNLO, etc.
  - The division is thus purely for our convenience as we organize the calculation of a perturbative correction.
- Real and virtual corrections are fundamentally linked by their infrared (IR, i.e. low energy) behavior.
- The calculation of loop amplitudes for virtual corrections is being covered by Simon Badger and Lorenzo Tancredi.
- I will focus on the real radiation corrections in these lectures.

Drell-Yan production

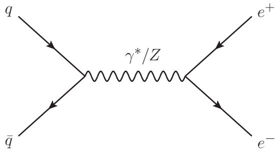
I will frequently use **Drell-Yan production** as a test process.

This is just the production of an electron-positron pair at a hadron collider.

At the partonic level at LO this is

$$q\bar{q} \to e^- e^+,$$

and proceeds through the exchange of an offshell photon γ^* or a Z-boson.



The matrix element for this is

$$\mathcal{M}(1,2,e^-,e^+) = \delta_{ij}\bar{v}(p_2)\gamma^\mu u(p_1)L_\mu(p_{e^-},p_{e^+}) \equiv \delta_{ij}\tilde{\mathcal{M}}(1,2,e^-,e^+),$$

where the factor $L_\mu(p_{e^-},p_{e^+})$ contains the lepton vertex and photon and Z-boson propagators, which are not relevant for our purposes.

LO amplitude squared for DY production

The LO matrix element for Drell-Yan production is

$$\mathcal{M}(1,2,e^-,e^+) = \delta_{ij}\bar{v}(p_2)\gamma^\mu u(p_1) L_\mu(p_{e^-},p_{e^+}) \equiv \delta_{ij}\tilde{\mathcal{M}}(1,2,e^-,e^+).$$

We will need its absolute value-squared in this lecture.

Squaring the color factor δ_{ij} gives the number of colors N_c:

$$|\mathcal{M}(1,2,e^-,e^+)|^2 = N_c|\tilde{\mathcal{M}}(1,2,e^-,e^+)|^2.$$

In more detail

$$\begin{aligned}\sum_{\text{hel.}} |\mathcal{M}(1,2,e^-,e^+)|^2 &= N_c \sum_{\text{hel.}} |\bar{v}(p_2)\gamma^\mu u(p_1) L_\mu(p_{e^-},p_{e^+})|^2 \\ &= N_c \sum_{\text{hel.}} \mathrm{Tr}\left\{\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu v(p_2)\right\} |L|^2_{\mu\nu}(p_{e^-},p_{e^+}) \\ &= N_c \mathrm{Tr}\left\{\hat{p}_2\gamma^\mu\hat{p}_1\gamma^\nu\right\} |L|^2_{\mu\nu}(p_{e^-},p_{e^+}),\end{aligned}$$

where we used the spinor sums in the final equality, and the notation $\hat{p} = p^\mu\gamma_\mu$ (you may be more familiar with the Feynman-slash notation).

Real corrections to Drell-Yan production

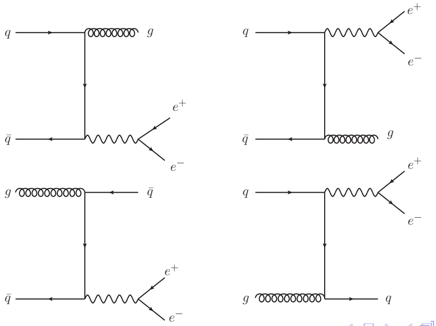
Real radiative corrections to DY production come from the emission of a gluon in the final state

$$q(p_1)\bar{q}(p_2) \to e^- e^+ + g(p_3)$$

and the partonic crossings

$$q(p_1)g(p_2) \to e^- e^+ + q(p_3) \qquad g(p_1)\bar{q}(p_2) \to e^- e^+ + \bar{q}(p_3).$$

(We should also include the $\bar{q}q$, gq and $\bar{q}g$ channels, but these can be obtained by a swap $p_1 \leftrightarrow p_2$.)



Real radiative corrections to Drell-Yan production

We will focus on the $q(p_1)\bar{q}(p_2) \to e^- e^+ + g(p_3)$ process.

To obtain the real radiation corrections we need to:

- **calculate** the tree-level amplitude $\mathcal{M}(1, 2, e^-, e^+, 3)$ for this process;
- **sum** the absolute value-squared of the amplitude over all possible helicities, $\sum_{\text{hel.}} |\mathcal{M}(1, 2, e^-, e^+, 3)|^2$;
- **average** over initial state colors and spins;
- **integrate** over the final state phase space.

This is just a tree-level amplitude, so this should be trivial, right?

Energy and angular scaling of phase space

To see why this is *not* trivial, let's look at the phase space for the radiated gluon. It is

$$[\mathrm{d}p_3] = \frac{\mathrm{d}^3 p_3}{2E_3(2\pi)^3} \sim \frac{\mathrm{d}E_3 E_3^2}{E_3} \mathrm{d}\theta_{13} \sin\theta_{13} \sim \mathrm{d}E_3 E_3 \; \mathrm{d}\theta_{13} \sin\theta_{13}$$

where we have looked at the scaling of the energy E_3 and of the angle θ_{13} between $\vec{p}_3$ and $\vec{p}_1$ ($\vec{p}_1$ is a reference vector about which $\vec{p}_3$ rotates).

We will return to the angular integral shortly; for now, let's look at the energy integral.

The limits of the energy integral are 0 and E_{max} (which is defined by energy conservation). Thus we have

$$\int_0^{E_{\mathrm{max}}} \mathrm{d}E_3 E_3 |\mathcal{M}(1,2,e^-,e^+,3)|^2 .$$

If $\lim_{E_3 \to 0} |\mathcal{M}(1,2,e^-,e^+,3)|^2 \sim E_3^{-2}$, then the integral **does not converge** on the required interval.

Computing the amplitude in the soft limit (1)

We therefore need to check what happens to the amplitude as $E_3 \to 0$, i.e. the **soft behavior** of the amplitude. The amplitude is

$$\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij} \bar{v}(p_2) \left[\gamma^\mu \frac{\hat{p}_1 - \hat{p}_3}{(p_1 - p_3)^2} \hat{\epsilon} - \hat{\epsilon} \frac{\hat{p}_2 - \hat{p}_3}{(p_2 - p_3)^2} \gamma^\mu \right] u(p_1) L_\mu(p_{e^-}, p_{e^+}),$$

where

- The two terms come from the two diagrams shown,
- T^a_{ij} is the color matrix for the quark and g_s is the strong coupling,
- ϵ is the polarization vector for the radiated gluon,
- The factor $L_\mu(p_{e^-}, p_{e^+})$ is the same as at LO,

Computing the amplitude in the soft limit (2)

$$\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij}\bar{v}(p_2)\left[\gamma^\mu \frac{\hat{p}_1-\hat{p}_3}{(p_1-p_3)^2}\hat{\epsilon} - \hat{\epsilon}\frac{\hat{p}_2-\hat{p}_3}{(p_2-p_3)^2}\gamma^\mu\right] u(p_1) L_\mu(p_{e^-},p_{e^+})$$

In the limit $E_3 \to 0$:

- $p_3 \to 0$ so we drop $\hat{p}_3$ in the numerators,
- The denominators $(p_1-p_3)^2 = -2p_1\cdot p_3 \sim E_3$ and similarly $(p_2-p_3)^2 = -2p_2\cdot p_3 \sim E_3$ so we keep these.

So we have

$$\lim_{E_3\to 0}\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij}\bar{v}(p_2)\left[\gamma^\mu\left(\frac{\hat{p}_1}{-2p_1\cdot p_3}\right)\hat{\epsilon} - \hat{\epsilon}\left(\frac{\hat{p}_2}{-2p_2\cdot p_3}\right)\gamma^\mu\right]u(p_1) \times L_\mu(p_{e^-},p_{e^+}).$$

Computing the amplitude in the soft limit (3)

Now use

$$\hat{p}_1\hat{\epsilon}u(p_1) = p_1^\mu\epsilon^\nu\gamma_\mu\gamma_\nu u(p_1) = p_1^\mu\epsilon^\nu(2g_{\mu\nu} - \gamma_\nu\gamma_\mu)u(p_1)$$
$$= (2p_1\cdot\epsilon - \hat{\epsilon}\hat{p}_1)u(p_1) = (2p_1\cdot\epsilon)u(p_1),$$

where the last equality follows from the Dirac equation $\hat{p}_1 u(p_1) = 0$. Similarly

$$\bar{v}(p_2)\hat{\epsilon}\hat{p}_2 = \bar{v}(p_2)(2\epsilon\cdot p_2 - \hat{p}_2\hat{\epsilon}) = 2\bar{v}(p_2)(\epsilon\cdot p_2),$$

using $\bar{v}(p_2)\hat{p}_2 = 0$.

Then

$$\lim_{E_3\to 0}\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij}\bar{v}(p_2)\left[\gamma^\mu\left(\frac{p_1\cdot\epsilon}{-p_1\cdot p_3}\right) - \left(\frac{p_2\cdot\epsilon}{-p_2\cdot p_3}\right)\gamma^\mu\right]u(p_1)$$
$$\times L_\mu(p_{e^-},p_{e^+})$$
$$= -g_s T^a_{ij}\left[\frac{p_1\cdot\epsilon}{p_1\cdot p_3} - \frac{p_2\cdot\epsilon}{p_2\cdot p_3}\right]\bar{v}(p_2)\gamma^\mu u(p_1)L_\mu(p_{e^-},p_{e^+})$$

Computing the amplitude in the soft limit (4)

Recall that the LO matrix element is

$$\mathcal{M}(1,2,e^-,e^+) = \delta_{ij}\bar{v}(p_2)\gamma^\mu u(p_1)L_\mu(p_{e^-},p_{e^+}) \equiv \delta_{ij}\tilde{\mathcal{M}}(1,2,e^-,e^+),$$

where $\tilde{\mathcal{M}}$ indicates the LO matrix element without the color factor δ_{ij}. This means we can write the soft limit of the gluon emission amplitude in terms of the **LO amplitude:**

$$\lim_{E_3\to 0}\mathcal{M}(1,2,e^-,e^+,3) = -g_s T^a_{ij}\left[\frac{p_1\cdot\epsilon}{p_1\cdot p_3} - \frac{p_2\cdot\epsilon}{p_2\cdot p_3}\right]\tilde{\mathcal{M}}(1,2,e^-,e^+).$$

Squaring the amplitude in the soft limit

We now need to square the amplitude. First we note

$$(p_i \cdot \epsilon)(p_j \cdot \epsilon^*) = \epsilon_\mu \epsilon^*_\nu p_i^\mu p_j^\nu = -p_i.p_j\,,$$

using the polarization sum $\sum \epsilon_\mu \epsilon^*_\nu = -g_{\mu\nu}$. Then we consider

$$\begin{aligned}\left|\frac{p_1 \cdot \epsilon}{p_1 \cdot p_3} - \frac{p_2 \cdot \epsilon}{p_2 \cdot p_3}\right|^2 =& \frac{(p_1 \cdot \epsilon)(p_1 \cdot \epsilon^*)}{(p_1 \cdot p_3)^2} + \frac{(p_2 \cdot \epsilon)(p_2 \cdot \epsilon^*)}{(p_2 \cdot p_3)^2} \\ &- \frac{(p_1 \cdot \epsilon)(p_2 \cdot \epsilon^*)}{(p_1 \cdot p_3)(p_2 \cdot p_3)} - \frac{(p_2 \cdot \epsilon)(p_1 \cdot \epsilon^*)}{(p_1 \cdot p_3)(p_2 \cdot p_3)} \\ =& \frac{2p_1 \cdot p_2}{(p_1 \cdot p_3)(p_2 \cdot p_3)},\end{aligned}$$

where I have used $p_1 \cdot p_1 = p_2 \cdot p_2 = 0$ in the first two terms.

Therefore

$$\lim_{E_3 \to 0} |\mathcal{M}(1,2,e^-,e^+,3)|^2 = g_s^2 \frac{N_c^2 - 1}{2} \frac{2p_1 \cdot p_2}{(p_1 \cdot p_3)(p_2 \cdot p_3)} |\tilde{\mathcal{M}}(1,2,e^-,e^+)|^2,$$

where the factor $(N_c^2 - 1)/2$ comes from squaring the T^a matrix.

Factorization in the soft gluon limit (1)

Recall that the square of the LO amplitude is

$$\left|\mathcal{M}(1,2,e^-,e^+)\right|^2 = N_c|\tilde{\mathcal{M}}(1,2,e^-,e^+)|^2,$$

where N_c is the number of colors, which comes from squaring the color factor δ_{ij}.

Therefore

$$\begin{aligned}\lim_{E_3\to 0}|\mathcal{M}(1,2,e^-,e^+,3)|^2 &= g_s^2\frac{N_c^2-1}{2N_c}\frac{2p_1\cdot p_2}{(p_1\cdot p_3)(p_2\cdot p_3)}|\mathcal{M}(1,2,e^-,e^+)|^2\\ &= \mathrm{Eik}(1,2,3)|\mathcal{M}(1,2,e^-,e^+)|^2,\end{aligned}$$

where

$$\mathrm{Eik}(1,2,3) = g_s^2 C_F\frac{2p_1\cdot p_2}{(p_1\cdot p_3)(p_2\cdot p_3)},$$

is the eikonal factor, and $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$.

Factorization in the soft gluon limit (2)

$$\lim_{E_3 \to 0} |\mathcal{M}(1,2,e^-,e^+,3)|^2 = \text{Eik}(1,2,3) |\mathcal{M}(1,2,e^-,e^+)|^2.$$

with

$$\text{Eik}(1,2,3) = g_s^2 C_F \frac{2 p_1 \cdot p_2}{(p_1 \cdot p_3)(p_2 \cdot p_3)}.$$

In the soft gluon limit, the amplitude-squared **factorizes** into:

- the amplitude-squared for the hard (i.e. energetic) process without the gluon, and
- an eikonal factor depending only on the color charges (C_F) and momenta of the hard partons (p_1 and p_2) and the momentum of the soft gluon (p_3).

We have shown this factorization in the soft gluon limit for Drell-Yan production, but in fact it is **universal**.

In the limit of a soft gluon emission, the amplitude-squared factorizes into a hard amplitude-squared without the additional gluon, and an eikonal factor which depends on the momenta and colors of the hard partons and the momentum of the soft gluon.

General expression for the soft gluon limit

For a generic QCD amplitude $\mathcal{M}$ with n partons with four-momenta $p_1, p_2, \ldots, p_n$ and a soft gluon with vanishing four-momentum q, the **soft gluon factorization formula** is[a]

$$|\mathcal{M}(p_1, p_2, \ldots, p_n; q)|^2 = -4\pi\alpha_s\mu^{2\epsilon} \sum_{i,j=1}^{n} \mathcal{S}_{ij}(q)\vec{T}_i \cdot \vec{T}_j \, |\mathcal{M}(p_1, p_2, \ldots, p_n)|^2 ,$$

where the general eikonal factor is

$$\mathcal{S}^{(i,j)}(q) = \frac{p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)},$$

and the square of color charges are $\vec{T}_i^2 = C_F$ if $i = q, \bar{q}$ and $\vec{T}_i^2 = C_A$ if $i = g$.

[a] S. Catani and M. Grazzini, Nucl. Phys. **B570** (2000) 287.

Applying the general soft factorization formula to DY production

If we apply this to DY production, we have

$$|\mathcal{M}(p_1,p_2,p_3)|^2 = -g_s^2 \left(\mathcal{S}_{12}(p_3)\vec{T}_q\cdot\vec{T}_{\bar{q}}\,|\mathcal{M}(p_1,p_2)|^2 + \mathcal{S}_{21}(p_3)\vec{T}_{\bar{q}}\cdot\vec{T}_q\,|\mathcal{M}(p_1,p_2)|^2\right),$$

using $4\pi\alpha_s = g_s^2$ (and ignoring the d-dimensional renormalization of α_s for now).

To calculate $\vec{T}_q\cdot\vec{T}_{\bar{q}} = \vec{T}_{\bar{q}}\cdot\vec{T}_q$, we use the fact that color conservation implies

$$\vec{T}_q + \vec{T}_{\bar{q}} = 0 \Rightarrow \left(\vec{T}_q + \vec{T}_{\bar{q}}\right)^2 = 0 \Rightarrow 2C_F + 2\vec{T}_q\cdot\vec{T}_{\bar{q}} = 0 \Rightarrow \vec{T}_q\cdot\vec{T}_{\bar{q}} = -C_F.$$

Then, using

$$\mathcal{S}^{(1,2)}(p_3) = \mathcal{S}^{(2,1)}(p_3) = \frac{p_1\cdot p_2}{(p_1\cdot p_3)(p_2\cdot p_3)},$$

we recover the previous expression:

$$|\mathcal{M}(p_1,p_2,p_3)|^2 = g_s^2 2C_F \frac{p_1\cdot p_2}{(p_1\cdot p_3)(p_2\cdot p_3)}\,|\mathcal{M}(p_1,p_2)|^2.$$

Factorization in the soft gluon limit (3)

The energy scaling of the eikonal factor encountered in DY production is

$$\mathrm{Eik}(1,2,3) = g_s^2 C_F \frac{2p_1 \cdot p_2}{(p_1 \cdot p_3)(p_2 \cdot p_3)} \sim \frac{1}{E_3^2},$$

so the energy integral for real emissions **does not converge!**

$$\int_0^{E_{\max}} \frac{\mathrm{d}E_3}{E_3} \to \infty$$

Indeed, looking at the general factorization formula, we observe that *every* term has a factor

$$\mathcal{S}^{(i,j)}(q) = \frac{p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} \sim \frac{1}{E_q^2},$$

so that the energy integral does not converge for soft gluon emissions in **any** process!

$$\int_0^{E_{\max}} \frac{\mathrm{d}E_q}{E_q} \to \infty$$

Angular phase space and collinear limit of the eikonal function

I will comment on the physical interpretation of these divergences and how to treat them later.
Now, I want to return to the gluon phase space and consider the angular phase space integral.

Recall that the gluonic phase space contained an integration over the angles

$$[\mathrm{d}p_3] \sim \mathrm{d}\theta_{13} \sin\theta_{13} \xrightarrow{\theta_{13}\to 0} \mathrm{d}\theta_{13}\,\theta_{13}.$$

We can immediately see there will be trouble if we consider the emission of a gluon that is both soft and collinear to p_1:

The soft limit gives rise to the eikonal factor which has a factor $p_1 \cdot p_3 = 2E_1E_3(1-\cos\theta_{13})$ in the denominator.

Applying the limit $\theta_{13} \to 0$ limit in addition to the soft limit, we have $p_1 \cdot p_3 \sim \theta_{13}^2 \Rightarrow |\mathcal{M}|^2 \sim (p_1 \cdot p_3)^{-1} \sim \theta_{13}^{-2}$ and the angular integral **also does not converge**,

$$\int_0^\pi \frac{\mathrm{d}\theta_{13}}{\theta_{13}} \to \infty$$

Collinear limit – Sudakov decomposition

So in the combined **soft-collinear limit**, both the **energy** and the **angular** integrals do not converge.

What happens if we consider the emission of a hard but collinear gluon?

To answer this question, we first apply a *Sudakov decomposition* for the momentum of the gluon

$$p_3 = x p_1 + y p_2 + p_\perp, \tag{1}$$

where $p_1 \cdot p_\perp = p_2 \cdot p_\perp = 0$ and $p_\perp = (0, p_\perp \sin\phi, p_\perp \cos\phi, 0)$. The gluon is onshell so

$$p_3^2 = 0 = 2xy(p_1 \cdot p_2) - p_\perp^2 \Rightarrow sxy = p_\perp^2, \tag{2}$$

using the center-of-mass energy $s = (p_1 + p_2)^2 = 2p_1 \cdot p_2$.
Using Eqs. (1) and (2) we get the scaling in the $\theta_{13} \to 0$ limit:

$$p_1 \cdot p_3 = y(p_1 \cdot p_2) \sim \theta_{13}^2 \Rightarrow y \sim \theta_{13}^2$$
$$p_2 \cdot p_3 = x(p_1 \cdot p_2) \sim 1 \Rightarrow x \sim 1$$
$$\Rightarrow sxy \sim s\,\theta_{13}^2 \;\Rightarrow p_\perp \sim \sqrt{s}\,\theta_{13}$$

Computing the amplitude in the collinear limit (1)

Recall that the $q(p_1)\bar{q}(p_2) \to e^- e^+ g(p_3)$ amplitude is

$$\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij} \bar{v}(p_2) \left[\gamma^\mu \frac{\hat{p}_1 - \hat{p}_3}{(p_1-p_3)^2} \hat{\epsilon} - \hat{\epsilon} \frac{\hat{p}_2 - \hat{p}_3}{(p_2-p_3)^2} \gamma^\mu \right] u(p_1) L_\mu(p_{e^-}, p_{e^+}).$$

Looking at the denominators, the first term scales $\sim \theta_{13}^{-2}$ while the second term scales $\sim x \sim 1$.

We will need to take the absolute value-squared $|\mathcal{M}(1,2,e^-,e^+,3)|^2$.

Squaring the first term gives a leading singularity $\sim \theta_{13}^{-4}$, while the interference between the first and second terms gives a subleading singularity $\sim \theta_{13}^{-2}$.

This seems to **contradict** our earlier statement that the leading singularity is $|\mathcal{M}(1,2,e^-,e^+,3)|^2 \sim \theta_{13}^{-2}$ in the soft-collinear limit – the leading singularity appears to be much more severe.

We shall see that using *physical polarization vectors for the gluons* gives a further scaling $\sim \theta_{13}^2$.

Thus the leading singularity is $\sim \theta_{13}^{-2}$ from the square of the first term, and the interference between the first and second terms is non-singular $\mathcal{O}(1)$.

Computing the amplitude in the collinear limit (2)

We will therefore only consider the first term

$$\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij}\bar{v}(p_2)\left[\gamma^\mu \frac{\hat{p}_1 - \hat{p}_3}{(p_1-p_3)^2}\hat{\epsilon}\right] u(p_1) L_\mu(p_{e^-},p_{e^+}) + \mathcal{O}(1),$$

and ignore the remaining contributions which are $\mathcal{O}(1)$.

Using the Sudakov decomposition we find

$$\lim_{\theta_{13}\to 0}\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij}\bar{v}(p_2)\left[\gamma^\mu \frac{(1-x)\hat{p}_1 - y\hat{p}_2 - \hat{p}_\perp}{(p_1-p_3)^2}\hat{\epsilon}\right] u(p_1) L_\mu(p_{e^-},p_{e^+}) + \mathcal{O}(1)$$

Since $y \sim \theta_{13}^2$ we can neglect the y term in the numerator as it gives an $\mathcal{O}(1)$ contribution.

Then using as before $\hat{p}_1\hat{\epsilon}u(p_1) = 2(p_1\cdot\epsilon)u(p_1)$ we have

$$\lim_{\theta_{13}\to 0}\mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij}\bar{v}(p_2)\left[\gamma^\mu \frac{2(1-x)p_1\cdot\epsilon - \hat{p}_\perp\hat{\epsilon}}{(p_1-p_3)^2}\right] u(p_1) L_\mu(p_{e^-},p_{e^+}) + \mathcal{O}(1).$$

Computing the amplitude in the collinear limit (3)

We then use $\epsilon \cdot p_3 = 0 \Rightarrow x\epsilon \cdot p_1 = -y\epsilon \cdot p_2 - \epsilon \cdot p_\perp$ and drop the y term as before

$$\lim_{\theta_{13}\to 0} \mathcal{M}(1,2,e^-,e^+,3) = g_s T^a_{ij} \frac{1}{(p_1-p_3)^2} \bar{v}(p_2) \left[\frac{-2(1-x)}{x} (p_\perp \cdot \epsilon)\gamma^\mu - \gamma^\mu \hat{p}_\perp \hat{\epsilon} \right] u(p_1)$$
$$\times L_\mu(p_{e^-}, p_{e^+}) + \mathcal{O}(1).$$

We have to take the absolute value squared of the amplitude, sum over all polarizations and take the the trace over the gamma matrices:

$$\lim_{\theta_{13}\to 0} \left|\mathcal{M}(1,2,e^-,e^+,3)\right|^2 = \frac{g_s^2}{(p_1-p_3)^2} \frac{N_c^2-1}{2} \frac{X}{(p_1-p_3)^2} |L_{\mu\nu}|^2 (p_{e^-}, p_{e^+}) + \mathcal{O}(1),$$

where

$$X = \mathrm{Tr}\left\{ \left[\bar{v}(p_2) \left(\frac{-2(1-x)}{x} p_\perp \cdot \epsilon \gamma^\mu - \gamma^\mu \hat{p}_\perp \hat{\epsilon} \right) u(p_1) \right] \right.$$
$$\left. \times \left[\bar{u}(p_1) \left(\frac{-2(1-x)}{x} p_\perp \cdot \epsilon^* \gamma^\nu - \hat{\epsilon}^* \hat{p}_\perp \gamma^\nu \right) v(p_2) \right] \right\}.$$

Computing the amplitude in the collinear limit (4)

Using the spinor sums

$$X = \mathrm{Tr}\Bigg\{ \left[\bar{v}(p_2) \left(\frac{-2(1-x)}{x} p_\perp \cdot \epsilon \gamma^\mu - \gamma^\mu \hat{p}_\perp \hat{\epsilon} \right) u(p_1) \right] \times \left[\bar{u}(p_1) \left(\frac{-2(1-x)}{x} p_\perp \cdot \epsilon^* \gamma^\nu - \hat{\epsilon}^* \hat{p}_\perp \gamma^\nu \right) v(p_2) \right] \Bigg\}$$

$$= \frac{4(1-x)^2}{x^2} (p_\perp \cdot \epsilon)(p_\perp \cdot \epsilon^*) \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\} + \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_\perp \hat{\epsilon} \hat{p}_1 \hat{\epsilon}^* \hat{p}_\perp \gamma^\nu\right\} + \frac{2(1-x)}{x} (p_\perp \cdot \epsilon) \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 \hat{\epsilon}^* \hat{p}_\perp \gamma^\nu\right\} + \frac{2(1-x)}{x} (p_\perp \cdot \epsilon^*) \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_\perp \hat{\epsilon} \hat{p}_1 \gamma^\nu\right\},$$

Using the sum over polarization vectors $\sum \epsilon_\mu \epsilon^*_\nu = -g_{\mu\nu}$ allows us to simplify the traces:

$$(p_\perp \cdot \epsilon)(p_\perp \cdot \epsilon^*) = -p_\perp^2$$

$$(p_\perp \cdot \epsilon) \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 \hat{\epsilon}^* \hat{p}_\perp \gamma^\nu\right\} = -p_\perp^2 \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\}$$

$$\mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_\perp \hat{\epsilon} \hat{p}_1 \hat{\epsilon}^* \hat{p}_\perp \gamma^\nu\right\} = -\mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_\perp \gamma^\rho \hat{p}_1 \gamma_\rho \hat{p}_\perp \gamma^\nu\right\} = 2\mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_\perp \hat{p}_1 \hat{p}_\perp \gamma^\nu\right\} = -2p_\perp^2 \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\}$$

Computing the amplitude in the collinear limit (5)

Thus

$$X = -2p_\perp^2 \frac{(1-x)^2+1}{x^2} \mathrm{Tr}\left\{\hat{p}_2\gamma^\mu\hat{p}_1\gamma^\nu\right\}.$$

Note that since $p_\perp \sim \theta_{13}$ and $x \sim 1$, we have $X \sim \theta_{13}^2$ – i.e. the additional scaling mentioned earlier.

This justifies considering *only* the leading singular term, as all other terms are non-singular $\sim \mathcal{O}(1)$ in the collinear limit.

The use of *physical gluon polarization vectors* in the sum $\sum \epsilon_\mu \epsilon_\nu^* = -g_{\mu\nu}$ was essential in computing his behavior for X.

We can use this result for the square of the amplitude in the collinear limit:

$$\begin{aligned}\lim_{\theta_{13}\to 0} \left|\mathcal{M}(1,2,e^-,e^+,3)\right|^2 &= \frac{g_s^2}{(p_1-p_3)^2}\frac{N_c^2-1}{2}\frac{X}{(p_1-p_3)^2}\left|L_{\mu\nu}\right|^2(p_{e^-},p_{e^+}) + \mathcal{O}(1) \\ &= g_s^2\frac{N_c^2-1}{2}\frac{-2p_\perp^2}{(p_1-p_3)^2}\frac{1}{(p_1-p_3)^2}\frac{(1-x)^2+1}{x^2} \\ &\quad\times \mathrm{Tr}\left\{\hat{p}_2\gamma^\mu\hat{p}_1\gamma^\nu\right\}\left|L_{\mu\nu}\right|^2(p_{e^-},p_{e^+}) + \mathcal{O}(1).\end{aligned}$$

Computing the amplitude in the collinear limit (6)

$$\lim_{\theta_{13}\to 0} \left|\mathcal{M}(1,2,e^-,e^+,3)\right|^2 = g_s^2 \frac{N_c^2-1}{2} \frac{-2p_\perp^2}{(p_1-p_3)^2} \frac{1}{(p_1-p_3)^2} \frac{(1-x)^2+1}{x^2}$$
$$\times \mathrm{Tr}\left\{\hat{p}_2\gamma^\mu\hat{p}_1\gamma^\nu\right\}\left|L_{\mu\nu}\right|^2(p_{e^-},p_{e^+}) + \mathcal{O}(1).$$

Now use the Sudakov decomposition $p_3 = xp_1 + yp_2 + p_\perp$ to write

$$(p_1-p_3)^2 = -2p_1\cdot p_3 = -2yp_1\cdot p_2 = -ys,$$

and recall that we had $p_\perp^2 = sxy \Rightarrow \frac{-2p_\perp^2}{(p_1-p_3)^2} = 2x$.
Then

$$\lim_{\theta_{13}\to 0} \left|\mathcal{M}(1,2,e^-,e^+,3)\right|^2 = g_s^2 \frac{N_c^2-1}{2} \frac{1}{(p_1-p_3)^2} 2\frac{(1-x)^2+1}{x(1-x)}$$
$$\times \mathrm{Tr}\left\{\hat{p}_2\gamma^\mu\hat{p}_1(1-x)\gamma^\nu\right\}\left|L_{\mu\nu}\right|^2(p_{e^-},p_{e^+}).$$

We have the correct scaling $|\mathcal{M}(1,2,e^-,e^+,3)|^2 \sim \theta_{13}^{-2}$ for the leading singularity, and have dropped the terms $\sim \mathcal{O}(1)$.

Factorization in the collinear limit (1)

The Sudakov decomposition also implies, in the collinear limit, $x = \frac{E_3}{E_1}$.

So let's consider a LO amplitude with the momentum p_1 rescaled as $p_1 \to (1-x)p_1 = \frac{E_1 - E_3}{E_1} p_1$.

This means that the energy of the quark is rescaled as $E_1 \to \frac{E_1-E_3}{E_1} E_1 = E_1 - E_3$. Calling this amplitude $\mathcal{M}(1-3, 2, e^-, e^+)$, we find (recalling the earlier expression for the LO amplitude-squared)

$$\left|\mathcal{M}(1-3,2,e^-,e^+)\right|^2 = N_c \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 (1-x) \gamma^\nu\right\} \left|L_{\mu\nu}\right|^2 (p_{e^-}, p_{e^+}).$$

Thus we find

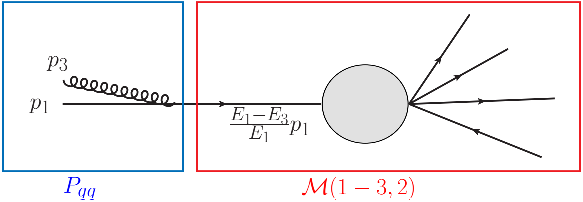
$$\begin{aligned}\lim_{\theta_{13}\to 0} |\mathcal{M}(1,2,e^-,e^+,3)|^2 &= g_s^2 C_F \frac{1}{(p_1-p_3)^2} 2 \frac{(1-x)^2+1}{x(1-x)} |\mathcal{M}(1-3,2,e^-,e^+)|^2 \\ &= -2g_s^2 \frac{1}{(p_1-p_3)^2} P_{qq}\left(\frac{E_1}{E_1-E_3}\right) |\mathcal{M}(1-3,2,e^-,e^+)|^2,\end{aligned}$$

where $P_{qq}(z) = C_F \frac{1+z^2}{1-z}$.

Factorization in the collinear limit (2)

$$\lim_{\theta_{13}\to 0} |\mathcal{M}(1,2,e^-,e^+,3)|^2 = -2g_s^2 \frac{1}{(p_1-p_3)^2} P_{qq}\left(\frac{E_1}{E_1-E_3}\right) |\mathcal{M}(1-3,2,e^-,e^+)|^2.$$

- In the collinear limit, the amplitude-squared **factorizes** into:
  - a hard process without the gluon and with a rescaled momentum for the quark, and
  - a splitting function $P_{qq}(z)$ which depends only on the color charges (C_F) and the energies of the two collinear partons, E_1 and E_3.
- The splitting function $P_{qq}\left(\frac{E_1}{E_1-E_3}\right)$ gives the probability for the collinear emission a gluon of energy E_3 from a "parent" quark with energy E_1, leaving a quark carrying a momentum $\frac{E_1-E_3}{E_1}p_1$ to enter the hard process.



Factorization in the collinear limit (2)

We have shown this factorization for the collinear emission of a gluon for Drell-Yan production, but it is also **universal:**

In the limit of a gluon emission which is collinear to a quark, the amplitude-squared factorizes into a splitting function giving the probability of the collinear emission of the gluon at a given energy by the quark, and a hard amplitude-squared involving the momentum of the quark after the gluon emission.

In the collinear limit, the scaling with the angle is

$$\lim_{\theta_{13}\to 0} |\mathcal{M}(1,2,e^-,e^+,3)|^2 \sim \frac{1}{(p_1-p_3)^2} \sim (1-\cos\theta_{13})^{-1} \sim \theta_{13}^{-2},$$

so the angular integral for real emissions **does not converge!**

$$\int_0^\pi \frac{\mathrm{d}\theta_{13}}{\theta_{13}} \to \infty$$

Because of the universality of factorization, this is true for real gluon emission corrections to any process!

Recap: energy and angular integrals in gluon emissions

We have seen that gluon emissions lead to the integrals

$$\int_0^{E_{\max}} \frac{dE_3}{E_3}; \qquad \int_0^{\pi} \frac{d\theta_{13}}{\theta_{13}}; \qquad \int_0^{\pi} \frac{d\theta_{23}}{\theta_{23}},$$

in the soft gluon limit $E_3 \to 0$ and the collinear limits $\theta_{13} \to 0$ and $\theta_{23} \to 0$, respectively. These integrals do not converge.

Note that it is only the **leading soft and collinear terms** which do not converge. Subleading terms in the soft and collinear limits give rise to the integrals

$$\int_0^{E_{\max}} dE_3; \qquad \int_0^{\pi} d\theta_{13}; \qquad \int_0^{\pi} d\theta_{23},$$

which **do** converge.

Other emissions

Before we address the problem of the non-converging soft and collinear integrals, let's look at what happens to other partons in the soft and collinear limits. For example, we know we have to consider the channel

$$q(p_1)g(p_2) \to e^-e^+q(p_3)$$

in the real corrections to Drell-Yan production. What happens if the final state **quark** becomes soft, $E_3 \to 0$?

The amplitude is

$$\mathcal{M}(1_q, 2_g, e^-, e^+, 3_q) = g_s T^a_{ij} \bar{u}(p_3) \left[\hat{\epsilon} \frac{\hat{p}_2 - \hat{p}_3}{(p_2 - p_3)^2} \gamma^\mu \right] u(p_1) L_\mu(p_{e^-}, p_{e^+}).$$

Setting $\hat{p}_3 \to 0$ in the numerator, squaring and summing over spins, we find

$$\frac{1}{(-2p_2 \cdot p_3)^2} \mathrm{Tr}\left\{ \hat{p}_2 \hat{\epsilon}^* \hat{p}_3 \hat{\epsilon} \hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu \right\} = -\frac{1}{(-2p_2 \cdot p_3)^2} \mathrm{Tr}\left\{ \hat{p}_2 \gamma^\rho \hat{p}_3 \gamma_\rho \hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu \right\}$$

using $\sum \epsilon_\mu \epsilon^*_\nu = -g_{\mu\nu}$.

Soft quark emission

Using $\gamma^\rho \hat{p}_3 \gamma_\rho = \gamma^\rho (2p_{3,\rho} - \gamma_\rho \hat{p}_3) = -2\hat{p}_3$ we have

$$
\begin{aligned}
-\frac{1}{(-2p_2 \cdot p_3)^2} \mathrm{Tr}\left\{\hat{p}_2 \gamma^\rho \hat{p}_3 \gamma_\rho \hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\} &= 2\frac{1}{(-2p_2 \cdot p_3)^2} \mathrm{Tr}\left\{\hat{p}_2 \hat{p}_3 \hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\} \\
&= 2\frac{1}{(-2p_2 \cdot p_3)^2} \mathrm{Tr}\left\{(2p_2 \cdot p_3 - \hat{p}_3 \hat{p}_2)\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\} \\
&= 2\frac{1}{(-2p_2 \cdot p_3)} \mathrm{Tr}\left\{\hat{p}_2 \gamma^\mu \hat{p}_1 \gamma^\nu\right\} \sim \frac{1}{E_3},
\end{aligned}
$$

where I have used $\hat{p}_2 \hat{p}_2 = p_2 \cdot p_2 = 0$.

Thus soft quark emission gives rise to a convergent integral – **there are no divergences associated with soft quark or antiquark emissions!**

Collinear emissions

There are, however, divergences in $q(p_1)g(p_2) \to e^-e^+ + q(p_3)$ that arise if the final state quark becomes **collinear** to the initial state gluon, i.e. $\theta_{23} \to 0$.
In this limit, the amplitude factorizes as before, with the rescaled quark entering the hard process. However, the splitting function changes. We can determine the form of the new splitting function by modifying P_{qq}.

We first consider the color factor of the splitting function. Since the real emission and LO initial states are now different (qg and $q\bar{q}$ respectively), we now need to take into account the *color averaging* of the initial states

$$\langle qg \rangle = \frac{1}{N_c(N_c^2-1)} \qquad \langle q\bar{q} \rangle = \frac{1}{N_c^2}.$$

We then need to multiply the color factor by the $\langle qg \rangle / \langle q\bar{q} \rangle = N_c/(N_c^2-1)$, so the color factor becomes $C_F N_c/(N_c^2-1) = 1/2 = T_R$.

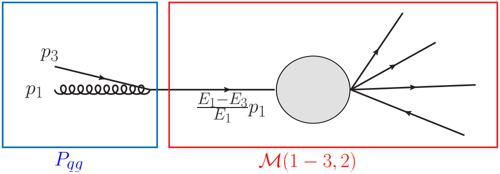
The argument of the splitting function changes by $E_1 \leftrightarrow E_3$, becoming $E_3/(E_3 - E_1)$.

Collinear emission of a final state quark

Putting this together, we can rewrite the splitting function for **an emitted quark becoming collinear to an initial state gluon** as

$$P_{qg}(z) = T_R(z^2 + (1-z)^2).$$

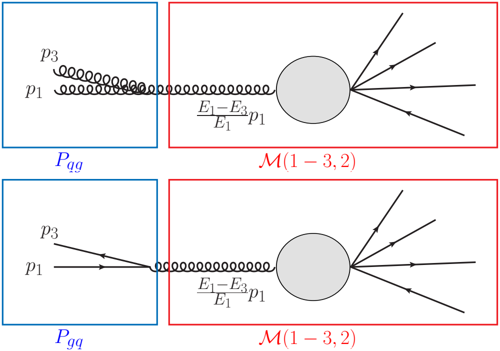
with $z = 1 - E_3/E_1$.



There are also splitting functions that don't appear in Drell-Yan but do arise in other processes, e.g. in $g(p_1)g(p_2) \to H + g(p_3)$ when $\theta_{13} \to 0$.
As before, in these collinear limits the amplitude factorizes into a hard process involving a parton with rescaled momentum, and a splitting function.

Other collinear emissions

- In general, for a splitting $a \to bc$ with c collinear to a and b entering the hard process, the associated splitting function is called P_{ba}.
- We have already seen a quark entering the hard process after the splittings $q \to qg$ and $g \to q\bar{q}$, with splitting functions P_{qq} and P_{qg} respectively.
- We can also have a gluon entering the hard process, after the splitting $g \to gg$ or $q \to gq$, with splitting functions P_{gg} and P_{gq}.



Handling infrared divergences (1)

We have now seen that infrared divergences:

- appear whenever emitted gluons become **soft**, or when a parton is emitted **collinear** to another parton,
- give rise to **universal factorizations**, characterized by **eikonal functions** for soft emissions and **splitting functions** for collinear emissions,
- lead to phase space integrals that **do not converge.**

Usually the appearance of infinities indicates that there is something wrong with our approach.

Perhaps because we are using **gluons** and **quarks** as degrees of freedom below the QCD scale Λ_{QCD}?

We could then try to introduce an *IR cutoff* at Λ_{QCD}:

$$\int_{\Lambda_{\mathrm{QCD}}}^{E_{\max}} \frac{\mathrm{d}E_3}{E_3}; \qquad \int_{\Lambda_{\mathrm{QCD}}}^{\pi} \frac{\mathrm{d}\theta_{13}}{\theta_{13}}; \qquad \int_{\Lambda_{\mathrm{QCD}}}^{\pi} \frac{\mathrm{d}\theta_{23}}{\theta_{23}},$$

which gives rise to terms $\sim \log(\Lambda_{\mathrm{QCD}})$ – i.e. our perturbative calculation is now sensitive to *non-perturbative effects*.

Handling infrared divergences (2)

This is also a problem as **we have no method for analytically calculating non-perturbative QCD effects from first principles.**

Thankfully, it is not the end of the story.

Recall that for a NLO calculation, we also need to include the *virtual contributions*. When we compute these, we see infrared divergences arising from loop integrals when the *loop momentum* becomes **soft** and/or **collinear** to another parton.

The infrared divergences are **guaranteed** to cancel when we sum the real and virtual corrections (Bloch-Nordsieck and Kinoshita-Lee-Nauenberg theorems).

This means that perturbative QCD corrections are *insensitive* to the IR physics.

The presence of IR divergences in the real and virtual corrections separately is an indication that this separation is not physical, it is something we impose as a way of organizing the calculation.

We will see the cancellation of poles for color singlet production *explicitly* in tomorrow's lecture.

Regularizing infrared divergences

To combine real emission and virtual corrections, we need a way of *regularizing* the divergences that appear in the phase space integrals so that they become manageable. With the virtual corrections in mind, a natural approach is to treat the dimensions of space-time as a non-integer $d = 4 - 2\epsilon$, with the aim of taking the limit $\epsilon \to 0$ at the end of the calculation.

We start with the d-dimensional phase space measure

$$[\mathrm{d}p] = \frac{|\vec{p}|^{d-2}\mathrm{d}|\vec{p}|\ \mathrm{d}\Omega_{d-1}}{2p_0(2\pi)^{d-1}} = \frac{E^{d-3}\mathrm{d}E\ \mathrm{d}\Omega_{d-1}}{2(2\pi)^{d-1}}, \tag{3}$$

where we have used, for a massless particle, $|\vec{p}| = p_0 = E$. The angular phase space is defined recursively

$$\mathrm{d}\Omega_{d-1} = \mathrm{d}\cos\theta(1 - \cos^2\theta)^{d/2-2}\mathrm{d}\Omega_{d-2}$$

and the solid angle is given by

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

where Γ is the Euler gamma function. We can easily check by setting $d = 4$ that we recover the usual four-dimensional phase space measure.

Soft and collinear factorization in d-dimensions (1)

In d-dimensions, the factorization expressions in the soft and collinear limits should include a **renormalization scale** $\mu^{2\epsilon}$ to balance the $E_3^{-2\epsilon}$ that now appears in the energy integration measure.
We already saw this $\mu^{2\epsilon}$ in the expression for the universal soft gluon factorization:

$$|\mathcal{M}(p_1, p_2, \ldots, p_n; q)|^2 = -4\pi\alpha_s\mu^{2\epsilon}\sum_{i,j=1}^{n} \mathcal{S}_{ij}(q)\vec{T}_i \cdot \vec{T}_j \, |\mathcal{M}(p_1, p_2, \ldots, p_n)|^2 \, ,$$

The splitting functions P_{ij} also acquire terms dependent on ϵ, e.g.

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z} + \epsilon(1-z)\right).$$

Evaluating soft and collinear integrals in d-dimensions

In the soft limit, the energy integral is now

$$\int_0^{E_{\max}} \mathrm{d}E_3 \frac{E_3^{1-2\epsilon}}{E_3^2} = \int_0^{E_{\max}} \frac{\mathrm{d}E_3}{E_3^{1+2\epsilon}} = -\frac{1}{2\epsilon} E_{\max}^{-2\epsilon} = -\frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0).$$

The collinear integral is

$$\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta (1-\cos^2\theta)^{d/2-2}}{1-\cos\theta} = \int_0^1 \frac{\mathrm{d}x}{(4x(1-x))^{\epsilon}} \frac{1}{x} = 2^{-2\epsilon} B(-\epsilon, 1-\epsilon)$$

where I have used $x = \frac{1-\cos\theta}{2}$ in the first equality, and the definition of the *Euler Beta-function* B in the second equality. Using $x\Gamma(x) = \Gamma(1+x)$

$$2^{-2\epsilon} B(-\epsilon, 1-\epsilon) = 2^{-2\epsilon} \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{2^{-2\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0),$$

So both the soft and collinear integrals can now be evaluated, and give rise to $1/\epsilon$ poles which capture the singularities in the real corrections.

Preview of next lecture

When we sum the real and virtual corrections, these $1/\epsilon$ poles are **guaranteed** to cancel against the $1/\epsilon$ poles that appear in the virtual corrections.

This leaves a finite result and allows us to take the limit $\epsilon \to 0$ to recover the usual four space-time dimensions.

However, summing the real and virtual corrections in such a way as to cancel the $1/\epsilon$ poles is not straightforward. This is due to the **different number of final state particles**: the virtual corrections inhabit an n-particle phase space while the real corrections inhabit an $(n+1)$-particle phase space.

The methods of manipulating the real corrections to allow the cancellation of the poles are called *subtraction schemes* and will be the topic of the next two lectures.