Scattering amplitudes II

Summary I

→ tree-level gluon scattering

→ colour ordering

☆ two counting methods

- both <u>recursive</u>
- off-shell recursion for ordered amplitudes

  ↳ Berends-Giele

→ Spinor-helicity

pol. vec.

$$A(p_i \cdot p_j\,,\ \varepsilon_i \cdot p_j\,,\ \bar{u}\gamma^\mu v, \ldots)$$

$$= A(\langle i j \rangle, [i j])$$

now $p_i^2 = 0$ manifest

$\not{p}\, v(p) = 0$ and $\bar{u}(p)\not{p} = 0$

in Weyl basis

$$(\sigma\cdot p)^{\dot\alpha\alpha}\, {}_{\alpha}|p\rangle = 0$$

$$(\bar\sigma\cdot p)_{\alpha\dot\alpha}\, {}^{\dot\alpha}|p] = 0$$

set the normalisation

$$(\bar\sigma\cdot p)_{\alpha\dot\alpha} = {}_{\alpha}|p\rangle [p|_{\dot\alpha} \qquad (*)$$

$$(\sigma\cdot p)^{\dot\alpha\alpha} = {}^{\dot\alpha}|p]\langle p|^{\alpha} \qquad (**)$$

raise and lower su(2) indices with the ε- tensor

$$|p]^{\dot\alpha} = [p|_{\dot\beta}\, \varepsilon^{\dot\beta\dot\alpha}$$

$$\varepsilon^{\dot\beta\dot\alpha} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Spinor products and identities

$$\langle p\, q\rangle = \langle p|^{\alpha}{}_{\alpha}|q\rangle$$

$$= \varepsilon^{\alpha\beta}{}_{\beta}|p\rangle_{\alpha}|q\rangle$$

$$= -\langle q\, p\rangle$$

Sim. $[pq] = -[q\, p]$

consequence $(*)$, $(**)$ is

- $\frac{1}{2}\langle p\, \bar\sigma^{\mu} p] = \frac{1}{2}[p\, \sigma^{\mu} p\rangle = p^{\mu}$

☆ $\langle p\, \bar{\sigma}^{\mu} q] \langle r\, \bar{\sigma}\, s]$

$$= -2 \langle p r \rangle [q s]$$

☆ $\langle p\, \not{q}\, r] = \langle p q \rangle [q r]$

if $q^2 = 0$

☆ $p_i \cdot p_j = \frac{1}{2}(p_i + p_j)^2$

$$= \frac{s_{ij}}{2} = \frac{\langle i j \rangle [j i]}{2}$$

☆ schouten identity

any 2-spinor can be written as a linear combination of 2 other 2-spinors.

$$|p\rangle = a|q\rangle + b|r\rangle$$

$$\Rightarrow \langle qp\rangle = a\cancelto{0}{\langle qq\rangle} + b\langle qr\rangle$$

$$\Rightarrow b = \frac{\langle qp\rangle}{\langle qr\rangle}$$

$$\langle rp\rangle = a\langle rq\rangle + b\cancelto{0}{\langle rr\rangle}$$

$$\Rightarrow a = \frac{\langle rp\rangle}{\langle rq\rangle}$$

$$|p\rangle = \frac{\langle rp\rangle}{\langle rq\rangle}|q\rangle + \frac{\langle qp\rangle}{\langle qr\rangle}|r\rangle$$

$$|p\rangle\langle rq\rangle = |q\rangle\langle rp\rangle - |r\rangle\langle qp\rangle$$

BCFW recursion relations

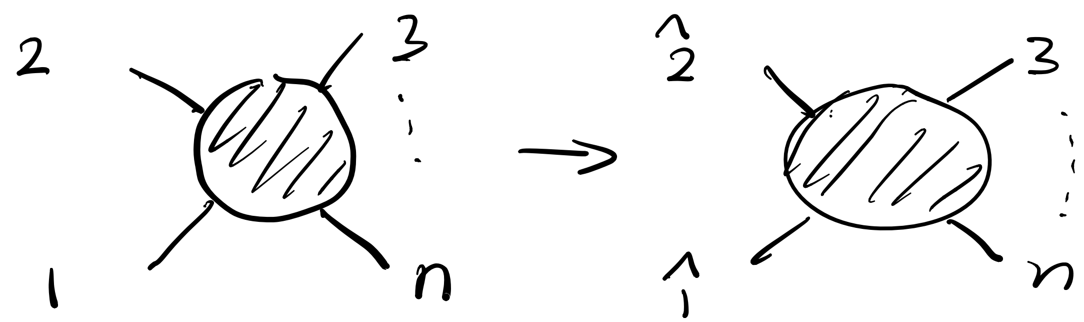
Britto - Cachazo - Feng - Witten (2005)

$$\frac{1}{2\pi i}\oint dz \frac{A(z)}{z} = A(0) + \sum_{\text{res } z_i} \frac{A(z_i)}{z_i}$$

z is a complex variable

$A(z)$ is an analytic continuation of A

One such complex deformation:



$$\hat{p}_1^\mu = p_1^\mu + z n^\mu$$

$$\hat{p}_2^\mu = p_2^\mu - z n^\mu$$

$$\hat{p}_2^2 = \tilde{p}_1^2 = 0 \Rightarrow n^2 = 0$$

$$n \cdot p_1 = 0$$

$$n \cdot p_2 = 0$$

$n = \langle 1 \, \bar{\sigma}^\mu \, 2]$ or

$n = \langle 2 \, \bar{\sigma}^\mu \, 1]$ will

solve these conditions.

- $\displaystyle\int \frac{dz}{z} A(z) = 0$ if

  $A(z)$ vanishes at ∞

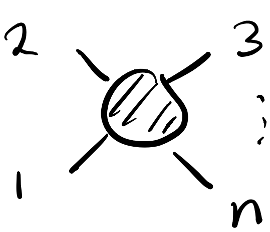
- $\displaystyle\operatorname*{Res}_{z_i} \frac{A(z_i)}{z_i}$ factorises

$$\text{Res}\ \frac{A(z_i)}{z_i} = \sum_h A(\hat{2}, 3 \cdots i, -\hat{P}_{2,i}^{-h}) \times \frac{1}{P_{2,i}^2} \times A(\hat{P}_{2,i}^{h}, i+1, \cdots, n, \hat{1})$$

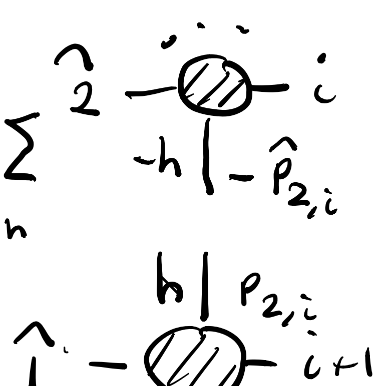
fixed z_i $\quad \hat{P}_{2,i}^{\mu} = P_{2,i}^{\mu} - z_i n^{\mu}$

such that $\hat{P}_{2,i}^2 = 0$

$$z_i = \frac{P_{2,i}^2}{2 P_{2,i} \cdot n}$$



$$= \sum_i \sum_h$$



$$\frac{1}{P_{2,i}^2}$$

h helicity $+$ or $-$

☆ we needed $A(\infty) = 0$

turns out in QCD this is possible ✓

→ depends on the helicities of the external legs.

☆ number of terms in the sum is <u>much</u> less than the ordered amplitudes

Let's look at some examples

all minus amplitudes

start from 3-point amplitudes

$$A(1^- 2^- 3^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

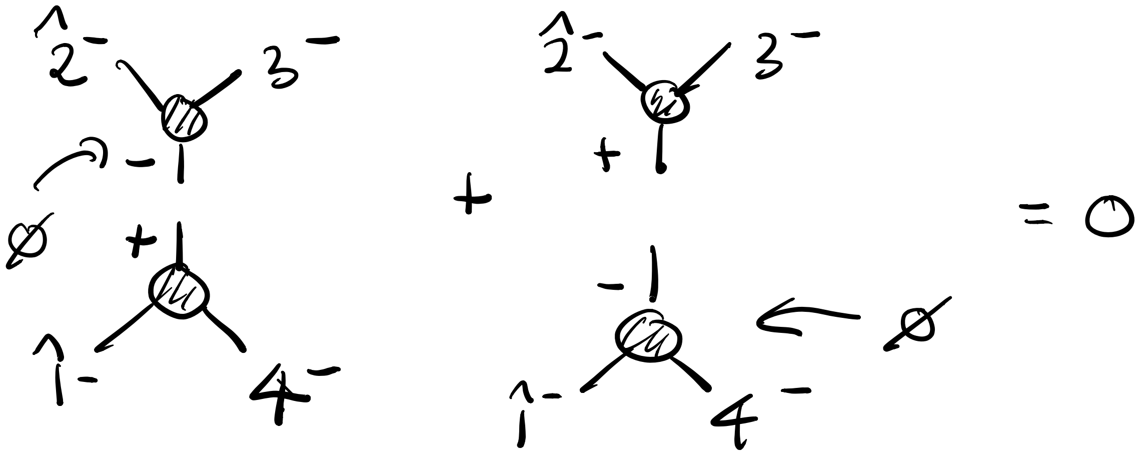
(can prove this using Feynman diagrams)

$$A(1^- 2^- 3^-) = 0$$

now feed this into the recursion relation.

- $A(1^- 2^- 3^- 4^-)$ shift

  1 and 2 using $\langle 1\bar{\sigma}^\mu 2] = n^\mu$



quickly show that

$$A(1^-, 2^- \cdots n^-) = 0$$

☆ how are the 3-gluon amplitudes defined?

$$s_{12} = \langle 12 \rangle [21]$$

$$p_1 + p_2 + p_3 = 0$$

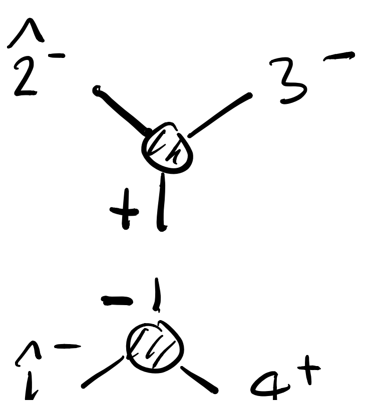
$$\Rightarrow \quad s_{12} = p_3^2 = 0$$

☆ BCFW is using complex momenta. This means that either $\langle 12 \rangle = 0$ or $[12] = 0$.

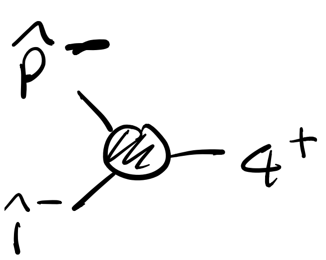
→ very important that 3-gluon amplitude $A(1^- 2^- 3^+)$ was only written with $\langle ij \rangle$

$$A(1^- 2^- 3^- 4^+)$$

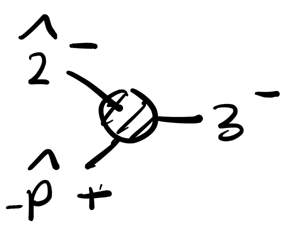
shift $n^\mu = \langle 1 \bar{\sigma}^\mu 2]$



deformation keeps $|\hat{1}\rangle = |1\rangle$



is unshifted



is shifted

e.g. $|\hat{2}\rangle = |2\rangle + z_3 |1\rangle$

can show using defn. from before that

$\Rightarrow \langle \hat{2}\, 3 \rangle = 0$ etc.

$$\Rightarrow A(\hat{2}^- 3^- -\hat{P}_{23}^+) = 0$$

this shows us that

$$A(1^- 2^- 3^- 4^+) = 0$$

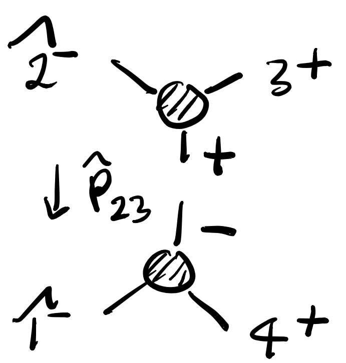
We can now quickly show that $A(1^- 2^- \cdots (n-1)^- n^+) = 0$

First non-zero amplitude is two -ve helicities and the rest +ve

$$A(1^- 2^- 3^+ \cdots n^+)$$

$$= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n 1 \rangle}$$

Consider $n=4$ with $n^\mu = \langle 1 \bar{\sigma}^\mu 2]$



$$= \frac{\langle \hat{1} \hat{P}_{23} \rangle^4}{\langle \hat{1} \hat{P}_{23} \rangle \langle \hat{P}_{23} 4 \rangle} \frac{1}{S_{23}}$$

$$\times \quad \frac{[3(-\hat{P}_{23})]^4}{[3(-\hat{P}_{23})][(-\hat{P}_{23})\hat{2}][\hat{2}3]} \qquad (1)$$

need to use $|-p\rangle = i|p\rangle$

$|-p] = i|p]$

the $++-$ amplitude can be obtained from $--+$ by changing $\langle \ \rangle$ to $[\ ]$

$$A(1^+2^+3^-) = \frac{[12]^4}{[12][23][31]}$$

(1) can be shown to be

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

using spinor-helicity identities.

- On-shell QCD amplitudes @ tree-level are extremely simple!

Remarks

- everything was built from 3-point amplitudes

  $\rightarrow$ no 4-gluon interaction!

- applies very generally to gauge theory amplitudes (and more) including W + QCD

Higgs + QCD

can also be done with BCFW.

Loop Level amplitudes

Unitarity method allows loop amplitudes to be constructed from on-shell tree-amplitudes.

S-matrix $S = 1 + iT$

$$1 = SS^\dagger \Rightarrow TT^\dagger = -i\left(T - T^\dagger\right)$$

Scattering amp. $A_{p \to p'}$

$$A_{p \to p'} = \langle p | T | p' \rangle$$

(NB <u>not</u> spinor helicity $|p\rangle$!)

- usual field theory notation.

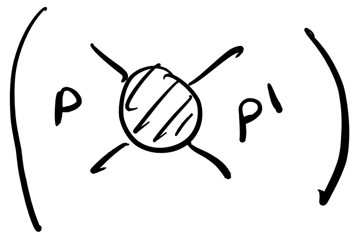
now use the complete set of states

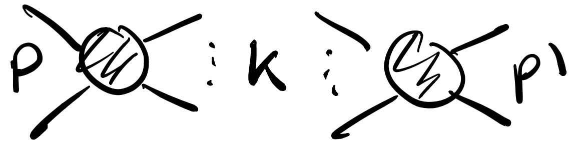
$$\sum\!\!\!\!\!\!\int |k\rangle\langle k| = 1$$

$$T T^{\dagger} = -i\left(T - T^{\dagger}\right)$$

$$\mathrm{Disc}_{p \to p'}\left(A_{p \to p'}\right) = \sum\!\!\!\!\!\!\int A_{p \to k}\, A_{k \to p'}$$

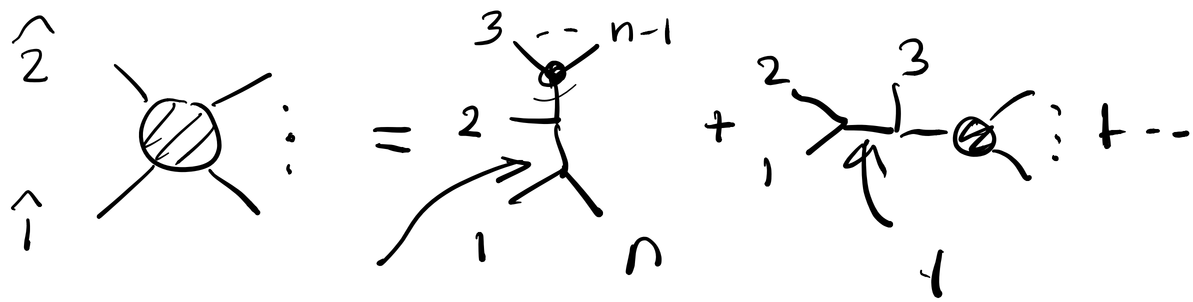
diagrammatically

$\mathrm{Disc}_{p \to p'}$ 

$= \displaystyle\sum\!\!\!\!\!\!\!\int_X$ 

expand this relation perturbatively to find relations between loops and trees.

Factorisation of BCFW residues



$$\frac{1}{p_{1n}^2} \qquad\qquad \frac{1}{p_{12}^2}$$

under shift $\quad 1 \to \hat{1} = 1 + z n$

$$\hat{2} = 2 - z n$$

$$P_{12} \to P_{12} \quad \text{no } z \text{ dep.}$$

$$\Rightarrow \text{no residue}$$

$$p_{1n}^2 \to p_{1n}^2 + z\, p_{1n} \cdot n$$

ref

particle n

$$A^\mu \cdot \varepsilon_\mu(p) = A$$

$$A^\mu \cdot p_\mu = 0$$

$$\varepsilon_+^\mu(p,n) = \frac{[p\,\sigma\, n\rangle}{\sqrt{2}\,[pn]}$$

$$\varepsilon_-^\mu(p,n) = -\frac{\langle p\,\bar{\sigma}\, n]}{\sqrt{2}\,\langle pn\rangle}$$

$$\Rightarrow \sum_h \varepsilon_h^\mu \varepsilon_{-h}^\mu = -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p\cdot n}$$

check amplitude is independent of ref vector n.

$$\Leftrightarrow \quad A^\mu \cdot p_\mu = 0$$

BCFW is gauge invariant by construction.